Lecture given by George Pierrakos. In this lecture we will cover

1. Characterization of 1D-domain.
2. Myerson’s optimal mechanism design.
3. Approximation in Bayesian mechanism design.

10.1 Characterization

Consider the case where we have \( n \) bidders and 1 item. The bidders have private values for the item \( v_1, v_2, \ldots, v_n \), and the mechanism is given by the allocation and payment functions.

\[
x_i : (v_1, \ldots, v_n) \rightarrow [0, 1]
\]

\[
p_i : (v_1, \ldots, v_n) \rightarrow \mathbb{R}^+
\]

We want the mechanism to satisfy the following constraints

- Incentive Compatibility (IC): \( x_i(v_i) v_i - p_i(v_i) \geq x_i(b_i) v_i - p_i(b_i) \) for any \( b_i \).
- Individual Rationality (IR): \( x_i(v_i) v_i - p_i(v_i) \geq 0 \)
- \( \sum_i x_i \leq 1 \) (probability of allocating the item \( \leq 1 \))

**Theorem 1.** (Characterization) A mechanism is IC+IR iff

1. \( x_i(v_i) \) is increasing in \( v_i \)
2. \( p_i(v_i) = x_i(v_i) v_i - \int_0^{v_i} x_i(z)dz \)
Proof. The only if part. Prove by picture, consider the various areas in the plot

\[ p_i(v_i) = A1 + A2, p_i(b_i) = A2, u_i(b_i) = A3 + A4, u_i(v_i) = A3 + A4 + A5. \]

IC is satisfied since \( u_i(v_i) \geq u_i(b_i) \), IR is also satisfied.

The if part. Using IC two times with bidder \( i \)’s true value being \( v_i \) and \( b_i \) respectively, we can write

\[ x_i(v_i) - p_i(v_i) \geq x_i(b_i)v_i - p_i(b_i) \]
\[ x_i(b_i)b_i - p_i(b_i) \geq x_i(v_i)b_i - p_i(v_i) \]

Adding the above two equations together, we get

\[ (x_i(v_i) - x_i(b_i))(v_i - b_i) \geq 0 \]

Thus we know \( x_i(v_i) \) is increasing with \( v_i \).

\[ u_i(b_i) = v_i x_i(b_i) - p_i(b_i) \]

Take derivative of the utility with respect to the bid \( b_i \), we get

\[ u_i'(b_i) = v_i x_i'(b_i) - p_i'(b_i) \]

IC suggests that whatever bidder \( i \)'s true value is, his utility should be maximized at that point, thus

\[ p_i'(z) = z x_i'(z) \quad \forall z \]

If the true value of bidder \( i \) is \( v_i \), we have

\[ \int_0^{v_i} p_i'(z)dz = \int_0^{v_i} z x_i'(z)dz \]
\[ p_i(v_i) - p_i(0) = [zx_i(z)]_0^{v_i} - \int_0^{v_i} x_i(z)dz \]
\[ p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0) \]

IR suggests \( p_i(0) = 0 \), thus we get \( p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)dz. \) \( \square \)

Notice when the image of \( x_i \) is \( \{0, 1\} \), the above theorem states that for each bidder \( i \) there exists a threshold \( t_i \) such that the bidder gets the item iff his value is at least \( t_i \), and the price he pays when he gets the item is \( t_i \).

### 10.2 Myerson’s mechanism (Optimal mechanism design)

For simplicity we will focus on deterministic mechanisms. Denote

\[ \text{Social Welfare (SW)} = \sum_i v_i x_i(v) \]
\[ \text{Revenue (Rev)} = \sum_i p_i(v) \]
It’s an ill-defined problem to maximize profit without any knowledge about users’ true utilities. Instead, we maximize expected profit assuming the value of bidder $i$ is drawn from known distribution with density function $f_i$, and the $f_i$’s are independent.

