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SYNCHRONIZATION OF SELF-OSCILLATIONS BY PARAMETRIC EXCITATION

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This paper is devoted to the problem of synchronization of symmetrically coupled self-oscillators exhibiting chaos by means of chaos control technique. We use a nonfeedback method of control, in particular, high frequency periodic modulation of the coefficient of coupling. The model of Chua's circuits coupled via capacity is considered in this work. We study the possibility of synchronization in dependence on amplitude and frequency of modulation for various values of the parameters of the self-oscillators. The dependence of the threshold values of the amplitude of the synchronizing influence on coefficient is presented.

1. Introduction

Recently, much of the interest in nonlinear dynamics has focused on the problem of synchronization of chaotic oscillations. Unfortunately, there is no common definition of this subject. Different authors use different approaches: from the classical view on synchronization such as the locking of frequencies [Anishchenko *et al.*, 1991, 1992] and phases [Rosenblum *et al.*, 1995] to the equality of oscillations in subsystems ($\mathbf{x}_1(t) = \mathbf{x}_2(t)$) [Fujisaka & Yamada, 1983; Pecora & Caroll, 1990]. Besides investigations of the self-synchronization problem, there are also tasks of forced-synchronization and synchronization by control [Lai & Grebogi, 1993; Chua *et al.*, 1993; Astakhov *et al.*, 1996a].

In this paper we investigate the effect of synchronization of symmetrically coupled identical self-oscillators applying a high frequency parametric excitation to the coupling element. The term "synchronization" is used here in a narrow sense as motions in the symmetric subspace ($\mathbf{x}_1(t) = \mathbf{x}_2(t)$).

The study of periodic parametric excitation to nonlinear oscillators is a traditional problem of classical and modern physics. The majority of works is devoted to resonance influence on an oscillator when the ratio between the own time scale of the system and the period of the external force is close to a rational number of a small order. Applications of periodic parametric perturbations for modification of chaotic dynamics were considered in the works [Lima & Pettini, 1990; Cicogna & Fronzoni, 1990; Fronzoni *et al.*, 1991]. It has been shown both theoretically and experimentally that resonant parametric perturbation can suppress chaotic behavior. Another interesting task is an application of high frequency parametric excitation when its frequency is much higher than the own characteristic frequency of the oscillator. This is the nonresonant case.

We propose to apply high frequency periodic modulation of the coupling coefficient to stabilize synchronous motions. The idea of this approach was induced by a classical task of mechanics: a pendulum with vibrating suspension. As is known, at certain values of amplitude and frequency of vibration it is possible to stabilize the upper equilibrium of the pendulum [Kapitza, 1951a, 1951b].

The task of the stability of symmetric motions can be reduced to the stability of the zero fixed point of some system. To illustrate this fact let us consider the system in the form:

$$\dot{\mathbf{x}}_1 = \mathbf{F}(\mathbf{x}_1) + \gamma(\mathbf{x}_2 - \mathbf{x}_1)$$
$$\dot{\mathbf{x}}_2 = \mathbf{F}(\mathbf{x}_2) + \gamma(\mathbf{x}_1 - \mathbf{x}_2)\,, \quad (1)$$

where the equation $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ describes an oscillator without coupling, γ is a matrix of coupling. Linearizing the system (1) near the symmetric subspace and using new variables:

$$\mathbf{u} = (\mathbf{x}_1 + \mathbf{x}_2)/2, \quad \mathbf{v} = (\mathbf{x}_1 - \mathbf{x}_2)/2\,,$$

we obtain the equations as follows:

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}) \quad (2)$$

$$\dot{\mathbf{v}} = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\bigg|_u - 2\gamma\right)\mathbf{v} \quad (3)$$

Here Eq. (2) describes the character of motions in the symmetric subspace. The stability of the zero equilibrium of Eq. (3) determines the stability of these motions.

As high frequency parametric excitation can change the stability of the equilibrium (in the case of the pendulum) we suppose that it can also change the stability of the symmetric motions. We applied the high frequency perturbation to the coupling parameter in the system of two coupled nonautonomous nonlinear oscillators with chaos [Astakhov *et al.*, 1996b] and found that such perturbation can stabilize the in-phase chaotic motions in this system. In this paper we present the effect of synchronization of chaotic oscillations in a self-oscillatory system by means of high frequency parametric modulation of the coupling coefficient.

