Chua’s Circuit Can Be Generated by CNN Cells

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Abstract—A new realization for the unfolded Chua’s circuit is introduced. It has been derived from the connection of three simple generalized cellular neural network (CNN) cells. The main theoretical implication of this result is that the CNN cell represents the primitive for realizing high complex dynamics. The circuit implementation of the introduced system and experimental results referring to the double scroll attractor are reported.

I. INTRODUCTION

Chua’s circuit is the simplest autonomous third-order nonlinear electronic circuit with a rich variety of dynamical behaviors including chaos, stochastic resonance, 1/f noise spectrum and chaos–chaos intermittency [1]–[3]. Moreover, Chua proved that the unfolded Chua’s circuit state equation, which is uniquely determined by 7 parameters, is topologically conjugate (i.e., equivalent) to a 21-parameter family of continuous odd-symmetric piecewise-linear equations in $\mathbb{R}^2$ [2].

From this result Chua’s circuit is considered the canonical circuit for studying chaos. Besides, in recent papers Chua’s circuit arrays have been investigated [1], [3].

The CNN cell is the core of the Cellular Neural Network (CNN) [3]–[6] and, in its classical definition [4], is a simple first-order nonlinear circuit. Chua’s idea was to use an array of simple, locally interconnected cells to build impressive analog signal processing systems (CNN).

In this paper the unfolded Chua’s circuit (also called Chua’s oscillator in the literature [1] for simplicity) has been obtained from an appropriate connection of three generalized CNN cells. This result has some important consequences. The main implication is that the CNN cell represents the building block for generating high complex dynamical behaviors.

In the following sections, some background concepts are first summarized, then the new theoretical results are introduced. Finally, the circuit implementation and experimental results are reported.

II. BACKGROUND

II.1. The Unfolded Chua’s Circuit

The unfolded Chua’s circuit (shown in Fig. 1(a)) state equations are [2]:

$$\frac{d}{dt}v_1 = \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)]$$
$$\frac{d}{dt}v_2 = \frac{1}{C_2}[G(v_1 - v_2) + i_3]$$
$$\frac{d}{dt}i_3 = -\frac{1}{L}(v_2 + Roi_3)$$

where

$$f(v_1) = G_3 v_1 + 0.5 \cdot (G_a - G_b) \cdot \begin{vmatrix} 0 & 1 \end{vmatrix}$$

The nonlinear resistor $N_r$ is the Chua’s diode and $f(v_1)$ is its characteristic. The difference among the classical Chua’s circuit and the unfolded Chua’s circuit consists of the resistor $R_0$, contained in the latter one.

It is often preferred to consider the equivalent dimensionless state equations:

$$\dot{x} = \alpha(y - h(x))$$
$$\dot{y} = x + y + z$$
$$\dot{z} = -\beta y - \gamma z$$

with

$$h(x) = m_1 x + 0.5 \cdot (m_0 - m_1) \cdot \begin{vmatrix} x + 1 \end{vmatrix} - \begin{vmatrix} x - 1 \end{vmatrix}$$

$x$, $y$, and $z$ being the state variables and $\alpha$, $\beta$, $\gamma$, $m_0$, $m_1$ the system parameters. With trivial algebra the relationships among these two representations are found:

$$x = v_1/E_1; \quad y = v_2/E_1; \quad z = i_3/(E G)$$
$$t = (G_c)/C_1; \quad m_0 = (G_a + G_b)/2; \quad m_1 = (G_a - G_b)/2; \quad \alpha = C_2/C_1; \quad \beta = C_2/(G L^2); \quad \gamma = (C_2 R_0)/(G L)$$

A zoo of more than 30 strange attractors generated by these equations can be found in [2]. It is nevertheless worth noting that a lot of attractors have been obtained with negative capacitors/inductors.

