Price of Anarchy

\[ P_{oa} = \max_{a \text{ equilibrium}} \frac{\sum c_i(a)}{\min \sum c_i(a)} \]

Assume minimization problem

Objective = sum of agents' objectives

Q: what equilibrium? (Nash, lower, correlated, etc.)

Why unweighted sum? (Context often supports it)

Why didn't economists study it?

(Atomic) Congestion Game

4 flows
1: a \rightarrow b
2: a \rightarrow c
3: b \rightarrow c
4: c \rightarrow b

E.g.

\[ \begin{array}{ccc}
\times & o & x \\
\times & x & \end{array} \]

congestion fun \( x = \{1,2,3,4\} \)

Pure Nash ex. exists: potential fun

\[ \phi(a) = \sum e \sum c_{e(i)} = e(j) \]

Choice of paths? resulting in flow \( f(a) \)

Note: \( \phi(a) - \phi(a') = c_i(a) - c_i(a') \)

Flow after defection of \( i \) \( \Rightarrow \) Nash eq always exists!
Example how two Nash eq.

Single hop → \( \sum_i c_i(a_i) = 4 \)

Double hop → \( = 10 \)

Also a Nash eq.!

PoA = 2.5. This is worst possible! (see below)

Note: It does make sense to want to optimize unweighted sum of time wanted, energy, etc.

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Non-atomic flow: A large population of agents of total volume \( \lambda \) want to go from \( s \) to \( t \) continuum of agents/drivers

Pigou's example

Only Nash eq: \( f \)

Optimum?

\[ \begin{align*}
\text{Cost} & = 1 \\
\min 1 - f + f^2 & \Rightarrow f = \frac{1}{2} \\
\text{Cost} & = \frac{3}{4} \\
\text{PoA} & = \frac{3}{4} \quad \text{Worst possible!}
\end{align*} \]
Brass's Paradox

\[ p.o.A = \frac{4}{3} \quad \text{coincidence?} \]

What do we know about optimum and Nash eq. flows in such networks?

Various kinds of costs:

\[ C_e(x) = x \cdot c_e(x) \]

Convex marginal cost:

\[ k_e^* = C_e'(x) = c_e(x) + x \cdot c_e'(x) \]

Thus, A Nash eq. flow is optimum flow in the sense that it creates that is, all used paths have equal cost, all unused paths worse.

In Pigozzi, Brass: \( x^\ast \text{ (Nash)} \) equalizes path lengths.

Thus, equilibrium flow iff global minimum

\[ \sum e \int_0^x c_e(x) \, dx = \Phi(x) \]

Proof: \( x \cdot c_e(x) \leftrightarrow \int_0^x c_e(x) \, dx \)

Comment: Equilibrium exists, its cost is unique (because cost is convex and unique)
Notice immediately: for some fixed class of congestion fans

\[
P_0 A \geq B = \max_{c(x), \alpha \geq 0} \frac{c(v)}{c(x) + c(v) + c(r)}
\]

**Proof:**

![Diagram](general_pigeon)

Note: for \( \varepsilon = \{a x + b : a, b \geq 0\} \), \( B = \frac{4}{3} \) (b?)

\[
B \geq \frac{f^* c(f^*)}{\hat{f} c(\hat{f}) + (f^* - \hat{f}) c(f^*)}
\]

or

\[
\hat{f} c(\hat{f}) + c(f^*)(f^* - \hat{f}) \geq \frac{1}{B} c(f^*) f^*
\]

because \( f^* \) is eq.

\[\text{QED}\]

\[\therefore B = P_0 A!\]

Amazing fact: PoA is determined by the class \( \varepsilon \), not by the network. Always attained at the trivial network!
**Atomic congestion game?**

**Lemma.** Suppose that \( \exists X^0, X^0 \forall s, s', i \)

\[
C_i(s'_i, s_{-i}) \leq \lambda \cdot C_i(s') + \mu \cdot C_i(s)
\]

then \( \text{PoA} \leq \gamma_{1-\mu} \)

**Proof.**

\[
\text{cost}(s^*) = \sum_i C_i(s^*) \leq \sum_i C_i(s^*) \Rightarrow \text{PoA} \leq \gamma_{1-\mu} \cdot \text{cost}(s^*) \cdot \text{QED.}
\]

To apply to atomic congestion:

\[
\sum_i C_i(x^{t+1}_i, x^t_i) \leq \sum_e (a_i(x_{e+1}) + b_e)x_e
\]

Now:

\[
\forall a, b, y, t: ay(t+1) + by \leq \frac{5}{3}(ay + by + \frac{1}{3}(at+bt))
\]

\[
\leq \frac{5}{3} \sum_e (a_{e}x_{e} + b_{e})x_{e}' + \frac{1}{3} \sum_e (a_{e}x_{e} + b_{e})x_{e}
\]

\[
\Rightarrow \text{PoA} \leq \frac{5/3}{1-1/3} = \frac{5}{2} \quad \text{(mathy triangle lower bound)}
\]

Also: holds not just \( \forall \) Nash eq, but also for no-regret play (boosting/experts) which also means for correlated equilibria.