2. The System under Consideration

As a model of coupled self-oscillators we consider two Chua's circuits coupled via a capacity. The

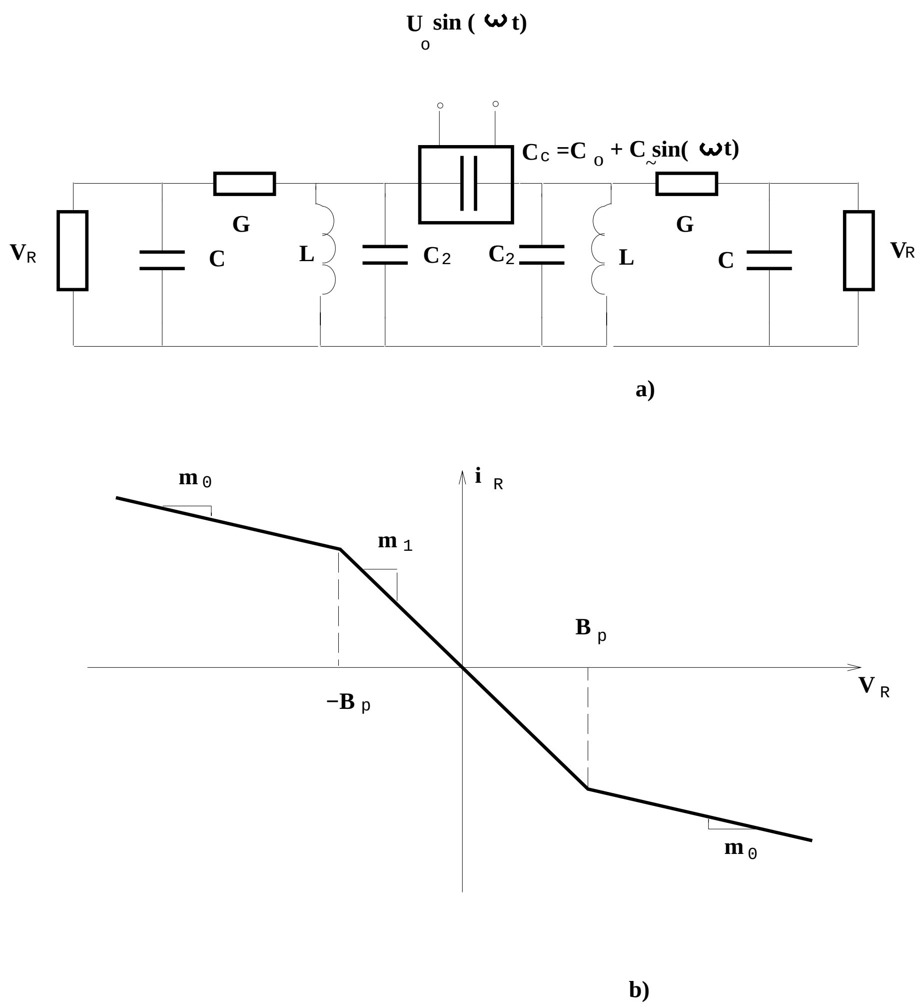

Fig. 1. The scheme of the circuit (a) and the characteristic of the nonlinear element (b).

capacity is parametrically modulated by external periodic force. The scheme of the coupled circuits is shown in Fig. 1. The normalized system of equations that describes this scheme has the following form:

$$
\begin{aligned}
\dot{x}_1 &= \alpha(y_1 - x_1 - f(x_1)) \\
\dot{y}_1 &= \frac{1+\gamma}{1+2\gamma}(x_1 - y_1 + z_1) + \frac{\gamma}{1+2\gamma}(x_2 - y_2 + z_2) \\
&\quad + \frac{\gamma_1}{1+2\gamma}(y_2 - y_1) \\
\dot{z}_1 &= -\beta y_1 \\
\dot{x}_2 &= \alpha(y_2 - x_2 - f(x_2)) \\
\dot{y}_2 &= \frac{1+\gamma}{1+2\gamma}(x_2 - y_2 + z_2) \\
&\quad + \frac{\gamma}{1+2\gamma}(x_1 - y_1 + z_1) + \frac{\gamma_1}{1+2\gamma}(y_1 - y_2) \\
\dot{z}_2 &= -\beta y_2\,,
\end{aligned}
\tag{4}
$$

