International Journal of Bifurcation and Chaos, Vol. 11, No. 10 (2001) 2607–2619


# SUPER PERSISTENT CHAOTIC TRANSIENTS IN PHYSICAL SYSTEMS: EFFECT OF NOISE ON PHASE SYNCHRONIZATION OF COUPLED CHAOTIC OSCILLATORS

VICTOR ANDRADE
*Department of Physics and Astronomy, University of Kansas, Lawrence, KS 66045, USA*

YING-CHENG LAI
*Departments of Mathematics, Electrical Engineering, and Physics Center for Systems Science and Engineering Research, Arizona State University, Tempe, AZ 85287, USA*

Received July 31, 2000; Revised October 30, 2000

A super persistent chaotic transient is typically induced by an unstable–unstable pair bifurcation in which two unstable periodic orbits of the same period coalesce and disappear as a system parameter is changed through a critical value. So far examples illustrating this type of transient chaos utilize discrete-time maps. We present a class of continuous-time dynamical systems that exhibit super persistent chaotic transients in parameter regimes of positive measure. In particular, we examine the effect of noise on phase synchronization of coupled chaotic oscillators. It is found that additive white noise can induce phase slips in integer multiples of $2\pi$'s in parameter regimes where phase synchronization is expected in the absence of noise. The average time durations of the temporal phase synchronization are in fact characteristic of those of super persistent chaotic transients. We provide heuristic arguments for the scaling law of the average transient lifetime and verify it using numerical examples from both the system of coupled Chua's circuits and that of coupled Rössler oscillators. Our work suggests a way to observe super persistent chaotic transients in physically realizable systems.

## 1. Introduction

Transient chaos is ubiquitous in nonlinear dynamical systems [Grebogi *et al.*, 1983a; Tél, 1991, 1996]. In such a case, dynamical variables of the system behave chaotically for a finite amount of time before settling into a final state that is usually not chaotic. A common situation for transient chaos to arise is where the system undergoes a *crisis* at which a chaotic attractor collides with the basin boundary separating it and another coexisting attractor [Grebogi *et al.*, 1983a]. After the crisis, the chaotic attractor is destroyed and converted into a nonattracting chaotic saddle. Dynamically, a trajectory then wanders in the vicinity of the chaotic saddle for a period of time before asymptoting to the other attractor. Chaotic transients of this sort are not super persistent in the sense that their average lifetimes scale with the system parameter only algebraically. Specifically, let $p$ be a system parameter and assume that as $p$ is increased a crisis occurs at the critical parameter value $p_c$. There is thus transient chaos for $p > p_c$. It is well established both theoretically [Grebogi *et al.*, 1987] and experimentally [Ditto *et al.*, 1989] that the average lifetime $\tau$ of the chaotic transients scales with the parameter variation, as follows

$$\tau \sim (p - p_c)^{-\gamma},\ p > p_c\,, \tag{1}$$

where $\gamma > 0$ is the algebraic scaling exponent.

There exists, however, a distinct class of chaotic transients that are super persistent in the following sense of scaling [Grebogi *et al.*, 1983b, 1985]

$$\tau \sim \exp\,[A(p - p_c)^{-\beta}],\ p > p_c\,, \tag{2}$$

where $A > 0$ is a constant, $\beta > 0$ is the scaling exponent, $p_c$ is a critical parameter value, and transient chaos occurs for $p > p_c$. We see that as $p \to p_c$, the lifetime of the transient behaves like $e^{+\infty}$, henceforth it is called *super persistent*. Physically, the scaling relation (2) means that as $p$ approaches $p_c$, the transient lifetime is significantly longer than that associated with "regular" chaotic transient characterized by (1). Because of the scaling (2), the asymptotic attractor of the system is practically unobservable for $p \gtrsim p_c$. Grebogi *et al.* [1983a, 1985] argue that super persistent chaotic transients are dynamically due to the unstable–unstable pair bifurcations. Briefly, suppose there is an unstable periodic orbit on the chaotic attractor and there is another orbit of the same period on the basin boundary. As $p$ is increased towards $p_c$, the two orbits *coalesce* and are destroyed simultaneously, leaving behind a "channel" in the phase space through which trajectories on the chaotic attractor can escape. Because of the opening of the channel, the chaotic attractor is converted into a chaotic transient, but because the channel created by the mechanism of unstable–unstable pair bifurcation is typically super narrow[1] [Grebogi *et al.*, 1983b, 1985], a trajectory initiated in the basin of the original attractor can spend a tremendous amount of time in the region where the original attractor lives, leading to a super persistent chaotic transient.[2] Recently, it is discovered that these extremely long chaotic transients can also occur in the context of riddling of chaotic systems [Lai *et al.*, 1996].

While the phenomenon of super persistent chaotic transients has been known since 1983, so far, examples illustrating it are mostly discrete-time maps [Grebogi *et al.*, 1983b, 1985; Lai *et al.*, 1996]. The purpose of this paper is to present a class of continuous-time dynamical systems that exhibit super persistent chaotic transients. The systems we consider are coupled chaotic oscillators, such as coupled chaotic electronic circuits, which can be implemented in laboratory experiments with relative ease. In particular, we investigate the effect of noise on chaotic phase synchronization, a delicate type of synchronization phenomenon recently discovered by Rosenblum *et al.* [1996, 1997]. Generally, when two chaotic oscillators are coupled together, synchronization in their dynamical variables (complete synchronization) can occur, but phase synchronization usually occurs at coupling strength much smaller than that required for complete synchronization. Briefly, if trajectories in each chaotic oscillator can be regarded as a rotation, then the phase angle of the rotation increases steadily with time: $\theta(t) = \omega t + \phi(t)$, where $\omega$ is the average rotation frequency and $\phi(t)$ is a term characterizing chaotic fluctuations which in general evolves diffusively in time: $\langle [\phi(t)]^2 \rangle \sim t$. As such, the rate of increase of phase can be modeled as a drift $\omega$ plus a zero mean chaotic process. In the absence of coupling, the phase angles of the two oscillators $\theta_1(t)$ and $\theta_2(t)$ are uncorrelated. That is, if one measures the difference $\Delta\theta(t) \equiv |\theta_1(t) - \theta_2(t)|$, one finds that $\Delta\theta(t)$ increases steadily with time. However, when a small amount of coupling is present, $\Delta\theta(t)$ can be confined within $2\pi$, while the amplitudes of the rotations are still completely uncorrelated. The bifurcation that leads to this phase synchronization is subsequently investigated [Pikovsky *et al.*, 1997; Rosa *et al.*, 1998; Lee *et al.*, 1998]. The ability for chaotic systems to have phase synchronization has implication to digital communication with chaos using the natural chaotic symbolic dynamics [Hayes *et al.*, 1993; Rosa *et al.*, 1997; Bollt & Dolnik, 1997;

[1]Why the channel is super narrow can be seen as follows. Let $T$ be the time required for a trajectory to pass through the channel. For $p \gtrsim p_c$, $T$ is large and scales with $(p - p_c)$ as: $T \sim (p - p_c)^{-\beta}$. In order for such a tunneling to occur, a trajectory on the chaotic set has to stay near the location of the channel for a time $T$. Due to ergodicity, the probability for a trajectory to stay near the particular location of the unstable periodic orbit for time $T$ is proportional to $e^{-\lambda T}$, where $\lambda > 0$ is the Lyapunov exponent of the chaotic set. The average lifetime of the transient is the inverse of this probability, which is thus proportional to $e^{\lambda T}$. Substituting the scaling of $T$ with $(p - p_c)$ gives the scaling relation (2).

[2]As we will see later, the same mechanism is responsible for the extremely long duration of the temporary phase synchronization between weakly coupled chaotic oscillators under the influence of noise.

Bollt *et al.*, 1997]. In such a case, it is highly desirable to suppress phase diffusions in chaotic communication channels to ensure proper timing for decoding.

