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CHAOS IN CONDITIONALLY RESET LINEAR SYSTEMS*

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This paper presents a novel and simple design of conditionally reset linear systems which exhibit chaotic behavior. The proposed systems are constructed from building blocks (e.g. reset integrator, nonlinear controller, summer and gain amplifiers) which can be either designed or found in the off-shelf electronics library. Throughout the paper, computer simulation results are used to show the stretching and folding mechanism for chaos generation in these systems.

1. Introduction

For the last few decades, significant progress has been made to define and explain what we observe as chaos in systems. Yet, the theory of chaos is still not able to answer the basic questions often posed by engineers, e.g. whether chaos is present in or absent from a specific circuit.[1] Among many existing complex chaotic systems, the Lorenz and Rössler dynamical systems have fascinated scientists for many years. However, understanding how chaos exists in such systems has been a challenging problem. Thus, the design of simple chaotic circuits is very important not only to understand the nature of chaos but also to develop sophisticated applications which actually use this behavior.[2]

The Chua's circuit (a third-order autonomous, dissipative electrical circuit) brought hope that simple chaotic systems can actually be designed. Several researchers have investigated and analyzed the Chua circuit family in depth.[3,4] For instance, the Chua's double scroll attractor, double hook, torus and other interesting attractors have been observed from several members of this family.[5,6] The colpitts oscillator[7] and the hysteresis chaos generator[8,9] are also among circuits in which a variety of dynamical behaviors including chaos have been reported. Brown[10] proposed a theoretical approach on how to decompose a highly complex chaotic

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system into a interconnected subsystem of lower order which can be easily studied. However, no implementation of such systems was given.

In this paper a novel and simple design of conditionally reset linear systems which exhibit chaos is presented. These systems are constructed from building blocks (e.g. reset integrator, nonlinear controllers, summers and gain amplifiers), which can be either designed or found in the off-shelf electronics library. Throughout the paper, computer simulation results obtained with the aid of SIMULINK,[11] are used to show the existence of the desired attractors. It is noteworthy to mention that the generalization of the Chua equations by Brown[10] has provided the basis of the implementation of these chaotic systems presented here.

Section 2 of this paper discusses the general procedure of designing a two-dimensional single scroll system using a reset integrator and a nonlinear controller. The problem of unstable attractors is also discussed. In Sec. 3, the order of the dynamical system is augmented to obtain a three-dimensional single scroll system. In Sec. 4, the building block approach is expanded to design chaotic systems that exhibit attractors similar to those observed experimentally in Chua's circuit. Different controller nonlinearities are used to obtain similar types of attractors.

2. Two-Dimensional Single Scroll System

Our aim in this section is to describe the implementation of a system that exhibits a two-dimensional single scroll. We will show later how this simple system evolves through some nonlinear blocks into a system exhibiting the Chua's double scroll attractor. The system specifications are such that the trajectory must scroll away from the equilibrium point and then resets itself to repeat the process.

We start by considering the following two-dimensional autonomous dynamical system which is described by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x(t) - b_1 \\ y(t) - b_2 \end{pmatrix} . \tag{1}$$

A more compact form is given by

$$\dot{X}(t) = AX(t) - b\,, \qquad X(0) = X_0 \tag{2}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{and} \quad X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} .$$

$X(0)$ is the initial condition. $X(t)$ is the trajectory. $\dot{X}(t)$ denotes the derivative of $X(t)$. The equilibrium point X_Q^+ of (1) is defined as the state at which $\dot{X}(t)$ is zero.

$$X_Q^+ = b$$

To meet the above specifications, we explore what is needed to generate a bounded two-dimensional single scroll. First, for the trajectory to scroll away from the equilibrium point, X_Q^+ must be unstable. That is the eigenvalues of A, (the roots λ of the characteristic equation, $det(\lambda I - A) = 0, I$ is the identity matrix) must be complex with a positive real part. Thus,

$$\lambda = \sigma \pm j\omega , \quad \text{where} \quad \sigma > 0 . \tag{3}$$

With these eigenvalues, the trajectories are pushed away from the equilibrium point while scrolling. This mechanism is called "*stretching*". The second item we need is a "*folding*" mechanism i.e. if the state trajectory satisfies a certain specified condition then the system resets itself. Thus, a reset function will be integrated as part of the complete system. Finally, the initial condition $X(0)$ must be selected so that the resultant trajectory is bounded. It should be pointed out that the above ingredients are necessary but may not be sufficient due to the selection of other parameters.