II.2. The CNN Cell Model

Chua and Yang introduced the Cellular Neural Network (CNN) in 1988 [4]. They defined the CNN cell as the nonlinear first order circuit shown in Fig. 1(b), $u_{ij}$, $y_{ij}$ and $x_{ij}$ being the input, the output and the state variable of the cell, respectively. The output is related to the state by the nonlinear equation:

$$y_{ij} = 0.5 \cdot \begin{vmatrix} x_{ij} + 1 \end{vmatrix} - \begin{vmatrix} x_{ij} - 1 \end{vmatrix}$$
III. NEW THEORETICAL RESULTS

III.1. The Generalized CNN Cell Model

A new generalized CNN cell model is introduced. It is described by the following dimensionless nonlinear state equation:

$$\dot{x}_j = -x_j + a_j y_j + G_e + G_s + i_j$$

(10)

where $j$ is the cell index, $x_j$ the state variable, $y_j$ the cell output given as:

$$y_j = 0.5 \cdot (|x_j + 1| - |x_j - 1|)$$

(11)

$a_j$ a constant parameter and $i_j$ a threshold value. In (10) the terms $G_e$ and $G_s$ are linear combinations of the outputs and state variables, respectively, of the connected cells being considered. This generalized model fits perfectly the mentioned Chua-Roska definition. Moreover, it differs from the original Chua-Yang definition just by the presence of $G_s$.

III.2. Main Result

The dynamic model of three fully connected generalized CNN cells in accordance with the state equation, (10), is:

$$\dot{x}_1 = -x_1 + a_{11} y_1 + a_{12} y_2 + a_{13} y_3 + \sum_{k=1}^{3} s_{1k} x_k + i_1$$

$$\dot{x}_2 = -x_2 + a_{21} y_1 + a_{22} y_2 + a_{23} y_3 + \sum_{k=1}^{3} s_{2k} x_k + i_2$$

$$\dot{x}_3 = -x_3 + a_{31} y_1 + a_{32} y_2 + a_{33} y_3 + \sum_{k=1}^{3} s_{3k} x_k + i_3$$

(12)

where $x_1$, $x_2$, and $x_3$ are the state variables, $y_1$, $y_2$, and $y_3$ the corresponding outputs. If:

$$a_{12} = a_{13} = a_{23} = a_{22} = a_3 = a_{21} = a_{31} = 0;$$

$$s_{13} = s_{31} = s_{22} = 0; \quad i_1 = i_2 = i_3 = 0;$$

(13)

Equation (12) becomes:

$$\dot{x}_1 = -x_1 + a_{11} y_1 + s_{11} x_1 + s_{12} x_2$$

$$\dot{x}_2 = -x_2 + s_{21} x_1 + s_{22} x_3$$

$$\dot{x}_3 = -x_3 + s_{32} x_2 + s_{33} x_3$$

(14)

Looking at (3) it can be seen how the unfolded Chua’s circuit equation is a particular case of (14). In fact, assuming:

$$a_1 = \alpha (m_1 - m_0); \quad s_{12} = 1 - \gamma; \quad s_{21} = s_{23} = 1;$$

$$s_{11} = 1 - \alpha \cdot m_1; \quad s_{12} = \alpha; \quad s_{32} = -\beta,$$

(15)

the unfolded Chua’s circuit (1) is obtained with $x_1, x_2,$ and $x_3$, respectively, equal to $x$, $y$, and $z$.

IV. GENERALIZED CNN CELL DESIGN AND IMPLEMENTATION

A circuit implementation, inspired by [4] and [6], for the new generalized model is presented. It is shown in Fig. 2(a). It is essentially constituted by three blocks:

1) $B_1$ forms the output nonlinearity. It exploits the natural output saturation of the amplifier $A_1$; so $R_7$ and $R_8$ are chosen such that $A_1$ output saturates when $|x_j| > 1$. The subsequent voltage divider ($R_9$ and $R_{15}$) is designed to scale the output voltage.
Observed phase portrait in $x_1$-$x_2$ plane

in the range $[-1,1]$. Hence the following design equations hold:

\begin{align}
R_4/R_1 &= V_{\text{sat,}A}/V_{\text{sat,}x} \\
R_5/R_0 &= R_{10}/(R_0 + R_{10})
\end{align}

where $V_{\text{sat,}A}$ is the output saturation voltage of $A1$, while $V_{\text{sat,}x}$ is its corresponding input voltage (i.e., in our case $V_{\text{sat,}x} = 1$). The input and output impedances of $B1$ are $R_5$ and the parallel of $R_0$ and $R_{10}$, respectively.

2) $B2$ is an inverting amplifier with unity gain ($R_6 = R_0$). Its input impedance is $R_0$ while the output impedance is 0.