The main point of this paper is that under the influence of noise, phase synchronization between two weakly coupled chaotic oscillators cannot be sustained forever, but the time that temporal phase synchronization lasts can be extremely long. In particular, we find that additive white noise, a type of noise that is encountered commonly in experimental situations, can induce phase slips in units of $2\pi$ between the coupled oscillators which would otherwise be synchronized in phase in the absence of noise. The average time duration between successive phase slips appears to obey the following scaling law with the noise amplitude $\epsilon$, which is similar to that of the super persistent chaotic transients

$$\tau \sim \exp\left(K\varepsilon^{-\alpha}\right), \qquad (3)$$

where $\alpha > 0$ is the scaling exponent depending on system parameters such as the coupling strength, and $K > 0$ is a constant. An implication is that in the presence of only small noise, the average time duration to observe phase synchronization can be extremely long due to the scaling (3). Phase synchronization is robust in this sense. Since phase synchronization occurs in general for continuous-time dynamical systems (flows) only [Rosenblum *et al.*, 1996], as we will demonstrate using numerical examples, it provides a natural setting for observing and testing super persistent chaotic transients in laboratory experiments. We mention that a brief account of the work has been published recently [Andrade *et al.*, 2000]. The present work emphasizes on the possibility of detecting super persistent chaotic transients by constructing numerical examples that can be implemented as systems of coupled electronic circuits in laboratory experiments and, in addition, a more detailed description of the theory is given.

The rest of the paper is organized as follows. In Sec. 2, we present numerical verification for the scaling law (3) using two systems: (1) coupled Chua's circuits [Matsumoto, 1984; Chua *et al.*, 1986; Chua & Lin, 1990; Chua & Tichonicky, 1991; Chua, 1992, 1993; Chua *et al.*, 1993; Madan, 1993; Chua, 1994; Chua *et al.*, 1995]; and (2) coupled Rössler oscillators [Rössler, 1976]. In Sec. 3, we give two heuristic theories for the scaling law (3). A conclusion is presented in Sec. 4.

## 2. Numerical Examples

### 2.1. *Coupled Chua's circuits*

The Chua's circuit is one of the paradigms for exploring various phenomena in chaotic dynamics [Matsumoto, 1984; Chua *et al.*, 1986; Chua & Lin, 1990; Chua & Tichonicky, 1991; Chua, 1992, 1993; Chua *et al.*, 1993; Madan, 1993; Chua, 1994; Chua *et al.*, 1995]. The circuit consists of four linear elements (two capacitors, one inductor and one resistor) and one nonlinear resistor, called Chua's diode. The circuit is described by the following set of three ordinary differential equations governing the voltages $V_1$ and $V_2$ across the two capacitors, respectively, and the current $i_L$ flowing through the inductor $L$:

$$\begin{aligned}
\frac{dV_1}{dt} &= \frac{1}{C_1}\left[\frac{1}{R}(V_2 - V_1) - f(V_1)\right], \\
\frac{dV_2}{dt} &= \frac{1}{C_2}\frac{1}{R}(V_1 - V_2 + i_L), \qquad (4)\\
\frac{di_L}{dt} &= -\frac{1}{L}(V_2 + Ri_L),
\end{aligned}$$

where $f(V)$ is a piecewise-linear function that describes the current–voltage relation of the nonlinear resistor

$$\begin{aligned}
f(V) = G_bV + \frac{1}{2}(G_a - G_b)& \\
\times\left[|V+E| - |V-E|\right],& \qquad (5)
\end{aligned}$$

as shown in Fig. 1. The circuit, mathematically represented by Eqs. (4) and (5), contains eight parameters: ($C_1$, $C_2$, $L$, $R$, $R_o$, $E$, $G_a$, $G_b$). Equations (4) and (5) are convenient for experimental

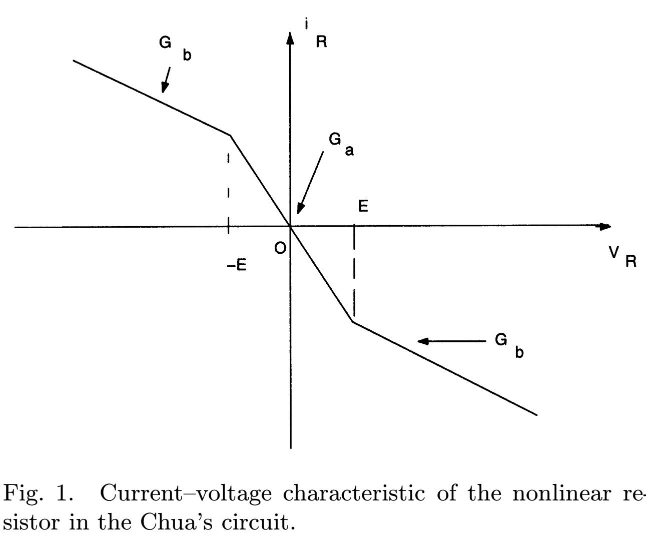



Fig. 1. Current–voltage characteristic of the nonlinear resistor in the Chua's circuit.

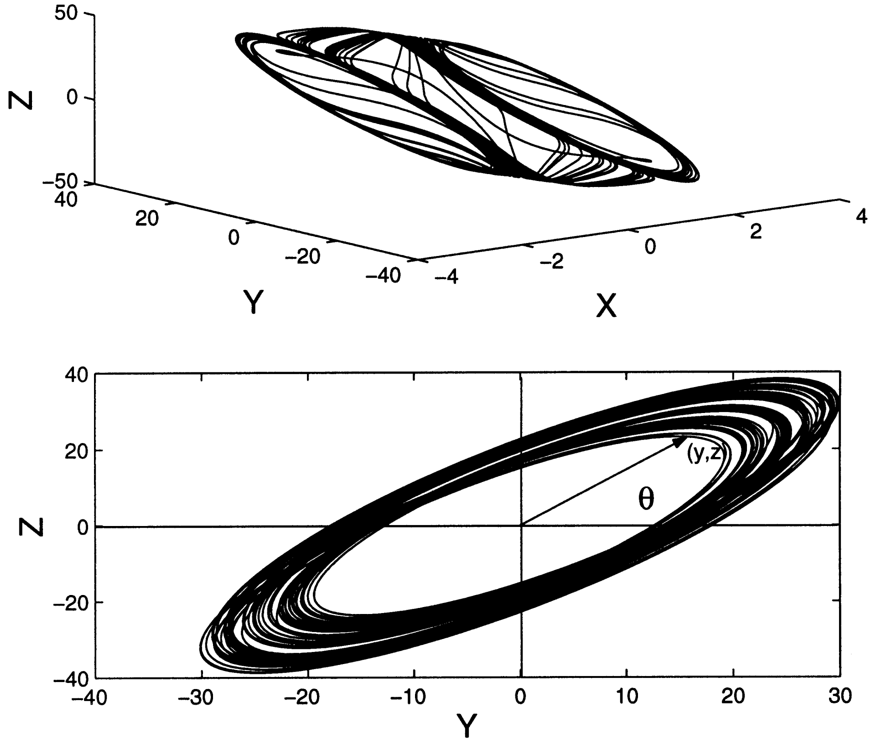



Fig. 2. Three-dimensional representation of the Chua's attractor and its projection on the $(y, z)$-plane.