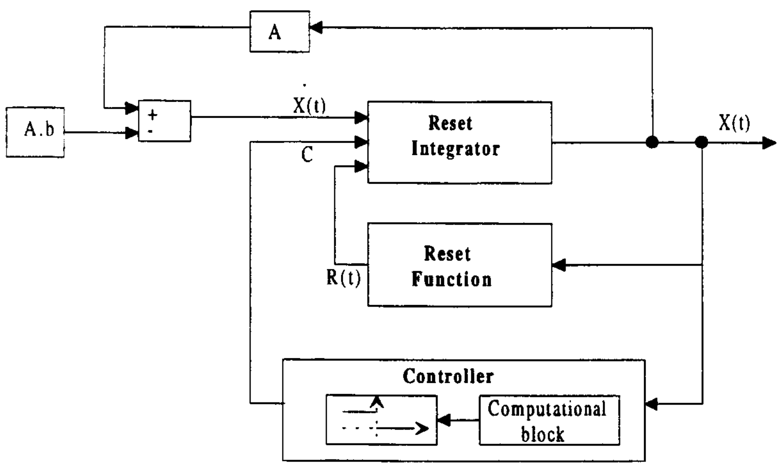

Fig. 1. Block diagram of a single scroll system.

The proposed system architecture which meets the above specifications is shown in Fig. 1. The system consists of three main blocks. The first and most important block is the "Reset Integrator" (RI) block. It has three inputs: the signal input $\dot{X}(t)$, the control input C and the reset signal $R(t)$. It operates as follows: Given an initial state $X(0)$, the function of the RI block is to integrate $\dot{X}(t)$ while $C = 0$. If $C = 1$, then the RI block resets itself to the current state specified by $R(t)$ and continues to integrate. The second block is the "Reset function" whose output $R(t)$ is fed to the RI block. For the system to reset itself every time at a different initial state, it is logical to select $R(t)$ to be a function of $X(t)$. The third block is the controller which consists of two cascaded blocks: a computational and a nonlinear one. The computational block receives an input $X(t)$ and transmits an output $F(X(t))$. Without loss of generality the function $F(X(t))$ is selected as follows:

$$F(X(t)) = k_1 x(t) + k_2 y(t) + k_3 \tag{4}$$

where k_1, k_2, k_3 are constant. The nonlinear block whose input is $F(X(t))$, generates an output C as follows. If $F(X(t)) < 0$ is satisfied then $C = 1$, otherwise $C = 0$. Thus, the controller function is to monitor when the reset process will occur. The proposed design and implementation approach is best illustrated by giving an example of the two-dimensional single scroll. The reader is referred to Ref. 10 for more details on how the systems parameters are selected in all examples presented here.

Example 1: *Consider the following dynamical system described by*

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 \\ 1.0 & 0.334 \end{bmatrix} \begin{pmatrix} x(t) - 0.4455 \\ y(t) + 0.0540 \end{pmatrix} . \tag{5}$$

The full implementation of this system is shown in Fig. 2(a). This system is characterized by the location of the equilibrium point $X_Q^+ = (0.4455, \ -0.0540)$, two complex eigenvalues $\lambda = 0.167 \pm j3.138$ and two complex eigenvectors which form an eigenplane E_c^+. Without loss of generality, the reset function is selected to be $R(t) = -X(t)$ whose implementation is achieved by the inverting amplifiers with unity gain. Finally the controller implementation, shown in Fig. 2(a) as a block, is fully detailed in Fig. 2(b). The controller consists of two cascaded blocks. The

