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A purely adaptive controller to synchronize and control chaotic systems

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Abstract

Following the work presented by the author in a previous paper, a model reference adaptive controller is proposed to control or synchronize chaotic systems. This is achieved by exploiting the boundedness of chaotic evolutions. The synthesis is carried out in two stages. Firstly, an adaptive controller containing a fixed gain linear action is presented and applied to the control of a Lorenz system. Then, when the linear term of the error equation is characterized by a Hurwitz matrix, the control law is further simplified to a purely discontinuous action whose amplitude is adaptively estimated. Finally, numerical results are presented for the case in which this simpler controller is used to synchronize two models of the Chua circuit, characterized by a Hurwitz linear matrix.

1. Introduction

The problem of controlling chaotic evolutions of nonlinear dynamical systems or of synchronizing two or more equivalent systems has been approached in several different ways [1]. The OGY method, for instance, first introduced in Ref. [2], has been successfully applied to achieve control in many different cases [3,4]. In the engineering context, several standard techniques have been used to solve the problem, as for instance linear state feedback [5] or stochastic control [6]. Adaptive control strategies have also been applied [7–9], showing that even standard or slightly modified control engineering methods can be applied to achieve the desired goal.

This paper is an extension of the work presented by the author in Ref. [7], in which an adaptive controller for chaotic systems was designed and tested. Here, the original adaptive scheme is further modified.

First, the requirement of knowing a continuous function, which upper-bounds the nonlinearity of the system to control, is removed. Then, any linear feedback action with fixed gains is omitted, in case the linear part of the error equation is characterized by a Hurwitz matrix. Therefore the original control law is modified into one in which the effort of achieving synchronization and control is left only to a pure adaptive term. Thus, the structure of the original controller is simplified removing the synthesis of a fixed gain linear action. As in other control techniques, knowledge of the linear part of the reference model is still required, but the control requires only boundedness of the nonlinearities involved. In contrast with other MRAC techniques, in this case the controller contains only a discontinuous contribution, whose amplitude is adaptively estimated. Moreover, in this

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case the assumption of bounded nonlinearities can be deduced from the fact that both the reference model and the plant are supposed to evolve along an attractor, hence their evolutions are bounded in a compact set of the phase space.

Finally, the method is applied, first, to control a Lorenz system, characterized by a model having a non-Hurwitz linear part and, then, to synchronize two models of the Chua circuit, described by a non-Hurwitz linear matrix. In addition to this the two systems chosen are characterized by different nonlinearities. The numerical results confirm the theoretical background developed in the paper.

2. The original controller

The adaptive strategy, presented in the paper mentioned above, was concerned with the problem of controlling and synchronizing chaos. Namely, given two systems,

$$\dot{x} = f(x,t) + Bu, \quad x \in \mathbb{R}^n,$$
$$\dot{y} = g(y,t), \quad y \in \mathbb{R}^n, \tag{1}$$

with $u \in \mathbb{R}^m, B \in \mathbb{R}^{n\times m}$, the problem consists of choosing an appropriate controller $u = u(t)$ in such a way as to have

$$\lim_{t\to\infty} |x(t) - y(t)| = 0. \tag{2}$$

Remark. As pointed out in Ref. [10], (1) and (2) can describe both the problems of controlling and synchronizing a chaotic system. If the reference model is chosen to be a chaotic system, identical to the plant, starting from different initial conditions, (2) describes a synchronization problem while if the reference model evolves along a periodic orbit a chaos control problem is described.

The strategy proposed in Ref. [7] can be outlined as follows.

First, the error equation is formed,

$$\dot{e}(t) = \dot{x}(t) - \dot{y}(t) = f(x,t) - g(y,t) + Bu. \tag{3}$$

An orthogonal projection operator $\Pi : \mathbb{R}^n \to \mathrm{Im}(B)$ is found so that (3) could be rewritten as

$$\dot{e}(t) = Le(t) + B[h(x,t) - l(y,t) + u],$$

where $Le(t)$ is the projection of $f(x,t) - g(y,t)$ on the complementary space of $\mathrm{Im}(B)$, which is assumed to be linear, and $h(x,t)$, $l(y,t)$ are the projection on $\mathrm{Im}(B)$ of $f(x,t)$ and $g(y,t)$ respectively.

