The Double Hook

PHILIPPE BARTISSOL AND LEON O. CHUA, FELLOW, IEEE

Abstract — This paper describes a new strange attractor exhibited by
the same dynamical equations governing Chua's circuit [1], [9] but with a
totally different parameter set. The main difference between the new
double hook attractor (see the orange contour in Fig. 11(a)) and the double
scroll is that the vector field of the new attractor has three real
eigenvalues at the origin, as opposed to the one real and the two complex eigenvalues
of the double scroll. We focus on the circuit realization of this attractor
and reconcile our experimental observations with theoretical predictions
and computer simulations of its structure.

I.  INTRODUCTION

The Double Scroll

The double scroll attractor [2] has been observed in
the circuit of Fig. 1(a), whose only nonlinear element
is a three-segment piecewise-linear resistor, with \( v - i \) characteristic as shown in Fig. 1(b). The dynamics of Chua's
Circuit are described by

\[
\begin{align*}
C_1 \frac{dv_{c1}}{dt} &= G(v_{c2} - v_{c1}) - g(v_{c1}) \\
C_2 \frac{dv_{c2}}{dt} &= G(v_{c1} - v_{c2}) + i_L \\
L \frac{di_L}{dt} &= -v_{c2}
\end{align*}
\]

(1.1)

where \( g(\cdot) \) is the piecewise-linear function in Fig. 1(b),
defined by

\[
g(v_R) = \begin{cases} m_R v_R + \frac{1}{2} (m_1 - m_0) |v_R + B_p| \\ + \frac{1}{2} (m_0 - m_1) |v_R - B_p| \end{cases}\]

(1.2)

Fig. 2 shows the double scroll attractor [2] observed by
solving (1.1) with

\[
\begin{align*}
1/C_1 &= 9, & 1/C_2 &= 1, & 1/L &= 7, & G &= 0.7, \\
m_0 &= -0.5, & m_1 &= -0.8, & B_p &= 1.
\end{align*}
\]

(1.3)

Manuscript received August 6, 1987; revised June 2, 1988. This work
was supported by the Office of Naval Research under Contract N00014-
86-K-0351 and by the National Science Foundation under Grant MIP-
8614000. The cost of color printing is supported by a University-
Industry grant. This paper was recommended by Associate Editor T.
Matsumoto.

P. Bartissol was with the Electronics Research Laboratory, University
of California, Berkeley, CA. He is now with the Ecole Nationale Supéreure
of L'Aeronaute et de l'Espace, Toulouse, France.

L. O. Chua is with the Department of Electrical Engineering and
Computer Sciences, University of California, Berkeley, CA 94720.
IEEE Log Number 882396.

Fig. 2. Double scroll computer simulation. Runge–Kutta integration
routine was iterated 10,000 times, with initial conditions: \( v_{c1}(0) = 0.15264, v_{c2}(0) = -0.02281, i_L(0) = 0.38127 \). (a) Projection onto the
\((i_L, v_{c2})\)-plane. (b) Projection onto the \((i_L, v_{c1})\)-plane. (c) Projection
onto the \((v_{c1}, v_{c2})\)-plane.
The Double Hook

The double hook attractor is a new strange attractor displayed by the same system (1.1) but with the following parameter values:

\[ C_1 = -0.0647, \quad C_2 = 0.3180, \quad L = -0.3005, \quad G = 0.539, \]
\[ m_0 = -0.5013, \quad m_1 = -1.3475, \quad B_p = 1. \quad (1.4) \]

The name "double hook" is chosen to call attention to the geometrical structure of the cross section through the origin (see the orange contour in Fig. 11(a)), which resembles two oppositely pointed "hooks," joined together on one side.

Fig. 3 shows the new attractor corresponding to this set of parameter values with an appropriate time scaling. This simulation has been obtained by computing the dimensionless form of (1.1).

Note that the most significant difference between the two parameter sets (1.3) and (1.4) is that the circuit giving the new attractor requires a negative capacitor and a negative inductor.

The realization of such negative devices is not easy. One technique for synthesizing negative capacitance, or negative inductance, is to use a Negative Impedance Converter (NIC) \[ ^3 \] and a positive-valued capacitance or inductance. One can realize an NIC in two different ways as shown in Fig. 4(a) and (b). In each case, the impedance seen across port 1 is \(-R_1/R_2 \)Z.