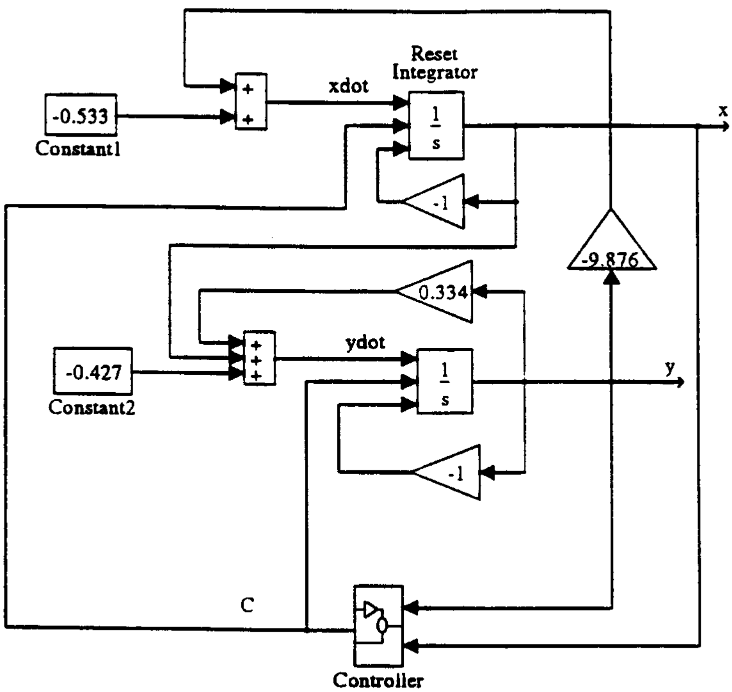

Fig. 2(a). Two-dimensional single scroll system.

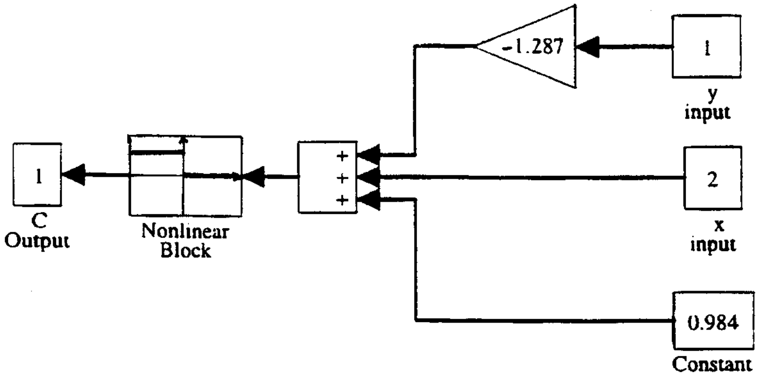

Fig. 2(b). Controller for the two-dimensional single scroll.

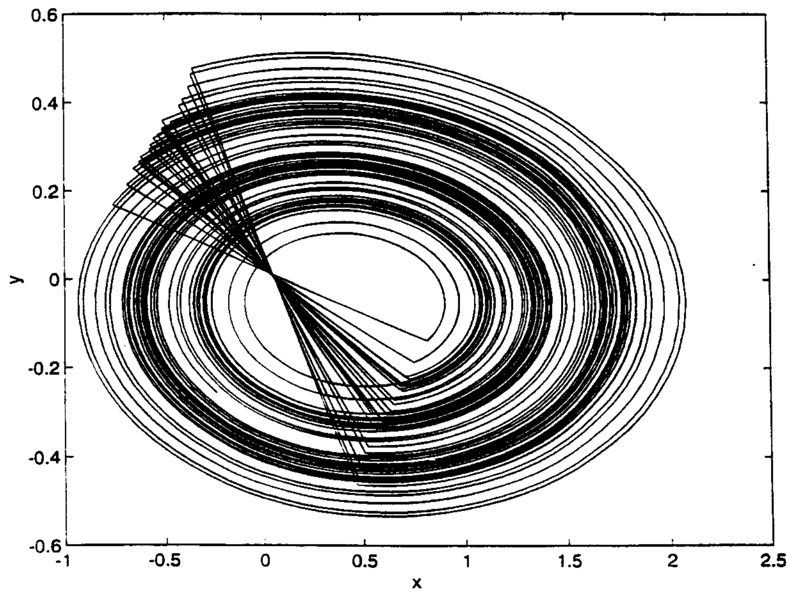

Fig. 2(c). A bounded two-dimensional single scroll.

computational block whose inputs are $x(t)$ and $y(t)$ and a constant generates an output $F(X(t))$ selected to be

$$F(X(t)) = x(t) - 1.287y(t) + 0.984\,. \tag{6}$$

Geometrically, equation $F(X(t)) = 0$ defines a line which separates two regions D^+ where $F(X(t)) \geq 0$, and D^- where $F(X(t)) < 0$. The computer simulation of the above system, shown in Fig. 2(c) reveals a bounded two-dimensional single scroll for the initial state $X(0) = (-0.015, -0.25)$. The global behavior of this system can be analyzed as follows. The trajectory starting at $X(0)$ in the region D^+ is repelled by the unstable equilibrium point X_Q^+ until it crosses into the region D^-. Once it is in the D^- region it is folded back to the D^+ region using the reset function. This process repeats itself and will lead to an attractor.