\( \text{(apply the inequality at each time t...)} \)
EXTENSIVE FORM GAMES

(We must include chess in our games...)

0 (Nature) gives player 1 a card
raise/fold
Player 2
passes/meets

Player 2

m -1,1
-2,2

Information set of player 2

Important restrictions on the trees: Perfect & info sets: Recall
not allowed

Can be reduced to the tree form. But there are issues
the reduction is exponential in general. And Nash eq.
is too good here!

What is a mixed strategy?

Q: Can you see why we can go the other way?
table form → extensive form

<table>
<thead>
<tr>
<th></th>
<th>m</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>SF</td>
<td>00</td>
<td>00</td>
</tr>
<tr>
<td>RF</td>
<td>-1</td>
<td>1/2</td>
</tr>
<tr>
<td>FR</td>
<td>1/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>RR</td>
<td>00</td>
<td>1/4</td>
</tr>
</tbody>
</table>

mixing all rows/cols
(expontential)

behavioral (mix at each info set)

sequence trick
(von Stengel, Kohker)
Sequence trick [Kohler+von Stengel 93]

Notation: Hi info sets for player i.
\[ \text{He} \times \text{Hi} \text{ hom } \text{Ah} \] (reusable actions)
Next: \[ \text{Hi} \times \text{Ah} \rightarrow \text{Hi u leaves?} \]
This is the "tree," valid because of perfect recall (depending on i).
Assign probabilities to the his (prob of getting there)
For the topmost he \( \text{Hi}, \) \( x_i(h) = \text{const} \) (depends on player 0)
\[ x_i(h) = \sum_{a \in \text{Ah}} x_i(\text{next}(h, a)) \] [linear!!]
We can now write the expected payoff of a mixed action profile:
\[ E[u_i] = \sum_{\ell \in \text{leaves}} u_i(\ell) \prod_{l=1}^{N} x_i(\ell) \]
Now we can solve by LP zero-sum game (like poker!)

<table>
<thead>
<tr>
<th>Nash is no good!</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Game graph]</td>
</tr>
</tbody>
</table>

Two Nash eq.
Upper-left is "empty threat!"
Solution: Subgame Nash eq.
Subgame:walrussubtree closed under info sets
SpN eq: A Nash eq. that is also a Nash eq. when
restricted to any subgame
No info sets: N eq. can be computed easily, and is
subgame perfect: by DP from the
leaves up (baldwards induction or
Ferrell's algorithm).
Info sets: Same idea. Compute Nash eq. of subtrees
bottom up. Replace subtrrees by payoffs
of Nash eq. on a leaf. Continue

Another solution: Trembling hand Nash eq.
A Nash eq. $x$ such that there is a sequence $(x_j)$
of e-fully mixed "Nash eq." (all strategies best
responses under the constraints $x_i \geq e^i > 0$),
such that $x_j \rightarrow x$.

Algorithm? Conjecture: Find e-fully mixed
Nash eq. with $e^i < 2^{-n^2}$. Round down/up
Third Fix Sequential Equilibrium

Game theorists consider it the "final" solution concept. For every info set a set of "beliefs" $R_h$ and a (behavioral) mixed strategy $x^h$; $R_h$ is a distribution over all paths to $h$ ("how did I get here?"). $x^h$ should be best response assuming $R_h$.

Furthermore: Sequence of $(x^j, \beta^j)$ such that $x^j \rightarrow x$, $\beta^j \rightarrow \beta$, $x^j$ fully mixed, $\beta^j$ derived from $x^j$ via Bayes rule...

Algorithm? Open

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Bayesian Games

Player $i$: actions $A_i$
types $T_i$
utility $u_i: \prod A_j \times \prod T_j \rightarrow \mathbb{R}$

Prior: $\pi_i \in \Delta \prod T_j$

Example: Vickrey auction!

Reduce to table form: action set $B_i: T_i \rightarrow A_i$, payoffs $u_i(b_1, \ldots, b_n) = E_{\pi_i} [u_i(b_j(t_j), t_j, j=1..n)]$
Theorem In Vickers, $b_i = \text{the identity function}$ is dominant.