where $f(x) = bx + 0.5(a-b)(|x+1| - |x-1|)$, $\alpha = C_2/C_1$, $\beta = C_2/(LG^2)$, $\gamma = \gamma_0 + \xi \sin \Omega\tau$, $\gamma_0 = C_{c0}/C_2$, $\xi = C_{c\sim}/C_2$, $\gamma_1 = \xi\Omega \cos \Omega\tau$, $a = m_1/G$, $b = m_0/G$, $x_{1,2} = (V_{c1})_{1,2}/B_p$, $y_{1,2} = (V_{c2})_{1,2}/B_p$, $z_{1,2} = (i_L)_{1,2}/GB_p$, $\dot{x} = dx/d\tau$, $\tau = tG/C_2$, $\Omega = C_2/G\omega$ (Fig. 1).

In-phase motions are typical for a system with coupling via a resistor in a wide range of its parameters, especially in the case of strong coupling [Anishchenko *et al.*, 1995], we use the coupling via a capacity in the model system. In this case the in-phase motions lose their stability right away after the first period-doubling bifurcation [Astakhov *et al.*, 1997].

Chua's circuit is a generator with 1.5 degrees of freedom. It demonstrates the transition to chaos through the cascade of period-doubling bifurcations. The evolution of chaos leads to a so-called double-scroll attractor [Chua *et al.*, 1986]. The system of coupled oscillators demonstrates more complicated dynamics. It has period-doubling, tori birth and breakdown, symmetry breaking and increasing bifurcations [Astakhov *et al.*, 1997]. The system is also characterized by multistability, when different stable regimes coexist in the phase space. In Fig. 2 the dashed regions represent the domain of stable symmetric motions. Outside it the in-phase oscillations are unstable and cannot be realized in experiments.

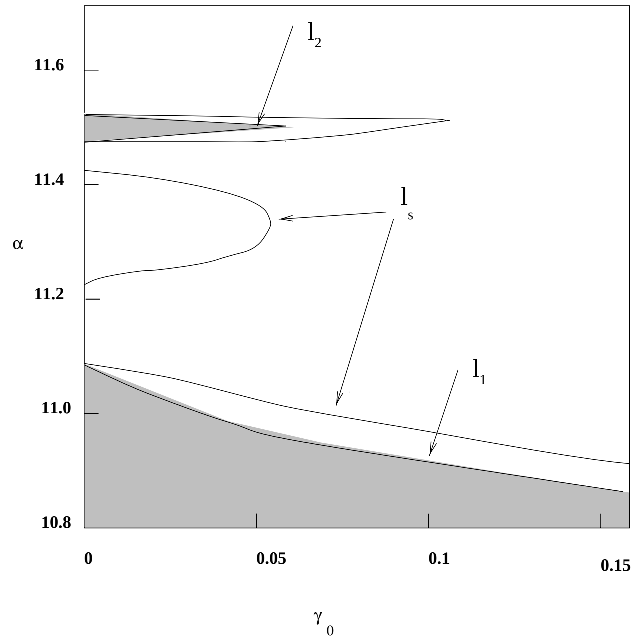

Fig. 2. Regions of stability of in-phase oscillations under control on the plane of the parameters $(\gamma_0 - \alpha)$.

3. Results of the Investigations

We investigated the possibility of synchronization in the system (4) applying high frequency parametric perturbation to the coupling capacity by means of numeric simulations. To obtain different regimes in the autonomous system we varied the parameters α and γ_0. Other parameters are fixed as $a = -8/7$, $b = -5/7$, $\beta = 22$. The ranges of amplitude and frequency values of the perturbation are limited to $[0, 0.25]$ and $[0, 30]$ intervals, respectively. Initial conditions are chosen in a small vicinity of the symmetric subspace.