If we change the initial state to $X(0) = (1.5, 0.75)$, we obtain an unbounded trajectory. This simple example shows how the system is sensitive to the initial state. In the next section, using the same design approach, we augment the order of the two-dimensional single scroll to build a three-dimensional single scroll.

3. Three-Dimensional Single Scroll System

As mentioned in the previous section, our ultimate goal is to show the steps that will lead to the implementation of a system whose trajectory is similar to the Chua's double scroll attractor. To achieve this objective, an intermediate step which consists of designing a three-dimensional single scroll system is needed. In this case, the order of the two-dimensional single scroll system of equations (1) is increased. This will lead to the following dynamical system.

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{pmatrix} x(t) - b_1 \\ y(t) - b_2 \\ z(t) - b_3 \end{pmatrix} . \tag{7}$$

Note that the additional block is completely decoupled from the two-dimensional single scroll block described in this previous section. The equilibrium point X_Q^+ will have two complex eigenvalues $\lambda = \sigma \pm j\omega$ with a positive real part σ and one real eigenvalue γ. The eigenspace is characterized by a complex eigenplane E_c^+ corresponding to the complex eigenvalues and one eigenvector E_r^+ corresponding to the real eigenvalue γ. In the E_c^+ eigenplane, the trajectory is in the stretching mode, i.e. if we start from an initial state $X(0) \in E_c^+$, then the trajectories are pushed away from the equilibrium point X_Q^+. If we start from some initial state $X(0) \notin E_c^+$, then for the trajectory to reach the E_c^+ eigenplane, the eigenvalue γ must be negative. If $\gamma > 0$, then the trajectory is going to be pushed away from the equilibrium point along the eigenvector E_r^+ and may not reach at all the eigenplane E_c^+. This will lead to an unstable system with no attractor.

The controller block is similar to the one used for the two-dimensional single scroll except that in the computational block, the constant k_3 is now replaced by z(t).

$$F(X(t)) = k_1 x(t) + k_2 y(t) + z(t) = K^T X(t) \tag{8}$$

where

$$K^T = \begin{pmatrix} k_1 & k_2 & 1 \end{pmatrix} , \quad X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} .$$

The implementation of the three-dimensional single scroll is illustrated in the following example.

Example 2: *Consider the following dynamical system which is an expansion of the previous example.*

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 & 0 \\ 1.0 & 0.334 & 0 \\ 0 & 0 & -3.9005 \end{bmatrix} \begin{pmatrix} x(t) - 0.4455 \\ y(t) + 0.0540 \\ z(t) - 0.983 \end{pmatrix} \tag{9}$$

The equilibrium point is $X_Q^+ = (0.4455, -0.0540, 0.983)$. The eigenvalues are $\lambda = 0.167 \pm j3.138$ and $\gamma = -3.9055$. The full implementation of this system is shown in Fig. 3(a). The reset function is selected to be $R(t) = -X(t)$. The detailed