implementation. By proper normalization of the three state variables $V_1$, $V_2$ and $i_L$, and of the time scale, the state equations of Chua's circuit can be specified with only five dimensionless parameters ($\alpha$, $\beta$, $\gamma$, $a$, $b$): $\alpha = C_2/C_1$, $\beta = R^2C_2/L$, $\gamma = RR_oC_2/L$, $a = RG_a$ and $b = RG_b$. The dimensionless form of the Chua's circuit is thus given by:

$$\begin{aligned}
\frac{dx}{d\tau} &= k\alpha[y - x - f(x)]\,,\\
\frac{dy}{d\tau} &= k(x - y + z)\,,\\
\frac{dz}{d\tau} &= k(-\beta y - \gamma z)\,,\\
f(x) &= bx + \frac{1}{2}(a-b)[|x+1| - |x-1|]\,,
\end{aligned} \tag{6}$$

where $x = V_1/E$, $y = V_2/E$, $z = i_L(R/E)$, $\tau = t/|RC_2|$, $k = 1$ if $RC_2 > 0$, and $k = -1$ if $RC_2 < 0$. A chaotic attractor of the Chua's circuit, obtained using the following set of parameter values: $\alpha = 0.07$, $\beta = 1.49$, $\gamma = -0.95$, $a = 18.20$, $b = -44.70$, $\tau = 10.32$, and $k = -1$, is shown in Fig. 2. The projection of the chaotic attractor in the three-dimensional phase space onto the $y - z$ plane (shown in the lower panel) indicates that the motion on the attractor resembles that of a rotation. The phase (angle) of the chaotic attractor can thus be defined properly because there is a unique center of rotation [Yalcinkaya & Lai, 1997]. To obtain the phase angle, it is convenient to rewrite Eq. (6) in the cylindrical coordinate ($x$, $r$, $\theta$)

$$\begin{aligned}
\frac{dx}{dt} &= k\alpha[r\cos\theta - x - f(x)]\,,\\
\frac{d\theta}{dt} &= \frac{-kx}{r}\sin\theta + k(1-\gamma)\cos\theta\sin\theta\\
&\quad - k(\sin^2\theta + \beta\cos^2\theta)\,,\\
\frac{dr}{dt} &= kx\cos\theta - kr(\gamma\sin^2\theta + \cos^2\theta)\\
&\quad + kr\cos\theta\sin\theta(1-\beta)\,.
\end{aligned} \tag{7}$$

We now consider the following system of two coupled, nonidentical Chua's circuits

$$\begin{aligned}
\frac{dx_{1,2}}{dt} &= k\alpha[y_{1,2} - x_{1,2} - f(x_{1,2})]\,,\\
\frac{dy_{1,2}}{dt} &= k(x_{1,2} - y_{1,2} + z_{1,2}) + C(y_{1,2} - y_{2,1})\,,\\
\frac{dz_{1,2}}{dt} &= k(-\beta y_{1,2} - \gamma_{1,2} z_{1,2})\,,
\end{aligned} \tag{8}$$

where a linear coupling term $C(y_{1,2} - y_{2,1})$ is applied to the $y$-equations and $C$ is the coupling parameter. Such a linear coupling scheme can be implemented easily in experiments. The parameters $\gamma_1$ and $\gamma_2$ are set at slightly different values so that the coupled circuits are not identical, to mimic an experimental situation where the circuits cannot be perfectly identical. In the cylindrical coordinate, Eq. (8) thus becomes the following

$$\begin{aligned}
\frac{dx_{1,2}}{dt} &= k\alpha[r_{1,2}\cos\,\theta_{1,2} - x_{1,2} - f(x_{1,2})]\,,\\
\frac{d\theta_{1,2}}{dt} &= \frac{kx_{1,2}}{r_{1,2}}\sin\,\theta_{1,2} + k(1-\gamma_{1,2})\sin\,\theta_{1,2}\cos\,\theta_{1,2} - k(\sin^2\,\theta_{1,2} + \beta\cos^2\,\theta_{1,2})\\
&\quad - C\left(\frac{r_{2,1}}{r_{1,2}}\sin\,\theta_{1,2}\cos\,\theta_{2,1} - \sin\theta_{1,2}\cos\,\theta_{1,2}\right),\\
\frac{dr_{1,2}}{dt} &= kx_{1,2}\cos\,\theta_{1,2} - kr_{1,2}(\cos^2\,\theta_{1,2} + \gamma_{1,2}\sin^2\,\theta_{1,2}) + kr_{1,2}\cos\,\theta_{1,2}\sin\,\theta_{1,2}(1-\beta)\\
&\quad + C(r_{2,1}\cos\,\theta_{1,2}\cos\,\theta_{2,1} - r_{1,2}\cos^2\,\theta_{1,2})\,. 
\end{aligned} \tag{9}$$

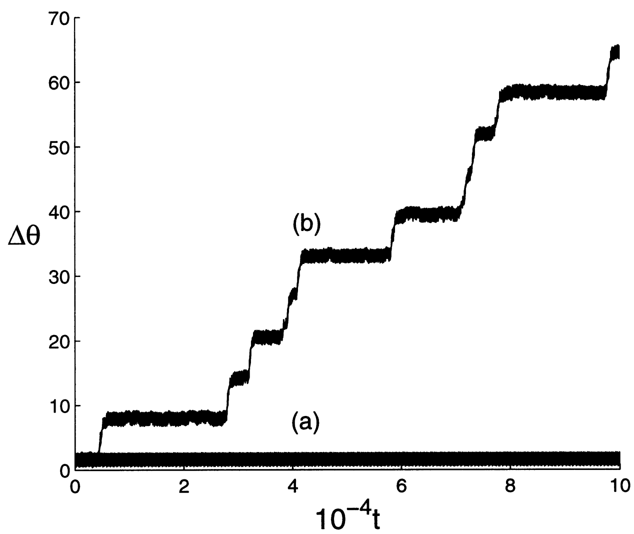



Fig. 3. For the system of two coupled Chua's circuits: phase synchronization without noise (lower trace); and $2\pi$-phase slips induced by noise of amplitude $\varepsilon = 10^{-3}$ (upper trace).

When there is no coupling ($C = 0$), the phase angles $\theta_1(t)$ and $\theta_2(t)$ are uncorrelated and, hence, the phase difference $\Delta\theta(t) = |\theta_2(t) - \theta_1(t)|$ increases steadily with time. When $C$ is increased through a critical value $C_p$ at which one of the two original null Lyapunov exponents becomes negative, phase synchronizatin occurs in the sense that the phase difference is bounded by a constant less than $2\pi$: $\Delta\theta(t) \le 2\pi$ [Rosenblum *et al.*, 1996]. For $\gamma_1 = -0.95$ and $\gamma_2 = -0.94$, we numerically obtain: $C_p \approx 0.01$. A typical situation of phase synchronization is shown in Fig. 3 (the lower trace) for $C = 0.011 \gtrsim C_p$, where we see that $\Delta\theta(t)$ is bounded within $2\pi$.

Under the influence of noise, phase synchronization can no longer be sustained ($\Delta\theta(t)$ is no longer bounded). To model noise, we add different realizations of the following terms $\varepsilon\sigma_{x,\theta,r}(t)$ to each one of the six variables[3] in the coupled Chua's system Eq. (7) at each step of integration, where $\varepsilon$ is the noise amplitude and $\sigma$'s are random variables uniformly distributed in $[-1, 1]$. The upper trace in Fig. 3 shows for $\varepsilon = 10^{-3}$, $\Delta\theta(t)$ versus $t$. We see that noise induces occasional phase slips in units of approximately $2\pi$ in $\Delta\theta(t)$. These phase slips are, however, rare and become extremely infrequent as the noise amplitude is decreased. Figure 4 shows, on a semi-logarithmic scale, histograms of the time intervals between successive phase slips for three values of the noise amplitude. These time intervals can be regarded as *transient time*. From Fig. 4, we see that approximately, the distributions of the transient times are exponential, indicating that average lifetimes of the transients can be defined for different noise levels. Figure 5 shows the average transient lifetime $\tau$ (averaged over 250 intervals) versus the rescaled noise amplitude $\varepsilon^{-\alpha}$, where $\alpha \approx 0.02$. The approximate linear behavior in Fig. 5 suggests the super persistent transient scaling relation (3). Thus, as $\varepsilon \to 0$, the average time

[3]In fact, adding noise to any subset of dynamical variables yields similar results, in particular, the scaling of the lifetime of the super persistent chaotic transients. The range of the noise that is convenient for numerically obtaining the scaling, however, tends to shift slightly when different subsets of dynamical variables are subject to noise.