We use the range of the parameters α and γ values that correspond to the regions of unstable synchronous motions. This region is shown by the shaded areas in Fig. 2. The line l_1 in Fig. 2 is the period-doubling bifurcational line for the original period-one in-phase cycle C^0. On this line the largest multiplier of the cycle becomes equal to -1. The in-phase cycle C^0 loses its stability and a stable out of phase cycle $2C^1$ appears. After this bifurcation in-phase oscillatory regimes do not exist in the phase space as stable, except the narrow region which is bounded by lines l_2 (see Fig. 2). With a further increase of the parameter α there is a transition to out-of-phase chaos. There are also routes to chaos through period-doubling bifurcations and tori breakdown depending on values of the coupling coefficient. The evolution of this system is

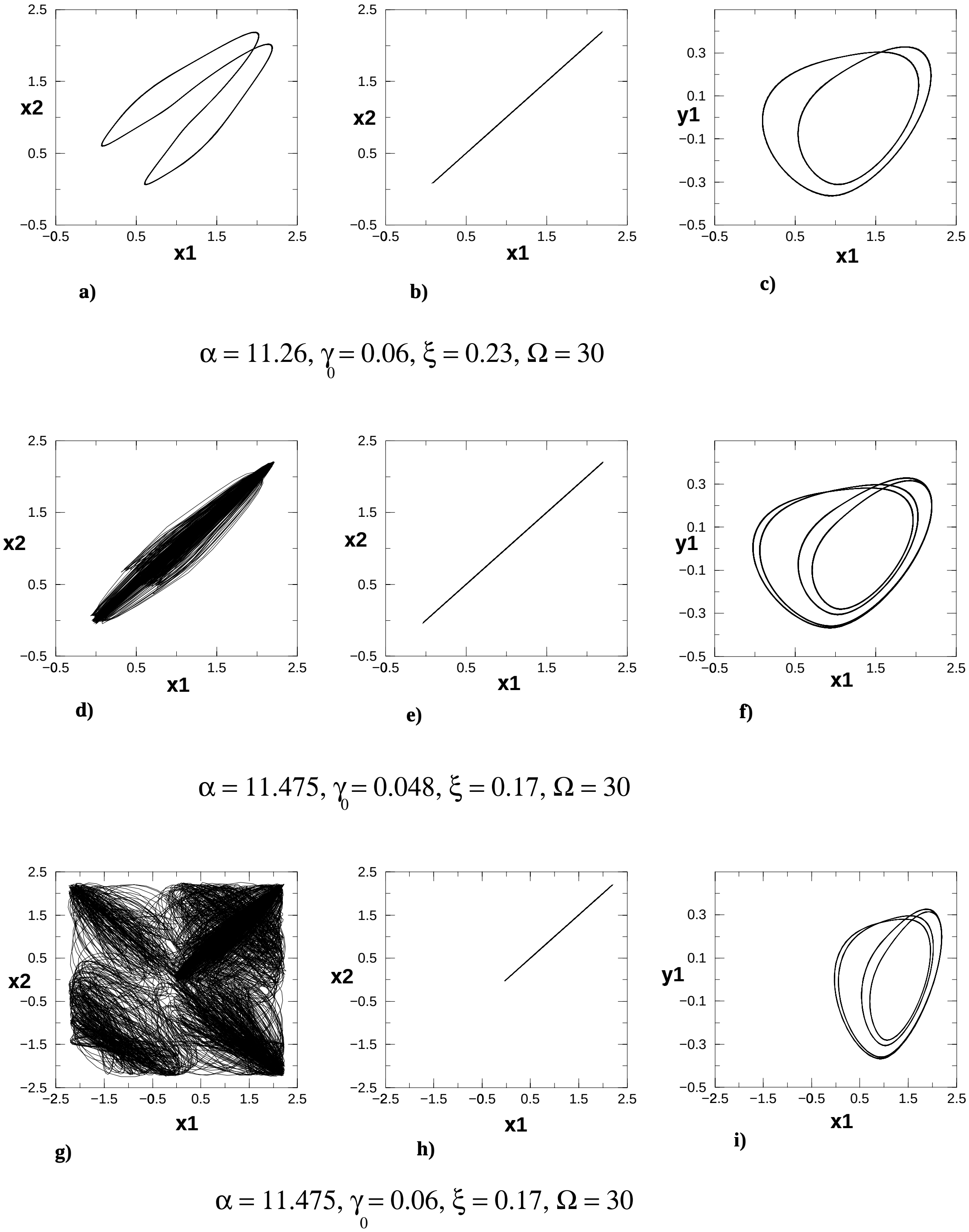

Fig. 3. Projections of phase portraits of different regimes without (a), (d), (g) and with parametric modulation (b), (c), (e), (f), (h), (i).

described in detail in the paper [Astakhov *et al.*, 1997].

The results of our numeric simulations show that there exists a region on the plane of the parameters γ_0 and α where the high frequency parametric perturbation stabilizes the in-phase motions which are unstable in the autonomous system. This region is limited by lines l_s (see Fig. 2). The regimes of the system behavior without excitation and with it are shown in Fig. 3 for different values of the parameters α and γ_0. The pictures on the left side of the figure represent the regimes of periodic

[Fig. 3(a)] and chaotic oscillation [Figs. 3(d) and 3(g)] that exist in the system without modulation of the parameter. On the right side of Fig. 3 we show the projections of phase portraits on the planes $(x_1 - x_2)$ [Figs. 3(b), 3(e) and 3(h)] and $(x_1 - y_1)$ [Figs. 3(c), 3(f) and 3(i)] which are observed when the parametric modulation is used. The resulting symmetric regime depends on the parameters of the single generator, i.e. the value of the parameter α. Chaotic oscillations appear at lower values of the parameter α in the coupled system than in the single oscillator [Astakhov *et al.*, 1997]. Therefore, the stabilization of symmetric motions leads to ordering of oscillations. The system transits from the chaotic regimes [Figs. 3(d) and 3(g)] to the periodic ones [Figs. 3(f) and 3(i)].