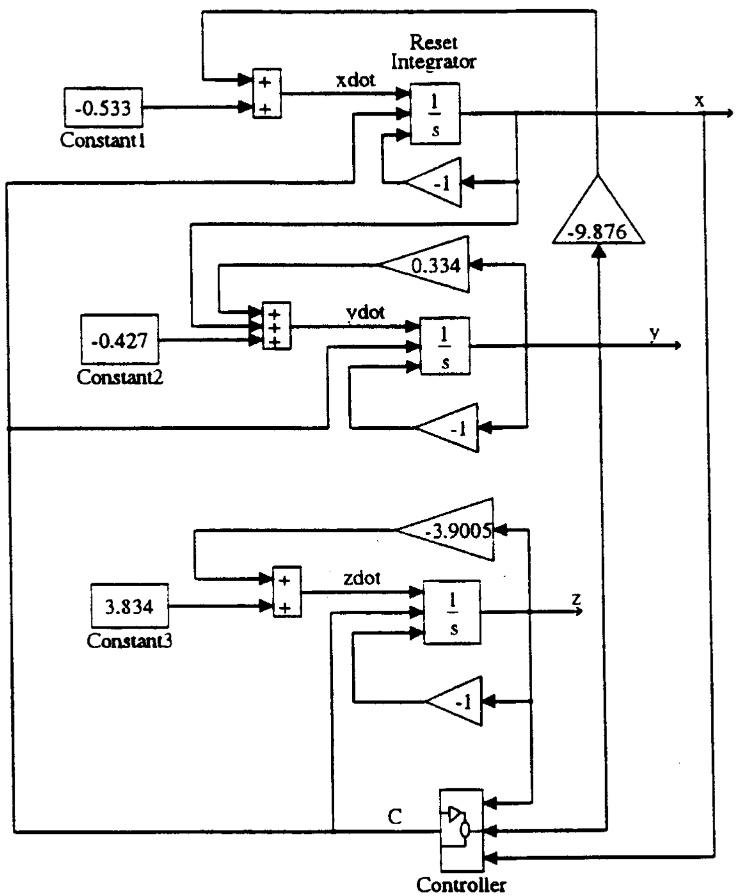

Fig. 3(a). Three-dimensional single scroll system.

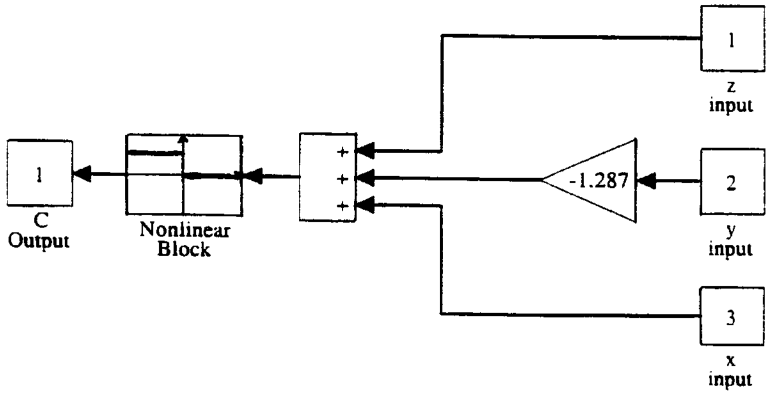

Fig. 3(b). Controller for the three-dimensional single scroll.

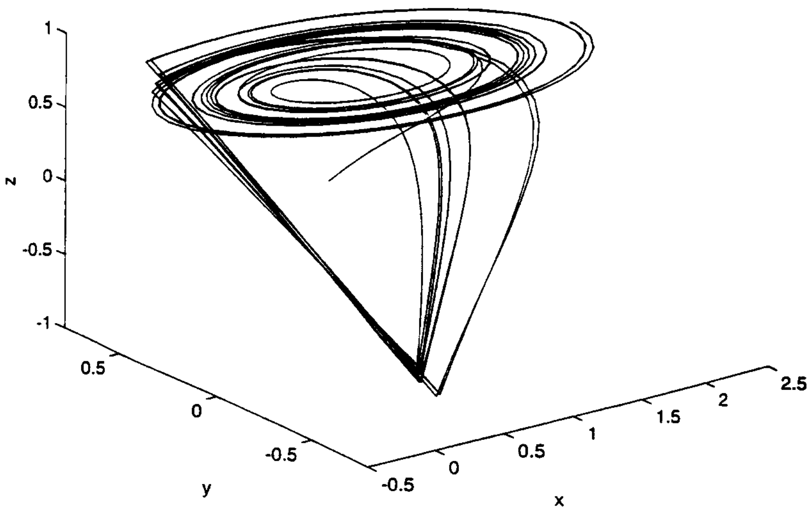

Fig. 3(c). A bounded three-dimensional single scroll.

implementation of the controller block is shown in Fig. 3(b). The computational block computes the function $F(X(t))$ described by

$$F(X(t)) = x(t) - 1.287y(t) + z(t)\,. \tag{10}$$

Figure 3(c) shows a bounded three-dimensional single scroll for the initial state $X(0) = (-0.015, -0.25, 0.5)$.