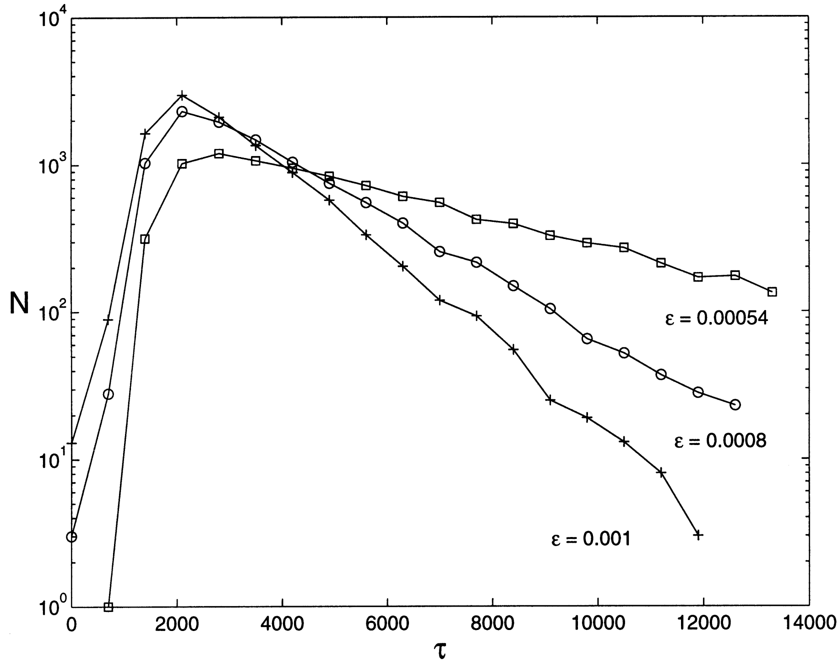



Fig. 4. For the system of two coupled Chua's oscillators: histograms of the transient time intervals for temporal phase synchronization at different noise levels.

interval to observe the $2\pi$-phase slips becomes extremely long so that in a practical sense, the chaotic oscillators can still be considered synchronized in phase.

The numerically observed scaling law, as in Fig. 5, is only indicative of the dynamical characteristic of the noise-induced phase slips. It is difficult to extend the range of numerical computations because of the extremely long transient behavior between the phase slips. For the parameter setting described above, we find that $\tau$ can be so prohibitively long that numerical computation of it becomes infeasible when the noise amplitude $\varepsilon$ is smaller than, say, $10^{-5}$. Nonetheless, the range of noise levels utilized in Fig. 5 extends to two orders of magnitude ($10^{-4.5} \lesssim \varepsilon \lesssim 10^{-2.5}$), indicating a robust scaling in (3).

## 2.2. *Coupled Rössler oscillators*

We consider the following system of two coupled Rössler oscillators which is utilized by Rosenblum [Rosenblum *et al.*, 1996] to first report phase synchronization

$$\begin{aligned} dx_{1,2}/dt &= -\omega_{1,2}y_{1,2} - z_{1,2} + C(x_{2,1} - x_{1,2})\,, \\ dy_{1,2}/dt &= \omega_{1,2}x_{1,2} + 0.15y_{1,2}\,, \\ dz_{1,2}/dt &= 0.2 + (x_{1,2} - 10.0)z_{1,2}\,, \end{aligned} \tag{10}$$

where $C$ is the coupling strength, and we choose $(\omega_1, \omega_2) = (1.015, 0.985)$ so that the two oscillators are slightly different. The Rössler chaotic attractor [Rössler, 1976] has the property that its $(x, y)$ variables represent a chaotic rotation with well-defined phase angles [Rosenblum *et al.* 1996, 1997],

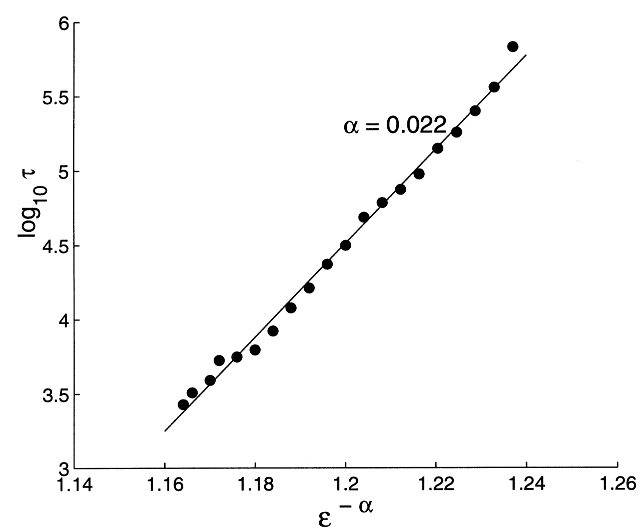



Fig. 5. For the system of two coupled Chua's oscillators at $C = 0.011$: $\log_{10}\tau$ versus $\varepsilon^{-\alpha}$, where $\alpha \approx 0.022$ is a fitting parameter. Each point represents the average over 250 time intervals.

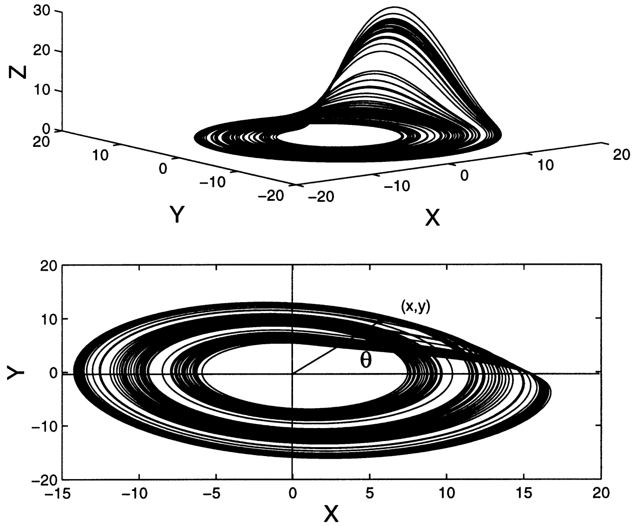



Fig. 6. Three-dimensional representation of the Rössler attractor and its projection onto the $(x, y)$-plane.

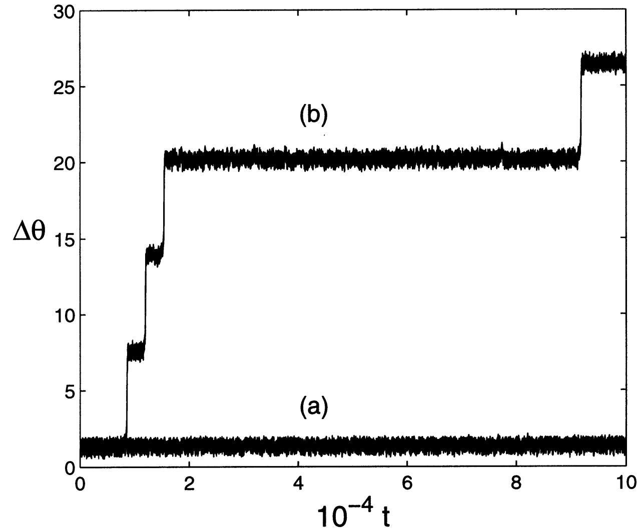



Fig. 7. For the system of two coupled Rössler oscillators: phase synchronization without noise (lower trace); and $2\pi$-phase slips induced by noise of amplitude $\varepsilon = 10^{-3}$ (upper trace).

as shown in Fig. 6. To compute the phase angles associated with the two oscillators, it is again convenient to use the polar coordinates $(r, \theta)$ to replace the $(x, y)$ coordinates. In the cylindrical coordinate $(r, \theta, z)$, the Rössler equations become

$$\begin{aligned}
\frac{dr_{1,2}}{dt} &= 0.15 r_{1,2} \sin^2 \theta_{1,2} \\
&\quad + [C(r_{2,1} \cos \theta_{2,1} - r_{1,2} \cos \theta_{1,2}) \\
&\quad - z_{1,2}] \cos \theta_{1,2} \,, \\
\frac{d\theta_{1,2}}{dt} &= \omega_{1,2} + 0.15 \sin \theta_{1,2} \cos \theta_{1,2} \\
&\quad - \frac{1}{r_{1,2}} [C(r_{2,1} \cos \theta_{2,1} - r_{1,2} \cos \theta_{1,2}) \\
&\quad - z_{1,2}] \sin \theta_{1,2} \,, \\
\frac{dz_{1,2}}{dt} &= 0.2 + (r_{1,2} \cos \theta_{1,2} - 10.0) z_{1,2} \,. \qquad (11)
\end{aligned}$$