3) $B3$ is the core of the cell. If the parallel of the input impedances of $B1$ and $B2$ is very high, compared with the output impedance of block $B3$ (that is $R_4/(1+j \omega R_4 C_4)$) then blocks $B1$ and $B2$ do not load the capacitor $C_4$. This is clearly true if $R_1 R_3/(R_1 + R_3) \gg R_4$. In this case, the generalized CNN cell state equation is:

\[ C_4 \frac{d}{dt} x_3 = -\frac{x_4}{R_4} + \frac{R_3}{R_4} V_1 + \frac{R_2}{R_2 R_4} V_2 \]

The impedances seen from the two inputs $-V_1$ and $-V_2$ are $R_1$ and $R_2$, respectively.

**Proposition 1:** Let us consider three generalized cells, and represent with $x_1$, $x_2$, $x_3$ each state variable, and with $y_1$, $y_2$, $y_3$ the corresponding outputs. If it is assumed that $V_1 = y_1$ and $V_2 = y_2$ for the first cell, $V_1 = x_1$ and $V_2 = x_2$ for the second one, and $V_1 = -y_2$ and $V_2 = x_3$ for the third one, as shown in Fig. 2(b), then the three connected cell equations are equivalent to system (14). Therefore, they generate Chua's circuit equation.

In fact (with the assumptions in hypothesis) it follows that (14) are the dimensionless versions of (17). By comparison between (14) and (17), the design of $B3$ is straightforward.

**V. EXPERIMENTAL RESULTS**

A double scroll attractor is observed in Chua's circuit dynamic if $\alpha = 9$, $\beta = 14.286$, $\gamma = 0$, $m_0 = -1/7$ and $m_1 = 2/7$. From expressions (15), the parameter values for the three CNN are $a_1 = 3.857$, $a_{31} = 2.91 = 9$, $a_{31} = -1.5714$, $a_{12} = 9$, $a_{22} = -14.286$. In accordance with the previous design considerations, suitable values for the cell components are:

- **cell 1:**
  - $R_1 = 13.2$ k$\Omega$; $R_3 = 5.7$ k$\Omega$; $R_5 = 20$ k$\Omega$;
  - $R_4 = 200$ k$\Omega$; $R_6 = 100$ k$\Omega$; $R_0 = 100$ k$\Omega$;
  - $R_7 = 74.8$ k$\Omega$; $R_8 = 970$ k$\Omega$; $R_9 = 27$ k$\Omega$;
  - $R_{10} = 2.22$ k$\Omega$; $C_1 = 51$ nF;
- **cell 2:**
  - $R_1 = R_2 = R_3 = R_6 = R_9 = 100$ k$\Omega$;
  - $R_4 = 1$ k$\Omega$; $C_2 = 51$ nF;
- **cell 3:**
  - $R_1 = 7.8$ k$\Omega$; $R_2 = R_3 = R_5 = R_6 = 100$ k$\Omega$;
  - $R_4 = 1$ k$\Omega$; $C_3 = 51$ nF;
- **power supply:** $V_{CC} = +15$ V; $V_{EE} = -15$ V.

**Remark:** In the previous paragraph the complete scheme of the generalized cell has been introduced. However, from the above considerations (in particular, see (14)), it can be noted that the second and third cell outputs $-y_3$ and $-y_3$ are not actually exploited in order to realize the unfolded Chua's circuit. Therefore, for these cells the nonlinear block is unnecessary. It can be replaced with a noninverting buffer in order to provide the variable $x_2$ value for the third cell. In Fig. 3, the phase portrait on the $x_1$-$x_2$ plane of the experimentally observed attractor for the circuit generated by the three cells is reported. It shows the classical double-scroll attractor on the $x$-$y$ plane.

**VI. CONCLUSION**

In this paper it has been proved that the unfolded Chua's circuit can be realized by three generalized CNN cells. In this way the cornerstone concept inside the CNN cell is emphasized, making it the primitive circuit for generating Chua's circuit; hence, a Chua's circuit array can be realized by a three-layer CNN. Another peculiarity of the proposed realization with respect to the classical one consists of using natural amplifier nonlinearities instead of the nonlinear Chua's resistor. Besides, with traditional realizations, some sets of the unfolded Chua's circuit parameters can be achieved only using negative capacitors/inductor, while this is not necessary with the scheme proposed in which just appropriate choices for positive resistors and, possibly, sign inversion of some cell signals are required.

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**REFERENCES**