However, using NIC’s with capacitors or inductors can cause problems. Besides stability and saturation problems, which one can solve respectively by inverting the polarity of the operational amplifiers and scaling the system, one also has to account for the frequency dependent behavior of a real NIC when used with a dynamical element, i.e., a capacitor, or an inductor.

II. PHYSICAL REALIZATION AND OBSERVATION

Circuit Realization

In order to avoid the problems associated with the physical realization of negative dynamical elements, we preferred to rewrite the equations in order to obtain a circuit using NIC’s exclusively associated with resistors, like the one used in obtaining the double scroll, which we reproduce in Fig. 5(a) for convenience. Let us start with (1.1) considering only positive parameter values, so that the equations corresponding to the circuitry of Fig. 1 can be rewritten as follows, with \( g(\cdot) \) of (1.2) unchanged:

\[ C_1 \frac{dv_1}{dt} = G(v_{C_1} - v_{C_2}) + g(v_{C_1}) \]

\[ C_2 \frac{dv_2}{dt} = G(v_{C_2} - v_{C_1}) + i_L \]

\[ L \frac{di_L}{dt} = v_{C_1}, \quad (2.1) \]
Defining the new variables

\[ i_{L_1} = i_L \]
\[ i_{L_2} = G \theta c_i \]
\[ v_c = G \theta c_i \]

and the new parameters

\[ L_1 = GL \]
\[ L_2 = \frac{C_2}{G} \]
\[ C = \frac{C_1}{G} \]

we obtain

\[
\frac{dv_c}{dt} = -i_{L_2} - g'(v_c) \\
L_2 \frac{di_{L_2}}{dt} = v_c - R(i_{L_2} - i_{L_1}) \\
L_1 \frac{di_{L_1}}{dt} = R(i_{L_2} - i_{L_1}) + R v_c \]

where \( R = 1 \) and \( g'(v_c) \equiv -v_c - g(v_c) \) has the form

\[
g'(v_c) = m_0 v_c + \frac{1}{2} (m_1 - m_0) |v_c + B_p| \\
+ \frac{1}{2} (m_0 - m_1) |v_c - B_p| \quad (2.5)
\]

Note that

\[
m_0' = -1 - m_0 \\
m_1' = -1 - m_1. \quad (2.6)
\]

Therefore, by a rescaling of the system, the parameter set (1.3) becomes, for the new equations (2.5):

\[
L_1 = 0.162, \quad L_2 = 0.590, \quad C = 0.120, \quad R = 1 \\
m_0' = -0.07, \quad m_1' = 1.5, \quad B_p = 1. \quad (2.7)
\]

(2.3) Since (2.4) with parameter set (2.7) is strictly equivalent to (1.1) with parameter set (1.3), the corresponding attractor is identical to that given in Fig. 3.

Note that the transformed circuit whose dynamics are described by (2.4) with elements defined by (2.5) and (2.7) contains only positive-valued dynamical elements. It can easily be built as shown in the schematic of Fig. 5(b).

This circuit requires only two active devices, one negative resistor and one piecewise-linear resistor, both of which can be easily constructed [4], especially because each is grounded at one terminal.

The left op amp is part of an NIC block, which inverts the sign of \( R_3 \). The piecewise-linear resistor is located on the right of the capacitor. Note that unlike in [4], the op amp in Fig. 5 is assumed to operate only in the linear regime. Here, the two diodes are responsible for realizing the piecewise-linear characteristics in Fig. 6(a).
Fig. 6. Constitutive relation of the nonlinear resistor. (a) Ideal piecewise-linear characteristic. (b) Oscilloscope trace of the actual nonlinear characteristic. Horizontal scale: 2 V/div. Vertical scale: 0.2 V/div.