Then, given a gain matrix $K \in \mathbb{R}^n$, such that $\widehat{L} = L - BK$ is a Hurwitz matrix, that is all its eigenvalues are in the left half of the complex plane, we solve the Lyapunov equation

$$P\widehat{L} + \widehat{L}^{\mathrm{T}}P + I = 0. \tag{4}$$

Finally, we exploit the fact that the reference model is evolving either on an attractor. Hence its evolution is bounded, i.e. $|l(y,t)| \leqslant W$, $W \in \mathbb{R}^+$, and we can form the controller

$$u(t) = -Ke(t) - k(t)[1 + \phi(x)]\|B^{\mathrm{T}}Pe\|^{-1}B^{\mathrm{T}}Pe. \tag{5}$$

In (5) $\phi(x)$ is a continuous function upper-bounding the nonlinearity of the system to control and $k(t)$ is adaptively estimated according to the law

$$\dot{k}(t) = [1 + \phi(x)]\|B^{\mathrm{T}}Pe\|. \tag{6}$$

Using an appropriate Lyapunov function, it is possible to prove that the error asymptotically decays to zero, while $k(t)$ tends toward a bounded value [7].

3. A modified adaptive approach

If we suppose, now, that the system to control is evolving in a chaotic regime (hence its trajectories are bounded to a compact set) and that its nonlinearity is upper bounded by a continuous function $\phi(x)$, we can then deduce that

$$|\phi(x)| \leqslant T, \quad T \in \mathbb{R}. \tag{7}$$

Hence, we can assume the nonlinearity of the system to control to be bounded.

In that case, the adaptive estimation law (6) can be modified to

$$\dot{k}(t) = \|B^{\mathrm{T}}Pe\|,$$

without losing either the global asymptotic stability of the origin of the error system (3) or the boundedness of $k(t)$.

Assuming (7) to be valid, we obtain the following result.

Theorem 1. Let $P \in \mathbb{R}^{n\times n}$ be the positive definite solution of (4) and let

$$\dot{k}(t) = \|B^{\mathrm{T}}Pe\|.$$

The controller

$$u(t) = -Ke(t) - k(t)\|B^{\mathrm{T}}Pe\|^{-1}B^{\mathrm{T}}Pe$$

guarantees that for every initial condition $(e(0), k(0)) = (e^0, k^0)$:

(1) $\lim_{t\to\infty} k(t) = k^* < +\infty$;
(2) $\lim_{t\to\infty} e(t) = 0$.

Proof. Consider the function

$$V(e,k) = e^{\mathrm{T}}Pe + \tfrac{1}{2}(W+T-k)^2. \tag{8}$$

We have that $V(e,k)$ is greater than zero for all $(e,k) \in \mathbb{R}^n \times \mathbb{R}$.

Moreover differentiating (8) we get

$$\begin{aligned}\dot{V}(e,k) &= \dot{e}^{\mathrm{T}}Pe + e^{\mathrm{T}}P\dot{e} - (W+T-k)\dot{k}\\ &\leqslant -\tfrac{1}{2}\|e\|^2 + k\|B^{\mathrm{T}}Pe\|^{-1}e^{\mathrm{T}}PBB^{\mathrm{T}}Pe\\ &\quad + e^{\mathrm{T}}PBh(x,t) - e^{\mathrm{T}}PBl(y,t)\\ &\quad - (W+T-k)\|B^{\mathrm{T}}Pe\|\\ &\leqslant -\tfrac{1}{2}\|e\|^2 + (\|l(y,t)\| + \|h(x,t)\|)\|B^{\mathrm{T}}Pe\|\\ &\quad - (W+T)\|B^{\mathrm{T}}Pe\|\\ &\leqslant -\tfrac{1}{2}\|e\|^2.\end{aligned}$$

Therefore, along the solution $(e(t), k(t))$

$$\dot{V}(e(t),k(t)) \leqslant -\tfrac{1}{2}\|e\|^2,$$

for almost all t.

Hence $(e(t), k(t))$ is bounded and the proof can be completed as in Ref. [7], without substantial modifications.

3.1. Example (controlling a Lorenz system)

Given two Lorenz systems with different parameter values, associated with two distinct attractors, an equilibrium point and a chaotic attractor respectively, we want to find an appropriate controller to make the chaotic Lorenz system to behave as the nonchaotic one.

First, we notice that the evolution of both the systems are bounded. Then, by looking at the structure of the Lorenz model

$$\dot{x} = \begin{pmatrix} -\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{pmatrix} x + \begin{pmatrix} 0\\ -x_1x_3\\ x_1x_2\end{pmatrix},$$

where $\sigma = 10, r = 28, b = \frac{8}{3}$, we decided to add the control only to the second state of the chaotic system, hence $B = [\,0\ 1\ 0\,]^{\mathrm{T}}$. Therefore, if we call σ', r', b' the parameters of the reference model (at the equilibrium point), the error equation (3) becomes

$$\dot{e}(t) = Le(t) + r(x(t), y(t)) + Bu,$$

where

$$L = \begin{pmatrix} -\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{pmatrix},$$
$$r(x,y) = \begin{pmatrix} -(\sigma+\sigma')y_1 + (\sigma+\sigma')y_2\\ (r+r')y_1 - 2y_2 - x_1x_3 + y_1y_3\\ -(b+b')y_3 + x_1x_2 + y_1y_2\end{pmatrix},$$
$$B = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