Harsanyi's Theorem (Justification of mixed Nash eq)
For almost all games $G$, there is a sequence of Bayesian games $G_j$ such that $G_j \rightarrow G$ and the equilibria of $G_j$ converge to the equilibria of $G$.
Payoffs of $G$: $u_{ij} + e_{ij} = \text{(perturbed)}$

Another special Case: Repeated Games
Repeat Prisoner's Dilemma $n$ times!
Fact: Defect $n$ times is the only Nash eq (backwards induction)

One path taken in the 1980's-1990's: Assume players are limited, e.g., automata with few states!

2 States: A possible strategy: undominated

- "tit-for-tat"
- "US foreign policy"
- "switchover D"
- "punish once"
\[ u_i(t) = \sum_{t=1}^{\infty} u_i(\alpha(t)) \cdot (1-\delta)^t \]

**Important Insight:** The Folk Theorem (Shapley, Hummeln, Maskin, ~1970's-80)

\[ \theta_i = \min_{\theta_i \in A_i} \max_{\theta_{-i}} u_i(\theta) \text{ in one-shot game} \]
Any rational \((= P/Q)\) point in \(IR > 0\) can be realized by repeated play (including C-C in P.D.).

Proof: Realize it by periodic play. If anybody defects, the rest will punish him!

(Objection: not subgame perfect. Subgame perfect version: punish \(n\) times, if anybody defects from that punish them \(n^2\) times, etc.)

For two players can be resolved.

(Because \(\Phi_1 = \text{min max} (-B, B)\))

But > 2 players? \(\Phi_i\) is NP-hard...


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Theorem: Finding an eq. in repeated game (i.e., any point in \(IR\)....) is as hard as in the one-shot game.

Proof. Take \(k\)-player game \(G\). Create \((k+1)\)-player game \(G'\) with Player \(k+1\) (the "kilotimer")

\[
\begin{align*}
  \phi_i(a, (y, b)) &= \begin{cases} 
  0 & \text{if } i \neq k+1, j \\
  u_j(b, a_{-i}) - u_j(y) & \text{if } i = k+1 \\
  u_i(a) - u_j(b, a_{-i}) & \text{if } i = j 
  \end{cases}
\end{align*}
\]

player, \(i \in \{j \in A_j\} \cup \{k+1\}\).
At equilibrium, $k$ players will play a Nash eq. of the one-shot game...

**Coalitional Games**

- $n$ players, $\nu: 2^{[n]} \to \mathbb{R}^+$
- $\nu(S)$: the worth of coalition $S$
- Assumptions: $\nu(\emptyset) = 0$, $\sum_{S \in \Pi} \nu(S) \leq \nu([n])$
- $\Pi$: partition of $[n]$ in games, atom of behavior = action of player

Here, what a coalition does

**Basic question:** How do the $n$ players split $\nu([n])$?

One solution concept: the core.

Importation: $x_i$ with $\sum x_i = \nu([n])$

or payoff profile feasible $S$-feasible: $\sum x_i = \nu(S)$

$x$ is in the core iff there is no "defection" coalition $S$ and $S$-feasible $y$ st $y_i > x_i$ $\forall i \in S$.

**Ex 1** $\nu([3]) = 1$, $\nu(S) = \alpha$ if $|S| = 2$, $\nu(S) = 0$ if $|S| \leq 1$.

**Ex 2** How should $n$ cities split road maintenance costs?

**Ex 3** Voting $\nu(S) = \leq 0$ either core $= \emptyset$ or there is a veto player.
Ex 4 Game \( G \), \( n \) players. Define \( v(S) = \min_{\pi \in \pi_i} \max_S \sum_{u_i \in S} u_i \) the bottom-line total utility players in \( S \) can guarantee for themselves.

Ex 5 A trip to \( n \) destinations. How do you apportion the surface to the \( n \) hosts?  

*Shapley value*

\[
S_i = \frac{1}{n!} \sum_{\pi \in S_n} \left( v(\{j: \pi(j) \leq \pi(i)\}) - v(\{j: \pi(j) < \pi(i)\}) \right)
\]

*Thm.* If the core is nonempty, \( S \in \text{core} \)

*Thm.* \( S \) is the unique value satisfying the balanced contributions property:

\[
S_i(G \{j\}) - S_i(G \{j\} - \{i\}) = S_j(G \{i\}) - S_j(G \{i\} - \{j\})
\]

*Also:* Symmetry, additivity, dummy player