Similar to the dynamics in the coupled Chua's circuits Eq. (9), when there is no coupling, the phase angles $\theta_1(t)$ and $\theta_2(t)$ are uncorrelated. Phase synchronization occurs when $C$ is increased through the critical value $C_p \approx 0.029$. The lower trace in Fig. 7 shows such a situation for $C = 0.03$, where $\Delta\theta(t)$ versus $t$ is plotted. Additive white noise induces phase slips in units of $2\pi$, as shown by the upper trace in Fig. 7, where the noise amplitude is $\varepsilon = 10^{-3}$. Figure 8 shows, on a semi-logarithmic scale, histograms of the transient lifetime of the temporal phase synchronization at four different noise levels, which again, suggest exponential distributions of the transient lifetimes so that an average lifetime can be defined properly for each noise level.[4] Figure 9 shows $\log_{10} \tau$ versus $\varepsilon^{-\alpha}$ for $10^{-3.5} \lesssim \varepsilon \lesssim 10^{-1.5}$ (again, approximately two orders of magnitude in $\varepsilon$) for three values of the coupling parameter close to $C_p$. All three data sets suggest the presence of super persistent chaotic transients characterized by (3).

### 2.3. Mechanism of $2\pi$-phase slips induced by noise

To qualitatively understand why noise can induce $2\pi$-phase slips, we make use of the idea of the lifted angle variable [Rosa *et al.*, 1998] and regard the state of phase synchronization as attractors in the extended angle space. In particular, we perform the following numerical experiments for both the coupled Chua and Rössler oscillators. First, we set $\varepsilon = 0$ and plot, in the coordinate $(r \equiv \sqrt{r_1^2 + r_2^2}, \Delta\theta)$, the attractors resulting from two different initial conditions with $0 < \Delta\theta < 2\pi$ and $2\pi < \Delta\theta < 4\pi$, respectively, as shown in Fig. 10(a) for the coupled

[4]We note that for small noise levels, the distributions exhibit an algebraic tail, as illustrated in the inlet of Fig. 8 for $\varepsilon = 10^{-3}$. The algebraic decay exponent is, however, larger than 2. Thus, an average lifetime can still be defined from the algebraic distribution.

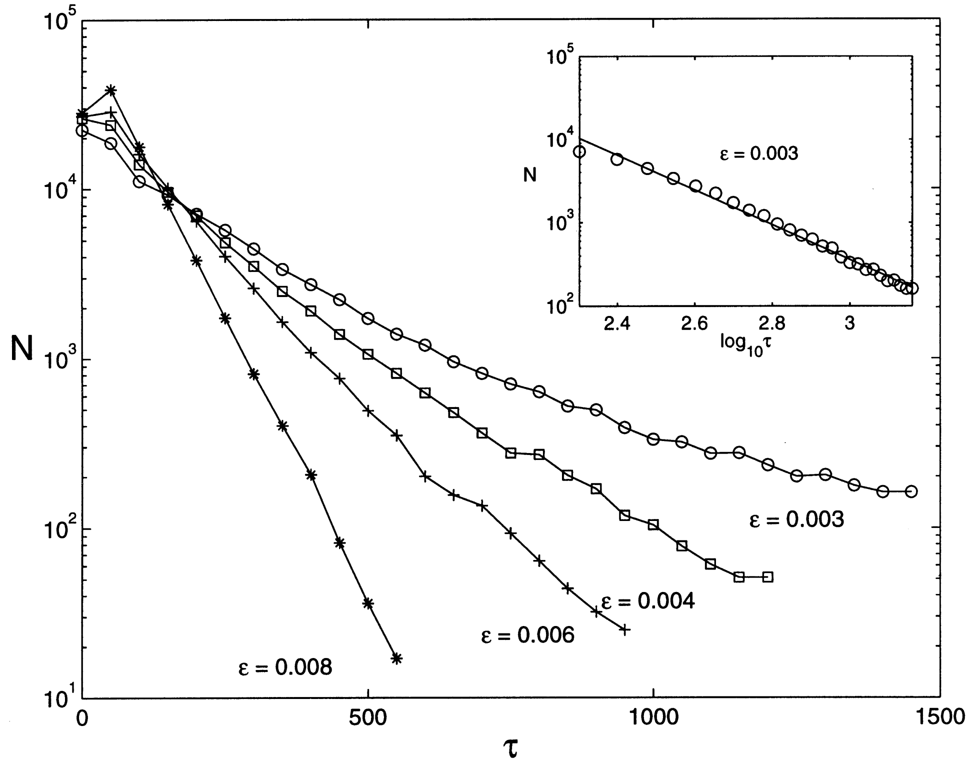



Fig. 8. For the system of two coupled Rössler oscillators: histograms of the transient time intervals for temporal phase synchronization at different noise levels.

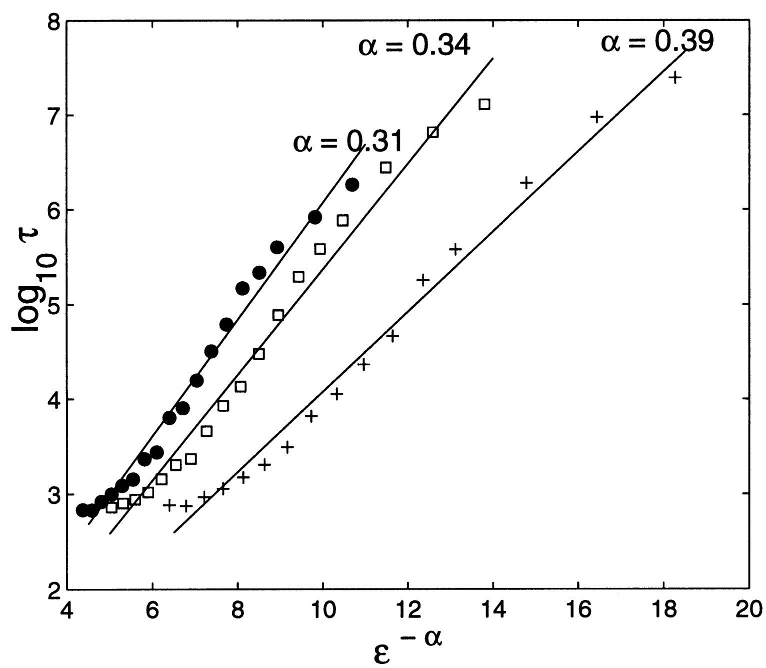
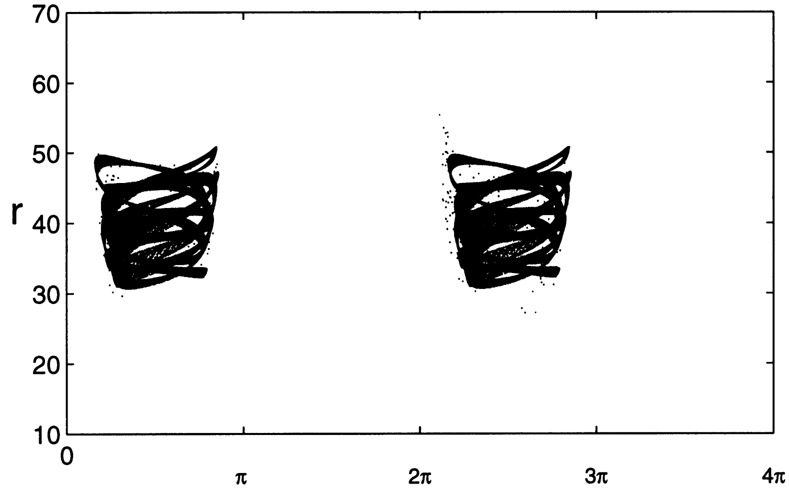
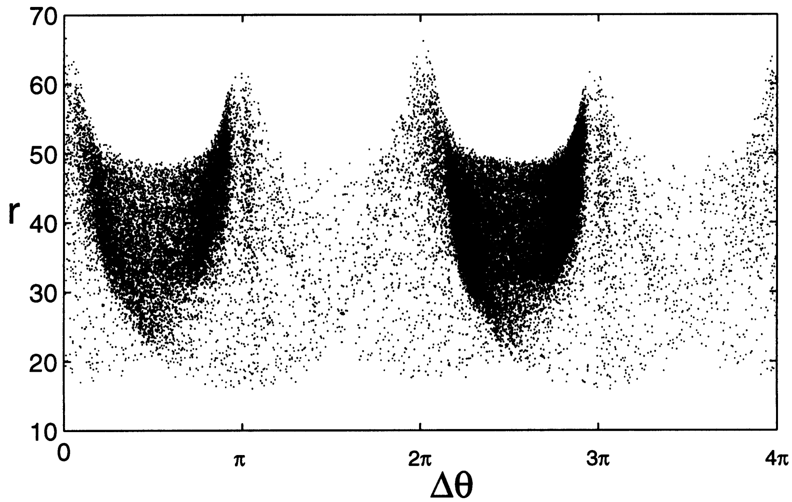