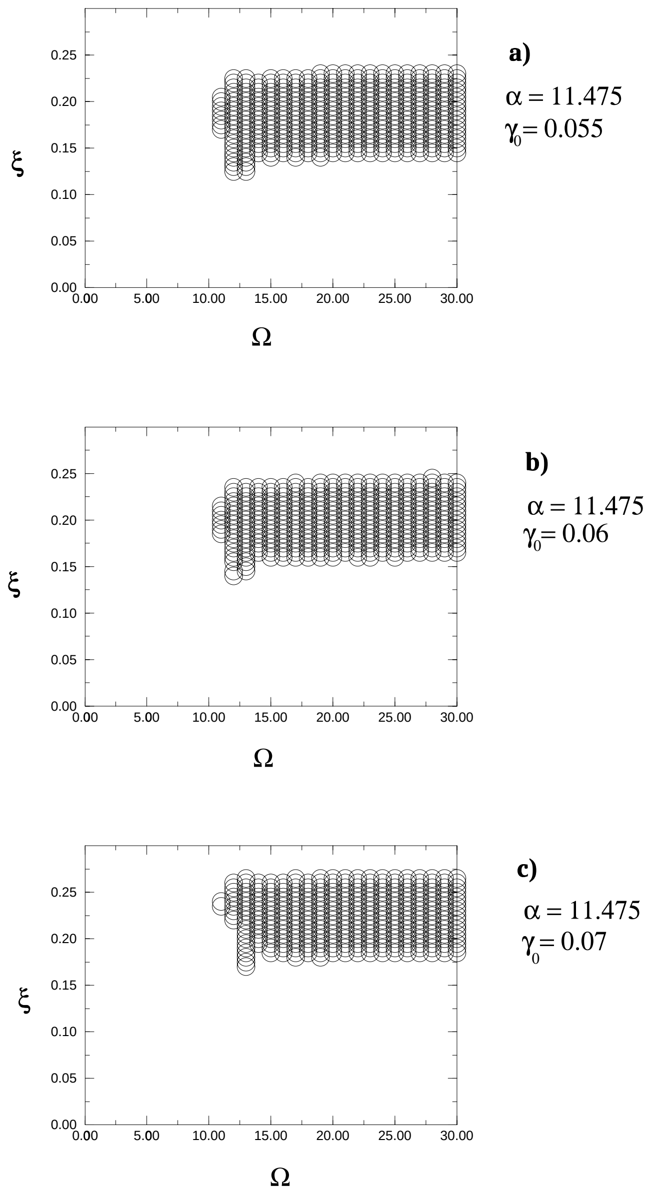

Fig. 4. Regions on the plane of the parameters $(\Omega - \xi)$ where synchronization takes place.

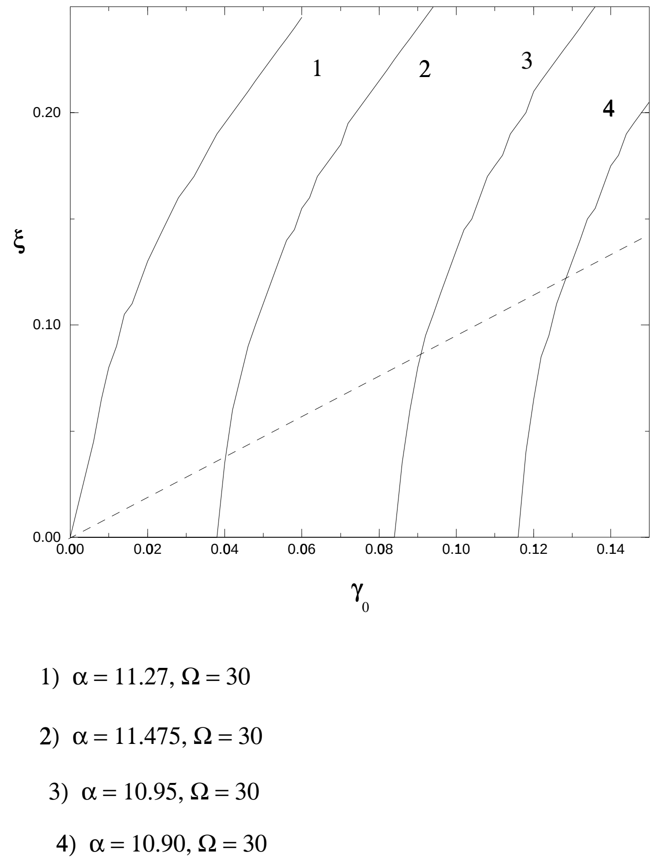

Fig. 5. Dependence of the minimal amplitude of the controlling influence on the coefficient of coupling.

In Fig. 4 there are regions on the parameters plane $(\Omega - \xi)$ where synchronization takes place. These cases correspond to the oscillatory regimes of the system (4) for different values of the coefficient of coupling. The form of the synchronization region has no resonant character. This region is the same as in the case of the coupled nonautonomous oscillators [Astakhov *et al.*, 1996b]. The dependence of the amplitude of synchronizing excitation on its frequency has threshold character. Synchronization is absent if Ω is less than some minimal value. But if Ω is larger than the threshold value, the minimal synchronizing amplitude does not depend on it. As seen from Fig. 4 the location of the regions of synchronization on the plane $(\Omega - \xi)$ depend on the coefficient of coupling. When we increase the coupling coefficient, the regions of synchronization shift up. The dependence of the minimal amplitude of the synchronizing excitation on the coefficient γ_0 for different regimes at $\alpha = 10.9$, 10.95, 11.27 and