Experimental Observations

After further time scaling and amplitude normalization to ensure that currents and voltages take on reasonable values, the final set of circuit parameter values for (2.4) is:

\[ L_1 = 16.2 \text{ mH, } L_2 = 59 \text{ mH, } C = 3 \text{ nF, } R = 2 \text{ k\Omega} \]

\[ m_0 = -0.035 \times 10^{-3}, \quad m_1 = 0.75 \times 10^{-3}, \quad B_0 = 0.2. \]

(2.8)

Fig. 6(a) gives the constitutive relation of the ideal piecewise-linear resistor and Fig. 6(b) shows the actual characteristics measured in the laboratory. The smooth transition of the curve near the breakpoint is due to the use of germanium diodes. As the required breakpoint voltage was so low, we had to use this type of diode, whose threshold is equal to 0.2 V, instead of the more ideal silicon diodes (threshold: 0.6 V). We also had to consider the internal resistance of the diode, in order to obtain suitable values for the nonlinear resistor. The values \( R_1 \) and \( R_2 \) in Fig. 5(b) can be easily calculated in order to obtain suitable values for \( m_0 \) and \( m_1 \). However, since \( m_0 \) is very low, the internal resistance of the diodes cannot be neglected. Experimentally, we used a variable resistor for \( R_1 \) and tweaked it until we obtained a suitable characteristic on the Negative Resistance Curve Tracer [5] (cf. Fig. 6(b)).

Note that the saturation characteristic of the op amp gives rise to the inevitable eventuality in \( g'(\cdot) \) (vertical segments in Fig. 6(b)). Unfortunately, the saturation occurs in a region very close to the attractor, i.e., \( v_c = 8 \text{ V} \), while the maximum usable range for \( v_c \) in Fig. 6(b) is also about 8 V. In order to prevent the attractor from crossing into this region of operation, we had to adjust further the value of the resistor \( R_1 \) in Fig. 5(b).

Fig. 7(a)–(c) shows various projections of the attractor observed from the circuit realization in Fig. 5(b). We were able to observe \( i_D \) by inserting a small series resistance followed by two buffers and a differential amplifier. We measured \( i_D \) by measuring the proportional voltage across \( R_2 \) (see Fig. 5(b)).

Fig. 8 shows the same attractor viewed from various perspectives using the 3-D-rotator described in [6]. Each picture represents the projection of the attractor from different positions.

Fig. 9(a)–(l) shows the various attractors that can be physically observed during the bifurcation process by tun-
Therefore, by rescaling,

\[
x = \frac{v_C}{B_p}, \quad y = R \frac{L_2}{B_p}, \quad z = R \frac{L_1}{B_p},
\]

\[
\tau = R \frac{L_1}{L_2}, \quad \alpha = \frac{L_2}{R^2C}, \quad \beta = \frac{L_2}{L_1}.
\]

Equation (2.4) is transformed into the following simpler dimensionless form

\[
\begin{align*}
\frac{dx}{d\tau} &= \alpha (-y - f(x)) \\
\frac{dy}{d\tau} &= x - y + z \\
\frac{dz}{d\tau} &= -\beta y
\end{align*}
\]  

(3.3)

where \( f(x) = Rg'(x) \),

\[
i.e., f(x) = \begin{cases} bx + a - b, & x \geq 1 \\ ax, & |x| \leq 1 \\ bx + b - a, & x \leq -1 \end{cases}
\]  

(3.4)

and

\[
a = Rm_1' \quad \text{and} \quad b = Rm_0'.
\]  

(3.5)

Again, Fig. 3 shows the attractor corresponding to (3.3) with the exact parameter set:

\[
\tau = \frac{1}{0.162}, \quad a = 0.590/0.120, \quad \beta = 0.590/0.162,
\]

\[
a = 1.5 \quad \text{and} \quad b = -0.07.
\]  

(3.6)

Note that (3.3) is symmetric with respect to the origin.

To find the equilibrium points, let us solve:

\[
\begin{align*}
y + f(x) &= 0 \\
x - y + z &= 0
\end{align*}
\]  

(3.7)

It follows from (3.7) that the three zeros of \( f(\cdot) \) determine the equilibrium points; namely,

\[
\begin{align*}
P^+ &= (k, 0, -k) \\
P^0 &= (0, 0, 0) \\
P^- &= (-k, 0, k)
\end{align*}
\]  

(3.8)

where \( k = (b - a)/a \) (with the parameter set of (3.6), \( k = 22.4 \), which, when compared to the breakpoint \( B_p = 1 \), explains the large size of the attractor).

Due to the nature of the nonlinearity \( f(\cdot) \), the space can be divided into three piecewise-linear subsets, each subset including one of the above equilibrium points, as follows:

\[
\begin{align*}
D_1 &= \{(x, y, z) : x \geq 1\} \\
D_0 &= \{(x, y, z) : |x| \leq 1\} \\
D_{-1} &= \{(x, y, z) : x \leq -1\}.
\end{align*}
\]  

(3.9)

In each region \( D_1, D_0, \) and \( D_{-1} \), the system (3.3) is linear.