At this end, it is relevant to point out that the linear matrix L is not Hurwitz, having an eigenvalue with positive real part. Therefore a linear gain matrix K is chosen in order to have a stable $\widehat{L} = (L - BK)$. Hence, the Lyapunov equation is solved and the controller (5) is synthesized. Figs. 1–3 show the evolutions of the three components of the error, the adaptively estimated gain $k(t)$ and the control input $u(t)$ respectively. As we can see, control is achieved after a relatively short transient, while the gain evolves towards a constant value and the control input decays to zero, after only a few switchings. Notice that the controller needs a fixed gain linear feedback in order to stabilize the linear part of the error system.

4. A purely adaptive action

The controller synthesized above consists of two different contributions: a linear feedback term and a

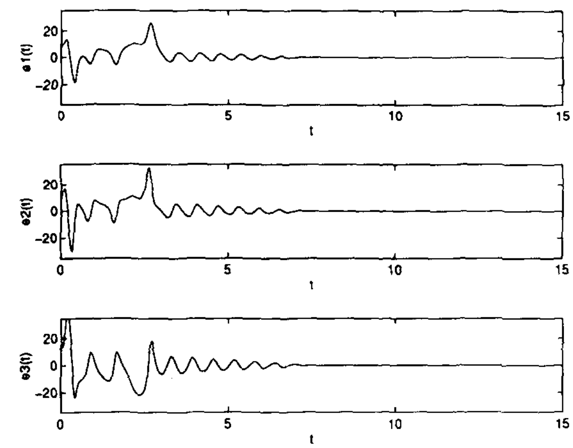

Fig. 1. Error dynamics for the controlled Lorenz system.

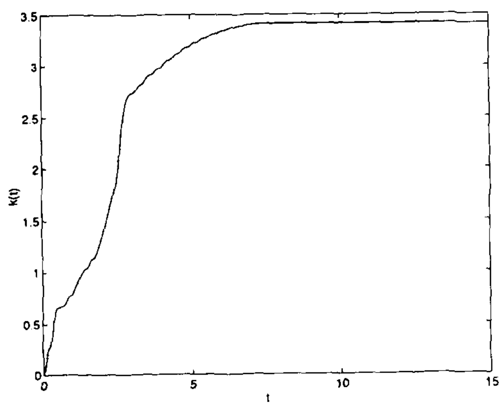

Fig. 2. Evolution of the gain.

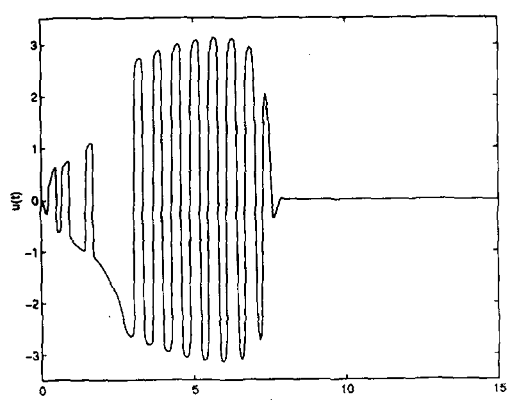

Fig. 3. Control input evolution.

discontinuous action whose amplitude is adaptively estimated.

If we now suppose that in the linear term of the error Eq. (3), the linear matrix L is a Hurwitz matrix (all eigenvalues having negative real parts), we can omit the linear feedback term, considering the controller

$$\dot{k}(t) = \|B^{\mathrm{T}}Pe\|,$$
$$u(t) = -k(t)\|B^{\mathrm{T}}Pe\|^{-1}B^{\mathrm{T}}Pe, \qquad (9)$$

which consists of a pure adaptive contribution.

If the matrix L is Hurwitz, we can prove that there exists a positive definite matrix P which satisfies the Lyapunov equation

$$PL + L^{\mathrm{T}}P + I = 0. \qquad (10)$$

and that P is its unique solution [11]. Therefore there is no need for a linear feedback action to stabilize the linear part of the error equation and the control (9), under this hypothesis, guarantees the claim of Theorem 1.

Remark. The control law (9) requires no specific knowledge of the nonlinearities of either the plant or the reference model except the fact that their evolutions are bounded, for example as a consequence of evolving on a chaotic attractor.

Preliminary numerical results by the author have shown that the controller (9) gives excellent results even when L is not Hurwitz. In that case, however, the matrix P cannot be obtained as the solution of the Lyapunov equation, but has to be chosen following a trial and error procedure.