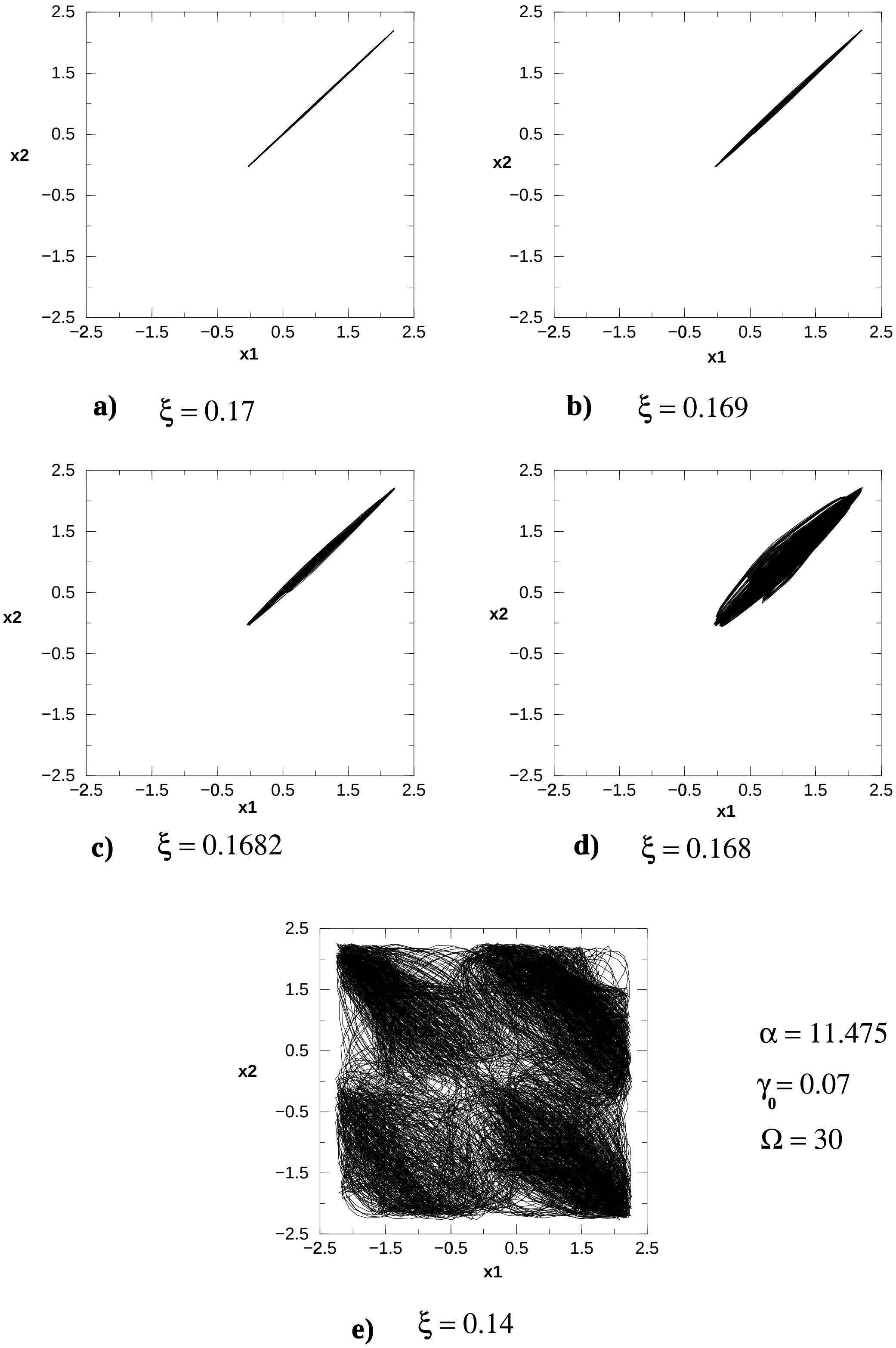

Fig. 6. Projections of phase portraits that illustrate the exit of the system from synchronous regime.

11.475 is shown in Fig. 5. If stable in-phase oscillations exist at the given value of the parameter α in some interval of the coefficient of coupling γ_0, the corresponding curve in Fig. 5 has zero part (curves 2, 3, 4) that relates to this stable region (see dashed regions in Fig. 2). Nonzero parts of the curves are practically parallel to each other. The dependences of the threshold value of the synchronizing amplitude of modulation on the coefficient of coupling γ_0 for different values of the parameter α are seen to coincide with each other up to horizontal shifts. It is interesting that the threshold value does not depend on the type of oscillations in the nonperturbed system. Curve 2 corresponding to chaotic oscillations in the nonperturbed system is located between curves 1 and 4 which correspond to periodic regimes.