Fig. 9. Scaling of super persistent transients of chaotic phase synchronization at different coupling parameters for the system of two coupled Rössler oscillators for $C = 0.03$ ("•"), $C = 0.0303$ ("□"), and $C = 0.0305$ ("+"). Shown are three data sets of $\log_{10}\tau$ versus the rescaled noise level $\varepsilon^{-\alpha}$. Each point is obtained by averaging 100 transient time intervals of temporal phase synchronization.



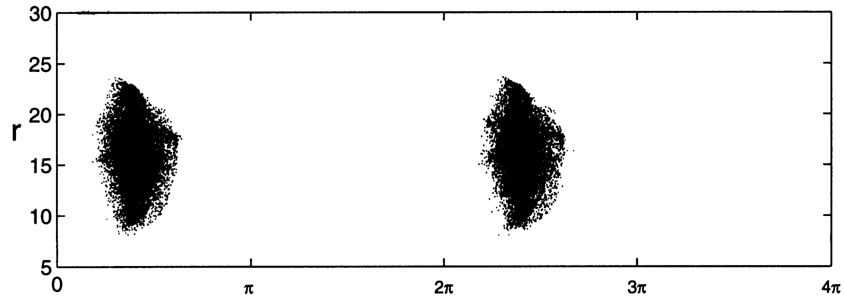

(a)



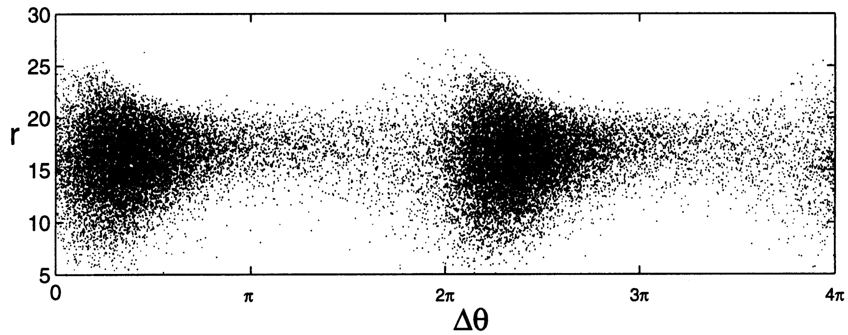

(b)

Fig. 10. Using the lifted phase variable $\Delta\theta$ for the system of two coupled Chua's oscillators: (a) two isolated phase-synchronized attractors in $0 < \Delta\theta < 2\pi$ and $2\pi < \Delta\theta < 4\pi$ in the absence of noise; (b) tunneling between the previously isolated attractors due to noise.

Chua's circuits and in Fig. 11(a) for the coupled Rössler oscillators. The variable $\Delta\theta$ is thus a *lifted* angle variable by which differences of integer multiple of $2\pi$'s are considered distinct. We see that initial conditions with $2\pi$ difference in $\Delta\theta$ result in attractors that live in different basins of attraction. Depending on the initial conditions, there is an infinite number of these attractors separated from each other by $2\pi$ in $\Delta\theta$. In the absence of noise, these attractors are completely isolated, corresponding to the situation of phase synchronization where $\Delta\theta$ remains within $2\pi$ if they start with a value less than $2\pi$. Next, we examine the influence of noise on the phase-space structure in Figs. 10(a) and 11(a), as shown in Fig. 10(b) for $\varepsilon = 10^{-3}$ and Fig. 11(b) for $\varepsilon = 10^{-2}$. We see that the basins of attraction of the previously isolated attractors in both cases are now connected. There is now a nonzero probability that a trajectory can switch to different attractors separated by $2\pi$ in $\Delta\theta$, corresponding to the $2\pi$-phase slips observed in Figs. 3 and 7. The switch occurs when the trajectory falls into an open "tunnel" connecting the basins. The widths of these tunnels must be exponentially small so that the probability




(a)




(b)

Fig. 11. Using the lifted phase variable $\Delta\theta$ for the system of two coupled Rössler oscillators: (a) two isolated phase-synchronized attractors in $0 < \Delta\theta < 2\pi$ and $2\pi < \Delta\theta < 4\pi$ in the absence of noise; (b) tunneling between the previously isolated attractors due to noise.

for the trajectory to fall into a tunnel is extremely small, leading to the scaling behaviors in Figs. 5 and 9. Figures 10 and 11 thus provide, qualitatively, the dynamical mechanism for super persistent chaotic transients described by Grebogi *et al.* [1983b, 1985].

## 3. Theories

Note that in Eq. (11), the scales of time variation of the amplitude variables $r_{1,2}(t)$ and phase variables $\theta_{1,2}(t)$ are generally distinct. Since, on average, we have $\theta_{1,2}(t) \sim \omega_0 t$, we see that the phase angles $\theta_{1,2}(t)$ are "fast" variables. The amplitudes $r_{1,2}(t)$ are slow variables because the Rössler chaotic trajectories have approximately a circularly rotational structure. Thus, one can average over rotations of the phase angles to separate out the dynamics of the slow variables. Letting $\theta_{1,2}(t) = \omega_0 t + \phi_{1,2}(t)$ and performing averaging in the time interval $t \in [0, 2\pi/\omega_0]$ yield [Pikovsky *et al.*, 1997]

$$\frac{d\Phi(t)}{dt} \approx 2\delta\omega + CG(r_1, r_2)\sin \Phi(t) + \textit{white noise term}\,, \qquad (12)$$

where $\Phi(t) \equiv \phi_2(t) - \phi_1(t) = \theta_2(t) - \theta_1(t)$, $\delta\omega \equiv \omega_1 - \omega_2$, and $G(r_1, r_2)$ is a function that depends on the chaotic amplitudes $r_{1,2}(t)$. Equation (12) thus describes the dynamics of a chaotically driven limit-cycle oscillator and is analogous to the equation describing a Brownian particle in a potential field [Stratonovich, 1963]. While the specific form of Eq. (12) is for the system of coupled Rössler oscillators, we notice the general feature of the phase-synchronization problem: *limit-cycle oscillator driven by chaos*. In the following we present two theories for the scaling law (3) of the super persistent transients of chaotic phase synchronization.