In the next section we expand this building approach to design a three-dimensional double scroll system with attractors similar to those observed experimentally in Chua's circuit.

4. Three-Dimensional Double Scroll System

The proposed architecture for a three-dimensional scroll system is shown in Fig. 4. It consists of three main blocks. It is constructed from a regular integrator (no reset function), a multiplier and a controller. In this system, the controller nonlinear block implements the sign function to create two symmetrical scrolls. The system is described by the following dynamical equations:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{pmatrix} x(t) - b_1 \cdot C \\ y(t) - b_2 \cdot C \\ z(t) - b_3 \cdot C \end{pmatrix} . \tag{11}$$

The signal C is generated by the controller which is built with the same computational block as in the three-dimensional single scroll system. However, the nonlinear block implements a "*sign*" function, i.e.

$$C = sign[F(X(t))]$$
$$F(X(t)) = k_1 x(t) + k_2 y(t) + z(t) = K^T X(t) .$$

To generate a Chua's double scroll attractor, we expanded the three-dimensional single scroll by introducing a second equilibrium point X_Q^- which is symmetrical to X_Q^+ with respect to the origin. X_Q^+ is located in the region D^+ while X_Q^- is located in the region D^-. The two regions D^+ and D^- are separated by a plane defined by the equation $F(X(t)) = 0$. The equilibrium point X_Q^+ will have two complex eigenvalues $\lambda = \sigma \pm j\omega$ with a positive real part σ and a negative real eigenvalue γ. Its eigenspace is characterized by a complex eigenplane E_c^+ and eigenvector E_r^+ corresponding to the real eigenvalue. Similarly the equilibrium point X_Q^- will have the same complex eigenvalues $\lambda = \sigma \pm j\omega$ with a positive real part σ and the same negative real eigenvalue γ. However, its eigenspace is characterized by a different complex eigenplane E_c^- and an eigenvector E_r^- corresponding to the real eigenvalue.

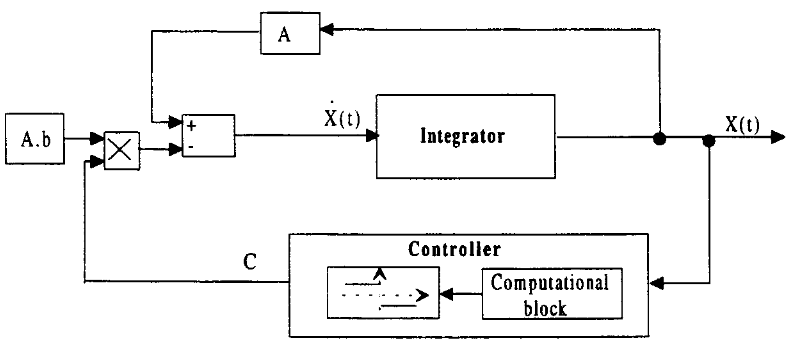

Fig. 4. Block diagram of a double scroll system.

A trajectory starting from some initial condition state $X(0)$ in the region D^+ will evolve along E_r^+ until it reaches the eigenplane E_c^+ where it spirals away from the equilibrium point X_Q^+. It is important to notice that a trajectory that originates from an initial state in D^+ will remain in D^+ until the sign of the controller output C changes, then it jumps to the region D^-. Once the trajectory crosses to D^-, it then evolves along E_r^- until it reaches the eigenplane E_c^- where it spirals away from the equilibrium point X_Q^-. Once the sign of the controller output changes, then it jumps back to D^+. The process repeats itself and will lead to the Chua's double scroll attractor.

The proposed implementation of the three-dimensional double scroll system is illustrated by the following example.