In the paper [Astakhov *et al.*, 1996b] it has been shown that in the case of coupled nonautonomous oscillators the stabilization of in-phase motions can be observed for rather small values of the amplitude of external force. But here we have a different case. Small perturbations are sufficient for synchronization only in a very narrow region on the parameters $(\alpha - \gamma_0)$ plane. The direct line in Fig. 5 bounds the regions of the parameter γ_0 for different values of the parameter α where the coefficient of modulation ξ/γ_0 is less than 1. Here we have an ordinary case of the parameter modulation. Outside those regions synchronization is achieved only for large values of the coefficient ξ when the value of the coupling capacity becomes negative at some time moments (though the average value of the capacity is positive). In this case construction of a real electronic circuit becomes problematic and Eq. (4) can be considered as an abstract mathematical model. Nevertheless, at some parameters values, when the threshold of the amplitude of the excitation is less than the coefficient γ_0, the modulation can be realized in experiments.

In Fig. 6 there are projections of phase portraits of oscillatory regimes that change each other when we gradually decrease the modulation amplitude ξ from the value which corresponds to in-phase oscillations. When the parameter ξ is decreased, the projection of the phase portrait on the plane $(x_1 - x_2)$ firstly expands [Figs. 6(b) and 6(c)] and at a certain value of ξ the system transits suddenly to another system attractor [Fig. 6(d)]. With further decreasing of the modulation amplitude we have transition to the attractor whose phase portrait is displayed in Fig. 6(e). The "thickness" of the attractors in Figs. 6(b) and 6(c) is a result of amplification of noise near the bifurcational point when we add the noise to the system. Without noise oscillations remain strictly in-phase until the bifurcation. On the other hand, the introduction of the external noise with any small intensity leads to the pictures as in Figs. 6(b) and 6(c).

Analyzing different projections of the phase portraits and time series of the oscillatory regimes presented in Figs. 6(d) and 6(e) we have found that they are very similar to the regimes that exist in the nonperturbed system. Of course, the motions in the system with the perturbation cannot be the same as in the autonomous system due to the presence of one more time scale $2\pi/\Omega$. Oscillations in this case can be presented as the sum of "rapid" motions with the characteristic time scale $2\pi/\Omega$ and "slow" motions with the own characteristic time scale of the autonomous system. As the period of the excitation is far from the own time scale of the system, the "amplitude" of such rapid vibrations is quite small relative to the "amplitude" of the slow motions. Averaging oscillations on the "rapid" time scale we obtain attractors that can be observed in the system without excitation [Fig. 6(d)], possibly at other values of the parameters [Fig. 6(c)]. The attractor presented in Fig. 6(e) corresponds to the double-scroll attractor and the attractor shown in Fig. 6(d) represents the regime of oscillations in the nonperturbed system, which is unstable for the given values of the parameters α and γ_0.

These observations give us the possibility to assume that high frequency parametric excitation does not induce new regimes in the system (up to this rapid vibration), but changes their stability properties. Nevertheless, the verification of this assumption demands additional investigations.

4. Conclusion

As a result of our investigations as presented in this paper we have obtained that in-phase oscillations can be stabilized by high frequency parametric excitation to the coupling element for the case of self-oscillatory systems with chaos. For synchronization the frequency of the perturbation must be 10–20 times higher than the own characteristic frequency of the system. The dependence of the amplitude of the synchronizing perturbation on its frequency has a threshold character. The minimal value of the amplitude of the synchronizing influence depends on the value of the coefficient of coupling and does not depend on the type of oscillations in the system.

The motions in the system with excitation can be considered as the sum of small "rapid" oscillations with the characteristic time scale $T = 2\pi/\Omega$ and "slow" oscillations with the own time scale of the system without excitation. The preliminary investigations have shown that at certain values of ξ and Ω the "slow" oscillations correspond to the unstable motions in the nonperturbed system.

We suppose that the suggested approach can be considered as one more method of synchronization of oscillations in coupled systems by external perturbation.

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