4.1. Example (synchronizing two Chua circuits)

The problem of synchronizing two identical Chua circuits starting from different initial conditions (see Ref. [12]) is solved following the strategy outlined above. Given two identical Chua circuits,

$$\dot{x} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{pmatrix} x + \begin{pmatrix} 10f(x) \\ 0 \\ 0 \end{pmatrix},$$

with $f(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x+1|)$, one is considered as the reference model, $\dot{y} = g(y)$, and

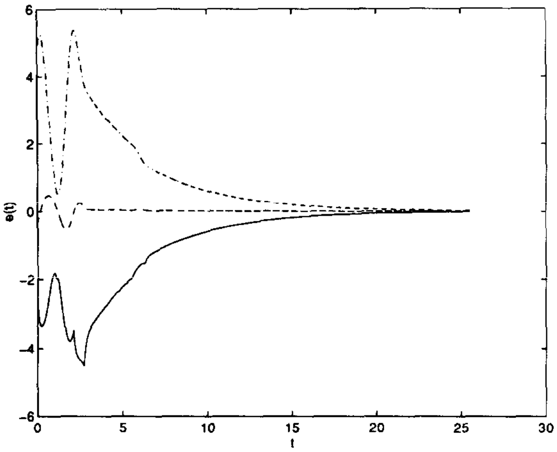

Fig. 4. Error dynamics of the controlled Chua circuit.

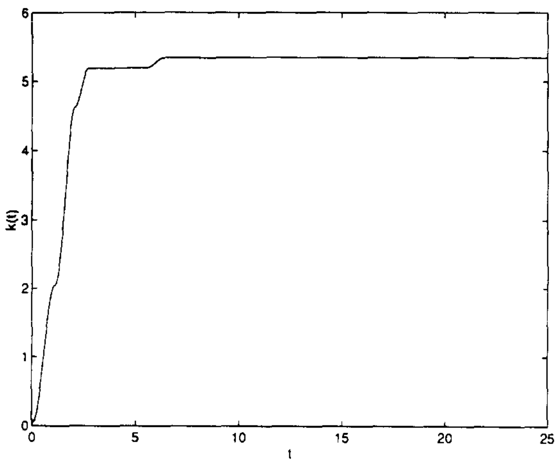

Fig. 5. Evolution of the gain.

the other as the system to control, $\dot{x} = f(x) + Bu$, the control being added only to the first state of the system. Hence the error equation is

$$\dot{e} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{pmatrix} e + \begin{pmatrix} 10[f(x) - f(y)] \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

In this case, the linear part of the error system is a Hurwitz matrix, and so we can apply the control (9).

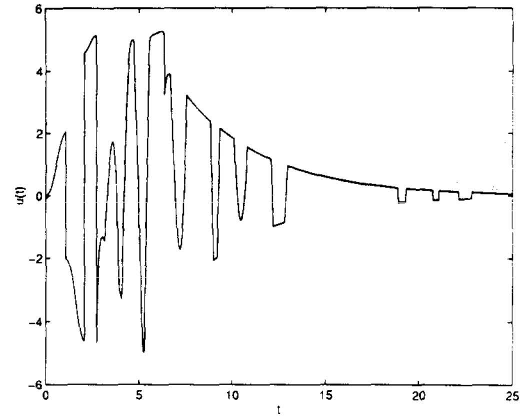

Fig. 6. Control input evolution.

The dynamics of the three component of the error system are shown in Fig. 4, while Figs. 5, 6 report the evolution of $k(t)$ and $u(t)$ respectively.

Even in this case synchronization is obtained via a dissipative control action; once the control goal has been achieved the control is switched off, leaving the systems evolving coherently and synchronously as required.

5. Conclusion

By exploiting the boundedness of chaotic evolutions, we have been able to simplify the adaptive controller structure, removing explicit knowledge of the nonlinearities of the systems involved from the controller. In addition to this, under certain conditions, the control law is further minimized and the controller is left with only a purely discontinuous action, whose amplitude is estimated adaptively. This shows that, to a certain extent, we can exploit chaos to make simpler the control of certain classes of nonlinear systems. It is only because of the bounded evolution of the system to control, that we were able to remove from the controller direct knowledge of the nonlinearities appearing in it. Furthermore, both the theoretical and numerical results seem to suggest that the adaptive scheme proposed in this paper can be successfully applied to a large number of chaotic systems, showing its flexibility and simple implementation. In particular, the model reference adaptive controller (in the two versions presented above) was able to achieve control and synchronization for two systems, the Lorenz system and a model of the Chua circuit, characterized by linear parts with different stability properties and nonlinearities of different types, acting on two of the three states in the Lorenz case and on the first state in the Chua circuits.

Nevertheless, many details still need to be investigated, for instance its robustness against parameter variations and external disturbances. These issues are left for further study.

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