Example 3: *Consider the following dynamical system which is an expansion of the previous example.*

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 & 0 \\ 1.0 & 0.334 & 0 \\ 0 & 0 & -3.9005 \end{bmatrix} \begin{pmatrix} x(t) - 0.4455 \cdot C \\ y(t) + 0.0540 \cdot C \\ z(t) - 0.9830 \cdot C \end{pmatrix} \tag{12}$$

The equilibrium points are $X_Q^+ = (0.4455, -0.0540, 0.9830)$ and $X_Q^- = (-0.4455, 0.0540, -0.9830)$. The eigenvalues are the same as in example 2, $\lambda = 0.167 \pm j3.138$ and $\gamma = -3.9055$. The full implementation of this system is shown in Fig. 5(a). The controller implementation, shown in Fig. 5(b), consists of two main blocks. The computational block computes first the function $F(X(t))$ and then the nonlinear block generates a signal $C = 1$ when $F(X(t)) \geq 0$ and $C = -1$ when $F(X(t)) < 0$.

$$F(X(t)) = x(t) - 1.287y(t) + z(t) \tag{13}$$

The computer simulation shown in Fig. 5(c), reveals a bounded three-dimensional Chua's double scroll attractor for the initial state $X(0) = (-0.015, -0.25, 0.5)$. A similar double scroll attractor can also be obtained by using another nonlinear block as shown by the following example.

Example 4: *Consider the previous dynamical system with a different controller nonlinear block described by the following sigmoid function:*

$$C = \frac{e^{\alpha F(X(t)} - 1.0}{e^{\alpha F(X(t)} + 1.0}.$$

Where $F(X(t))$ is the same as in the previous example, and $\alpha = 30.0$. Figure 6 shows the Chua's double scroll attractor with initial conditions $X(0) = (-0.015, -0.25, 0.50)$.

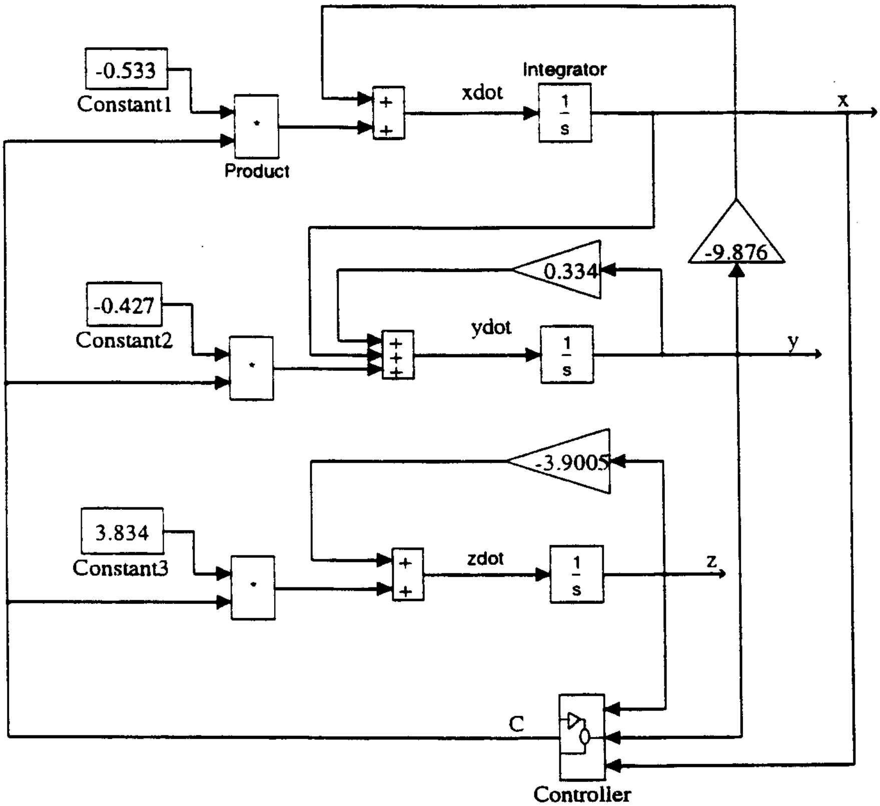

Fig. 5(a). Three-dimensional double scroll system.

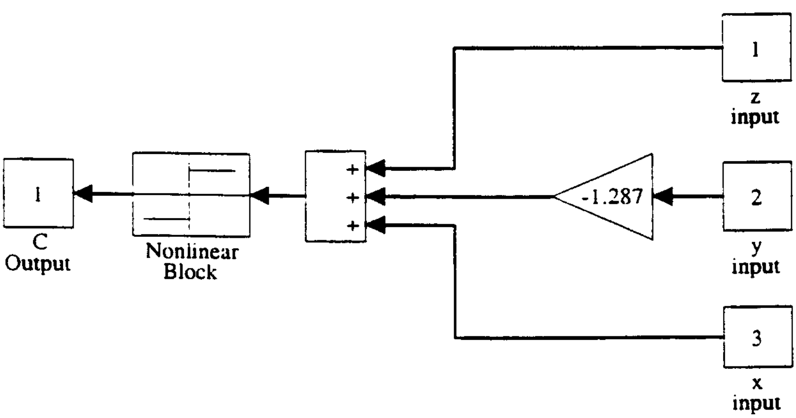

Fig. 5(b). Controller for the three-dimensional double scroll.