In \( D_1 \) and \( D_{-1} \), (3.3) has one real negative eigenvalue:

\[
\tau_p \approx -0.991 \quad \text{(stable)}
\]

and two complex-conjugate eigenvalues:

\[
\sigma_p \pm j\omega_p = 0.168 \pm j1.112 \quad \text{(unstable)}.
\]
Fig. 9. Closed orbits and chaotic attractors observed near the double hook by tuning \( L_4 \) (projections on the \((v_1, i_2)\)-plane). This set of pictures gives an idea of the bifurcation process (our observation terminates after Fig. 9(i) because of saturation in one of the op amps). (a) \( L_4 = 28.0 \) mH. (b) \( L_4 = 24.0 \) mH. (c) \( L_4 = 21.5 \) mH. (d) \( L_4 = 20.7 \) mH. (e) \( L_4 = 20.0 \) mH. (f) \( L_4 = 19.9 \) mH. (g) \( L_4 = 19.8 \) mH. (h) \( L_4 = 19.4 \) mH. (i) \( L_4 = 19.0 \) mH. (j) \( L_4 = 18.7 \) mH. (k) \( L_4 = 18.65 \) mH. (l) \( L_4 = 18.6 \) mH.
Three real eigenvalues are associated with region $D_0$, one positive:

$$\gamma_0 = 1.279 \quad \text{(unstable)}$$

and two negative:

$$\gamma_2 \approx -3.310 \quad \text{and} \quad \gamma_3 \approx -6.344 \quad \text{(stable)}.$$

Let us examine the structure formed by the principal planes, lines and points, which sustain the attractor.

**Geometric Structure**

Define (see Fig. 10):

- $E^s(P^+)$: the stable eigenspace of region $D_1$ \(\text{(Dim = 1)}\)
- $E^u(P^+)$: the unstable eigenspace of region $D_1$ \(\text{(Dim = 2)}\)
- $E^{u1}(0)$: the unstable eigenspace of region $D_1$ associated with $\gamma_{01}$ \(\text{(Dim = 1)}\)
- $E^{s2}(0)$: the stable eigenspace of region $D_1$ associated with $\gamma_{02}$ \(\text{(Dim = 1)}\)
- $E^{s3}(0)$: the stable eigenspace of region $D_1$ associated with $\gamma_{03}$ \(\text{(Dim = 1)}\)
- $F_{12}(0)$: the subset generated by $E^{u1}(0)$ and $E^{s2}(0)$ \(\text{(Dim = 2)}\)

$F_{13}(0)$: the subset generated by $E^{u1}(0)$ and $E^{s3}(0)$ \(\text{(Dim = 2)}\)

$F_{23}(0)$: the subset generated by $E^{s2}(0)$ and $E^{s3}(0)$ \(\text{(Dim = 2)}\)

$U_1 = \{(x, y, z): x = 1\} = D_1 \cap D_0$ \(\text{(Dim = 2)}\)

$U_{-1} = \{(x, y, z): x = -1\} = D_{-1} \cap D_0$ \(\text{(Dim = 2)}\).

Lines and points on the plane $U_1$:

- $L_0 = E^s(P^+) \cap U_1$
- $L_1 = F_{23} \cap U_1$
- $L_2 = F_{12} \cap U_1$
- $L_3 = F_{13} \cap U_1$
- $L_4 = \{(x, y, z): x = -1 \text{ and } y + f(x) = 0\}$
- $A = L_0 \cap L_1$
- $B = L_0 \cap L_3$
- $C = L_2 \cap L_3$
- $D = L_1 \cap L_3$.

Note that $L_4$ clearly separates the plane $U_1$ into two regions, one where $dx/dt > 0$ (upgoing trajectory) and another where $dx/dt < 0$ (downgoing trajectory).