### 3.1. *Basin tunneling theory*

We construct the following model of two-dimensional maps incorporating the general dynamical features of phase synchronization [Pikovsky *et al.*, 1997]

$$\begin{aligned} x_{n+1} &= f(x_n)\,, \\ \Phi_{n+1} &= \varepsilon + pg_1(x_n)\Phi_n + g_2(x_n)\Phi_n^2 \\ &\quad + g_3(x_n)\Phi_n^3 \end{aligned} \qquad (13)$$

where $x \in R^N (N \geq 1)$, $\Phi \in R^M (M \geq 1)$, and $f(x)$ is a chaotic map in which the variable $x$ models the chaotic amplitudes in Eq. (12), $\varepsilon \gtrsim 0$ models the combination of the small noise and the slight parameter mismatch between the two coupled chaotic oscillators, $g_{1,2,3}(x)$ are smooth functions, and $p$ is a parameter that is proportional to the coupling strength. Assume that $f(x)$ generates a chaotic attractor with an infinite number of unstable periodic orbits embedded in it, and phase synchronization occurs for $p > p_c$. In the $\Phi$-direction, these periodic orbits can be stable or unstable. For $p \gtrsim p_c$, all periodic orbits are stable in the $\Phi$-direction in the absence of noise, so $\Phi$ remains approximately constant (phase synchronization). Under the influence of noise a bifurcation occurs. That is, some of the periodic orbits embedded in the chaotic attractor, usually of low period, become unstable in the $\Phi$-direction. Since these periodic orbits are already unstable in the chaotic attractor, they become repellers in the two-dimensional phase space, as such, a set of "tongues" opens at the locations of these periodic orbits, allowing the trajectory to escape from one approximately constant $\Phi$ state to another ($2\pi$-phase slips). The sizes of the tongues are exponentially small [Grebogi *et al.*, 1985; Lai *et al.*, 1996], which accounts for the extremely long time duration between the successive $2\pi$-phase slips. Let $\lambda > 0$ be the maximum Lyapunov exponent of the $x$ chaotic attractor and let $L_u = e^{\lambda} > 1$, which is the expanding eigenvalue of an infinitesimal vector in the $x$-direction, and let $T$ be the time for a trajectory to tunnel through one of the tongues. We have, for the typical size of the opening of the tongue, the following

$$\delta < \frac{1}{(L_u)^T} \sim e^{-\lambda T} . \tag{14}$$

The probability that a trajectory falls into the tongue is proportional to the size of the opening $\delta$ and the average time for a trajectory to fall into the tongue is then

$$\tau \sim \frac{1}{\delta} \sim e^{\lambda T} \tag{15}$$

which is the average time between the successive phase slips.

The tunneling time $T$ can be estimated by noting that when $T$ is large, the map equation in $\Phi$ in Eq. (13) can be approximated as: $d\Phi/dt \approx \varepsilon + [pg_1(x) - 1]\Phi + g_2(x)\Phi^2 + g_3(x)\Phi^3$, which yields

$$T \approx \int_0^{2\pi} \frac{d\Phi}{\varepsilon + [pg_1 - 1]\Phi + g_2\Phi^2 + g_3\Phi^3} . \tag{16}$$

The dependence of $T$ on $\varepsilon$ is thus determined by the specific functions $g_{1,2,3}(x)$. For instance, since we know that most periodic orbits embedded in the $x$ chaotic attractor are stable in the $\Phi$-direction, we have $pg_1(x) \lesssim 1$ and, hence

$$T \approx \int_0^{2\pi} \frac{d\Phi}{\varepsilon + g_2\Phi^2 + g_3\Phi^3} . \tag{17}$$

If, we have $g_2(x) \approx 0$ and $g_3(x) \approx 1$, then

$$T \approx \int_0^{2\pi} \frac{d\Phi}{\varepsilon + g_2\Phi^2 + g_3\Phi^3} \sim \varepsilon^{-2/3} . \tag{18}$$

We thus obtain

$$\tau \sim e^{C\lambda\varepsilon^{-2/3}} . \tag{19}$$

The exponent 2/3 is a consequence of the cubic dependence: $\Phi^3$. If, $g_2(x) \approx 1$ and $g_3(x) \approx 0$, we have

$$T \approx \int_0^{2\pi} \frac{d\Phi}{\varepsilon + g_2\Phi^2} = C\varepsilon^{-1/2} , \tag{20}$$

which gives

$$\tau \sim e^{C\lambda\varepsilon^{-1/2}} . \tag{21}$$

In general, we expect $T \sim \varepsilon^{-\alpha}$, where $\alpha > 0$, which when substituted in Eq. (15), gives the scaling relation (3).

### 3.2. *Statistical mechanical approach*

Qualitatively, Eq. (12) models the motion of a classical particle in the following potential field that is approximately periodic in $\Phi$

$$\begin{aligned} V(\Phi) = C &\left( \frac{\partial\Phi}{\partial r_1}\dot{r_1} + \frac{\partial\Phi}{\partial r_2}\dot{r_2} \right) \cos\Phi \\ &- 2(\delta\omega)CG(r_1, r_2) \sin\Phi \\ &+ \frac{1}{4}C^2G^2(r_1, r_2) \cos 2\Phi . \end{aligned} \tag{22}$$

When the coupling strength is large enough, the potential function $V(\Phi)$ possesses an infinite number of local minima separated by $2\pi$ in the phase variable $\Phi$ [Stratonovich, 1963]. The chaotic amplitude factor $G(r_1, r_2)$ and its partial derivatives model the fluctuations of the minimum potential values. When these minima are present, a particle starting near one of the local minima is trapped in its vicinity forever in a noiseless situation. The explanation for noise-induced $2\pi$-phase slips in chaotic phase synchronization is then as follows. From our potential model, in the regime of phase synchronization ($C > C_p$), the depths of the potential wells are larger than the maximum amplitude of the chaotic fluctuations. Therefore, in the absence of noise, a particle trapped in one of the potential wells cannot move to the adjacent wells. Under the influence of noise, when the combined amplitude of noise and chaotic fluctuations exceeds the depth of the potential well, $2\pi$-phase slips can occur. This is similar to the case before phase synchronization ($C < C_p$) where the depth of the potential well is smaller than the amplitude of the chaotic fluctuations. In this sense, intuitively, the phenomenon of noise induced $2\pi$-phase slips is similar to that observed before phase synchronization [Lee *et al.*, 1998]. The parameter regime slightly before phase synchronization where the phase slips are rare is called *nearly synchronous* regime [Rosenblum *et al.*, 1996].

The scaling of the transient lifetime can thus be derived based on the above mechanism. The probability for noise to kick a particle from a potential minimum to its adjancent one is given by

$$P \sim e^{-\Delta E/T}\,,$$

where $\Delta E$ is the typical height of the potential barrier that separates neighboring minima and $T$ is the "temperature" that is determined by the noise. Typically, we have $T \sim \varepsilon^{\alpha}$, where $\alpha > 0$. The average time for a $2\pi$-phase jump to occur is thus given by: $\tau \sim 1/P \sim \exp(\Delta E \varepsilon^{-\alpha})$, which is the scaling law Eq. (3).

As the coupling parameter $C$ is increased, the height of the potential barrier $\Delta E$ decreases and, as a result, it is "easier" for noise to kick a particle from the region of a potential minimum. The transient lifetime thus decreases in general, as shown in Fig. 9. To measure the scaling in a laboratory experiment, it is thus desirable to use a relatively large value of coupling parameter.

## 4. Conclusion

In summary, we have studied the effect of small random noise on phase synchronization of coupled chaotic oscillators. Under the influence of noise, indefinite phase synchronization is no longer possible. Instead, $2\pi$-phase slips between the oscillators occur. When the noise amplitude is small, these phase slips are extremely rare. Thus, we expect to be able to observe phase synchronization for a long time in well-controlled laboratory experiments where noise is small. We give theoretical arguments and numerical evidence suggesting that the average time during which temporal phase synchronization can be maintained scales with the noise level in a way that is completely analogous to that of the super persistent chaotic transients. The paradigm of chaotic phase synchronization thus provides a natural setting for observing super persistent chaotic transients in laboratory experiments [Zhu *et al.*, 2001].