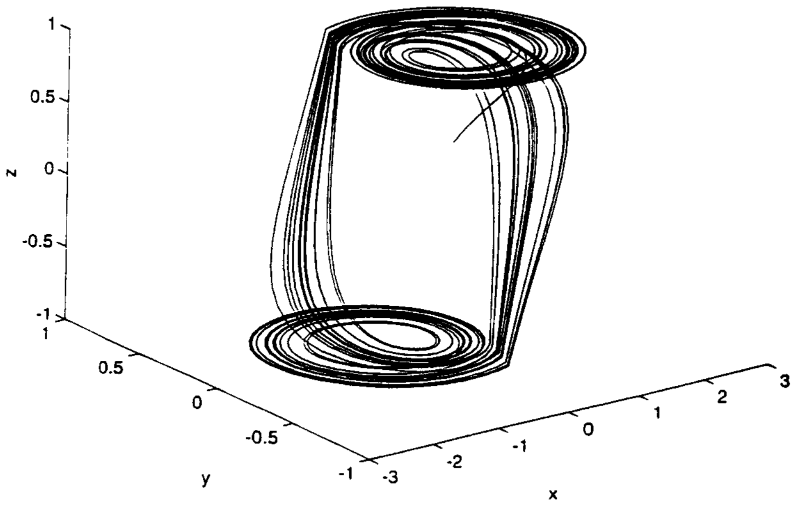

Fig. 5(c). A bounded three-dimensional Chua's double scroll attractor using a sign nonlinear controller block.

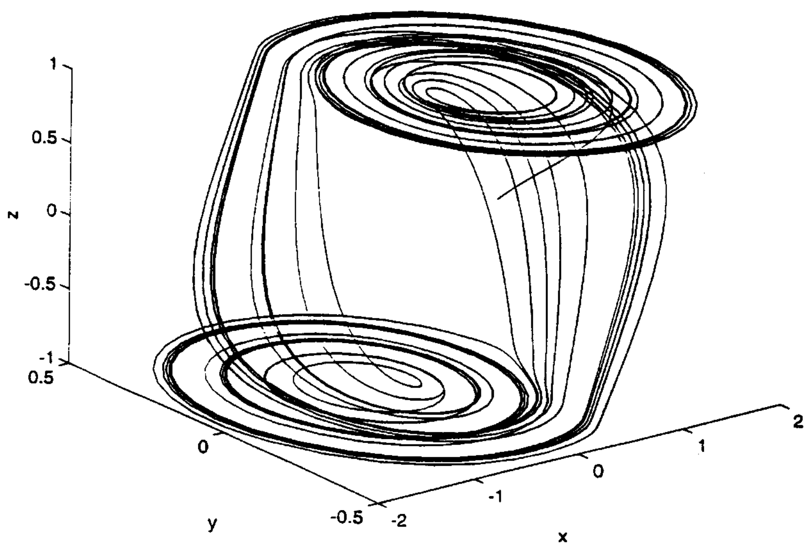

Fig. 6. A bounded three-dimensional Chua's double scroll attractor using a sigmoid nonlinear controller block.

5. Conclusions

In this paper we have proposed a new and simple implementation of conditionally reset linear systems which exhibit chaotic behavior. These systems are constructed from building blocks (e.g. reset integrator, nonlinear controllers, summers and gain amplifiers), which can be either designed or found in the off-shelf electronics library.

The design approach presented here will enable us to implement complex chaotic systems which can be used in many applications.[2] We investigated the use of different controller nonlinearities and have shown by computer simulation the existence of attractors. We are currently trying to overcome the following problems: (1) existence of an analytic condition which guarantees the presence of strange attractors, and (2) theoretical generalization of the design procedure.

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