Let us try to describe the trajectories near $U_1$, basing the analysis on Fig. 11(a) and (b).
Fig. 11. Cross sections at the boundary. (a) Cross section at $x = 1$ and projection of the trajectory onto the same plane $x = 1$. The points represent upgoing trajectories while the crosses denote downgoing trajectories. The "Z-shape" contour in orange is the double hook attractor. The blue (resp. green and purple) lines represent the projection of the part of the trajectory located in the upper (resp. middle and lower) region and $I_1$, $I_2$ and $I_3$ are shown in yellow. (b) Cross section at $x = 1$ (the points represent upgoing trajectories while the crosses denote downgoing trajectories).
Fig. 12. Various cross sections parallel to the \((y, z)\)-plane. (a) Location of the cross sections. (b) \(A\): cross section at \(x = 0\), \(B\): cross section at \(x = 5\), \(C\): cross section at \(x = 10\), \(D\): cross section at \(x = 15\), \(E\): cross section at \(x = 20\), \(F\): cross section at \(x = 25\), \(G\): cross section at \(x = 30\), \(H\): cross section at \(x = 35\), \(I\): cross section at \(x = 40\), \(J\): cross section at \(x = 45\). All of them have the same scaling and the same number of points (2000).
Let $\phi^t$ be the flow generated by (3.3) and pick an initial condition $x_0 \in E^{n}(P^*)$ in a neighborhood of $P^*$. As in the double scroll, the flow $\phi(x_0)$ starts wandering away from $P^*$ in a counterclockwise spiral and hits $U_0$ on $L_0$ (since $E^{n}(P^*)$ is invariant). Let $x_1$ be the intersection point.

If $x_1 = A$, then the trajectory goes directly to the origin (since $A \in E_{23}$, which is stable).

If $x_1 \in \{AB\}$, then the trajectory approaches the origin following $F_{23}$, but goes up again before reaching the plane $x = 0$ ($\phi^t(x_1)$ has a component on $F_{23}$ and a component on $E^{n}(0)$; where the latter is directed towards $D_1$). Going back to $U_1$, the flow is bounded by $F_{12}$, and later by $F_{13}$. The closer $x_1$ is to $A$, the deeper $\phi^t(x_1)$ descends. On Fig. 11 (b) the segment $\{AB\}$ maps into the points between $B$ and $C$.

If $x_1 \in \{AF\}$, then the trajectory descends again towards 0, but, this time, enters the region $D_{24}$.

The segment-like set of points between $E$ and $D$ can easily be explained as follows. Coming from $D_{24}$, the flow hits $U_{24}$ from below, forming a segment symmetrical to $\{AB\}$ on $U_0$, with respect to the origin. As the $D_0$ region is very small in comparison with $D_1$ and $D_{24}$, the previous segment is mapped into a segment-like set on $U_0$.

What happens to the set of points between $E$ and $D$, however, remains unclear. We assume that a point $G$ separates $ED$ into two parts. The first one $\{EG\}$ is made of points heading for $P^*$, meanwhile $\{GD\}$ is mapped into the set of points between $D$ and $C$ (so that the trajectory reaches $D_0$ again), and then into a very small region around $C$ (probably $C$ itself).

Fig. 12 shows various cross sections of the attractor calculated numerically at $x = 5k$, $k = 0$ to 9. One can observe how flat the attractor is, especially at the ends. The cross section at $x = 0$ gave the name of “double hook” to the attractor.

Fig. 13 shows the same cross sections observed in the circuit realization using the 3-D rotator described in [6]. With these cross sections we were able to observe the trajectory located under the plane of each cross section. The correspondence with the digital computations in Fig. 12 is quite remarkable.

IV. CONCLUSION

We have described the circuit realization of the double hook attractor, a new strange attractor using the double scroll system of equations. The double hook occurs when the eigenvalues at the origin are all real. We have presented experimental evidence to confirm the theoretical predictions and computer simulations of the structure of this attractor. The bifurcation sequence appears to differ considerably from the period-doubling route to chaos observed in the double scroll. Further study in this area may produce some interesting results. A comprehensive analysis of the piecewise-linear geometry and the normal form equations associated with the double hook is given in [7].

ACKNOWLEDGMENT

The authors would like to thank R. Tokunaga of Waseda University, Tokyo, Japan, for his very useful suggestions on the experimental work and M. P. Kennedy for his valuable comments.

REFERENCES


Philippe Bartissol, photograph and biography not available at time of publication.

Leon O. Chua (S'60–M'62–SM'70–F'74), for photograph and biography please see page 880 of the July 1988 issue of this TRANSACTIONS.