We remark that there are also other mechanisms for phase slips which are purely deterministic. There are at least two situations, as follows: (1) coupled nonidentical chaotic oscillators in parameter regimes immediately preceding phase synchronization [Lee *et al.*, 1997; Park *et al.*, 1999], and (2) a periodically driven chaotic system with a broad distribution of intrinsic time scales [Zaks *et al.*, 1999]. In the first case, the origin of phase slips is similar to that due to noise, as discussed in this paper. In particular, consider the statistical mechanical picture described in Sec. 3.2 where phase synchronization is characterized by a permanent confinement of particles in one of the potential wells, which occurs when the magnitude of chaotic fluctuations are smaller than the depths of the potential wells. In parameter regimes close to phase synchronization, however, the instantaneous chaotic fluctuations can occasionally be larger than the depths of the wells, thereby inducing $2\pi$-phase slips. In time scales larger than the intrinsic correlation time of the chaotic oscillators, chaotic fluctuations can be regarded as noise. Thus, the origin of the phase slips is completely identical to what is described in this paper and, hence, we expect to observe transient phase synchronization that is super persistent. In the second case where the unstable periodic orbits embedded in the chaotic oscillators have distinct time scales, the phase of a chaotic variable tends to follow that of the driving force for long time, but such a synchronization is usually interrupted by phase shifts that occur in short time scales. It is argued [Zaks, *et al.*, 1999] that the shift is in fact another short-lived synchronized state, which is caused by the passage of the chaotic trajectory near the long unstable periodic orbits whose frequencies are locked by external force in ratios different from 1 : 1. As such, the distribution of the lifetime of the transient phase synchronization depends on how often distinct unstable periodic orbits are visited by a typical trajectory, which can possibly follow a scaling that is different from that of super persistent transients. This problem is interesting and certainly warrants further investigation.

## Acknowledgments

This work is supported by AFOSR under Grant No. F49620-98-1-0400 and by NSF under Grant No. PHY-9722156.

## References

Andrade, V., Davidchack, R. & Lai, Y.-C. [2000] "Noise scaling of phase synchronization of chaos," *Phys. Rev.* **E61**, 3230–3233.

Bollt, E. & Dolnik, M. [1997] "Encoding information in chemical chaos by controlling symbolic dynamics," *Phys. Rev.* **E55**, 6404–6413.

Bollt, E., Lai, Y.-C. & Grebogi, C. [1997] "Coding, channel capacity, and noise resistance in communicating with chaos," *Phys. Rev. Lett.* **79**, 3787–3790.

Chua, L. O., Komuro, M. & Matsumoto, T. [1986] "The double scroll family (I, II)," *IEEE Trans. Circuits Syst.* **33**, 1073–1118.

Chua, L. O. & Lin, G. N. [1990] "Canonical realization of Chua's circuit family," *IEEE Trans. Circuits Syst.* **37**, 885–902.

Chua, L. O. & Tichonicky, I. [1991] "1-D map for the double scroll family," *IEEE Trans. Circuits Syst.* **38**, 233–243.

Chua, L. O. [1992] "The genesis of Chua's circuit," *Archiv. Elektronik Übertragungstechnik* **46**, 250–257.

Chua, L. O. [1993] "Global unfolding of Chua's circuit," *IEICE Trans. Fundamentals Electron. Commun. Comput. Sci.* **E76-A**, 704–734.

Chua, L. O., Wu, C. W., Huang, A. & Zhong G.-Q. [1993] "A universal circuit for studying and generating chaos (I, II)," *IEEE Trans. Circuits Syst.* **40**, 732–761.

Chua, L. O. [1994] "Chua's circuit: Ten years later," *Int. J. Circuits Th. Appl.* **22**, 279–305.

Chua, L. O., Pivka, L. & Wu, C. W. [1995] "A universal circuit for studying chaotic phenomena," *Phil. Trans. R. Soc. Lond.* **A353**, 65–84.

Ditto, W. L. *et al.* [1989] "Experimental observation of crisis-induced intermittency and its critical exponent," *Phys. Rev. Lett.* **63**, 913–916.

Grebogi, C., Ott, E. & Yorke, J. A. [1983a] "Crises, sudden changes in chaotic attractors, and transient chaos," *Physica* **D7**, 181–200.

Grebogi, C., Ott, E. & Yorke, J. A. [1983b] "Fractal basin boundaries, long-lived chaotic transients, and unstable-unstable pair bifurcation," *Phys. Rev. Lett.* **50**, 935–938.

Grebogi, C., Ott, E. & Yorke, J. A. [1985] "Super persistent chaotic transients," *Ergod. Th. Dyn. Syst.* **5**, 341–372.

Grebogi, C., Ott, E., Romeiras, F. & Yorke, J. A. [1987] "Critical exponents for crisis-induced intermittency," *Phys. Rev.* **A36**, 5365–5380.

Hayes, S., Grebogi, C. & Ott, E. [1993] "Communicating with chaos," *Phys. Rev. Lett.* **70**, 3031–3034.

Lai, Y.-C., Grebogi, C., Yorke, J. A. & Venkataramani, S. C. [1996] "Riddling bifurcation in chaotic dynamical systems," *Phys. Rev. Lett.* **77**, 55–58.

Lee, K. J., Kwak, Y. & Lim, T. K. [1998] "Phase jumps near a phase synchronization transition in systems of two coupled chaotic oscillators," *Phys. Rev. Lett.* **81**, 321–324.

Madan, R. [1993] *Chua's Circuit: A Paradigm for Chaos* (World Scientific, Singapore).

Matsumoto, T. [1984] "A chaotic attractor from Chua's circuit," *IEEE Trans. Circuits Syst.* **CAS-31** (12), 1055–1058.

Park, E.-H., Zaks, M. A. & Kurths, J. [1999] "Phase synchronization in the forced Lorenz system," *Phys. Rev.* **E60**, 6627–6638.

Pikovsky, A. S., Osipov, G., Rosenblum, M. G., Zaks, M. & Kurths, J. [1997] "Attractor–repeller collision and eyelet intermitency at the transition to phase synchronization," *Phys. Rev. Lett.* **79**, 47–50.

Rosa, E., Hayes, S. & Grebogi, C. [1997] "Noise filtering in communication with chaos," *Phys. Rev. Lett.* **78**, 1247–1250.

Rosa, E., Ott, E. & Hess, M. H. [1998] "Transition to phase synchronization of chaos," *Phys. Rev. Lett.* **80**, 1642–1645.

Rosenblum, M. G., Pikovsky, A. S. & Kurths, J. [1996] "Phase synchronization of chaotic oscillators," *Phys. Rev. Lett.* **76**, 1804–1807.

Rosenblum, M. G., Pikovsky, A. S. & Kurths, J. [1997] "Phase synchronization of chaotic oscillators by external driving," *Physica* **D104**, 219–238.

Rössler, O. E. [1976] "An equation for continuous chaos," *Phys. Lett.* **A57**, 397–400.

Stratonovich, R. L. [1963] *Topics in the Theory of Random Noise* (Gordon and Breach, NY).

Tél, T. [1991] "Transient chaos," *Directions in Chaos*, Vol. 3, ed. Hao, B.-L., Feng, D. H. & Yuan, J. M. (World Scientific, Singapore).

Tél, T. [1996] "Transient chaos: A type of metastable state," in *STATPHYS 19*, ed. Hao, B.-L. (World Scientific, Singapore).

Yalcinkaya, T. & Lai, Y.-C. [1997] "Phase characterization of chaos," *Phys. Rev. Lett.* **79**, 3885–3888.

Zaks, M. A., Park, E.-H., Rosenblum, M. G. & Kurths, J. [1999] "Alternating locking ratios in imperfect phase synchronization," *Phys. Rev. Lett.* **82**, 4228–4231.

Zhu, L., Raghu, A. & Lai, Y.-C. [2001] "Experimental observation of superpersistent chaotic transients," *Phys. Rev. Lett.* **86**, 4017–4020.