$$\max \mathbb{E}[\sum_i p_i]$$

$$= \int_{v \in T} \sum_i p_i(v)f_1(v_1)f_2(v_2) \cdots f_n(v_n) dv$$

**Example 1.** Consider the auction with 2 bidders and 1 item. $v_1,v_2$ are independent, uniform over $[0,1]$.

$$\text{Vickrey: } \mathbb{E}[\sum_i p_i] = \mathbb{E}[\min(v_1, v_2)] = \frac{1}{3}$$

Vickrey($\frac{1}{2}$) (Vickrey with reserve price $\frac{1}{2}$): run a normal Vickrey auction as if there is a third bidder who bids $\frac{1}{2}$, and if it turns out that the third bidder wins, the auctioneer keeps the item.

$$\begin{cases} 
\mathbb{E}[p_1 + p_2 | v_1, v_2 \geq \frac{1}{2}] \mathbb{P}[v_1, v_2 \geq \frac{1}{2}] = \frac{3}{4} = \frac{1}{3} \\
\mathbb{E}[p_1 + p_2 | v_1 > \frac{1}{2} > v_2 \vee v_2 > \frac{1}{2} > v_2] \mathbb{P}[v_1 > \frac{1}{2} > v_2 \vee v_2 > \frac{1}{2} > v_2] = \frac{1}{22} = \frac{1}{4} \\
\mathbb{E}[p_1 + p_2 | v_1, v_2 < \frac{1}{2}] = 0
\end{cases}$$

Adding the three cases together we have $\mathbb{E}[p_1 + p_2] = \frac{5}{12} > \frac{1}{3}$. With a reserve price of $\frac{1}{2}$, we can do better than Vickrey in terms of profit maximization.

Consider the general situation with $n$ bidders and 1 item

$$v_1 \sim f_1, \ldots, v_n \sim f_n \text{ (independent)}$$

$$\text{Rev}(v) = \sum_i \int_{v_{-i}} f_{-i}(v_{-i}) \left( \int_{v_i} f_i(v_i)p_i(v)dv_i \right) dv_{-i}$$

SW takes a similar form except $p_i(v)$ is replaced by $v_ix_i(v_i)$.

$$\text{SW}(v) = \sum_i \int_{v_{-i}} f_{-i}(v_{-i}) \left( \int_{v_i} f_i(v_i)v_ix_i(v_i)dv_i \right) dv_{-i}$$

High level idea of Myerson’s mechanism: Reduce expected Rev maximization to expected SW maximization in some "virtual" space.

Virtual valuation: $\phi_i(v_i)$

**Goal:** Find $\phi_i(v_i)$ s.t. $\text{Rev}(v) = \text{SW}(\phi(v))$. It suffices for the virtual valuations to satisfy

$$\int_{v_i} f_i(v_i)p_i(v)dv_i = \int_{v_i} f_i(v_i)v_ix_i(v_i)\phi_i(v_i)dv_i \quad \forall i, v_{-i}$$

From our Characterization theorem (for deterministic mechanisms), we for know each bidder there is a threshold $z_i$ (fixing $v_{-i}$). Then the above condition becomes

$$\int_{z_i}^{1} f_i(v_i)z_idv_i = \int_{z_i}^{1} f_i(v_i)\phi_i(v_i)dv_i \quad \forall i, v_{-i}$$
Take derivative w.r.t \( z_i \) on both sides, we get

\[
\int_{z_i}^{1} f_i(v_i) dv_i - z_i f_i(z_i) = -f_i(z_i) \phi_i(z_i)
\]

\[
\phi_i(z_i) = z_i - \frac{\int_{z_i}^{1} f_i(v_i) dv_i}{f_i(z_i)}
\]

Virtual valuation:

\[
\phi_i(v_i) = v_i - 1 - \frac{F_i(v_i)}{f_i(v_i)}
\]

Myerson’s Auction: Input \( f_1, \ldots, f_n, b_1, \ldots, b_n \)

1. Compute \( \phi_1(b_1), \ldots, \phi_n(b_n) \)

2. Get the allocation and price function \((x, p)\) from VCG with bids \( \phi_1(b_1), \ldots, \phi_n(b_n) \)

3. Output \((x', p') \leftarrow (x, \phi^{-1}(p))\)

Example 2. (Revisit) \( n \) bidders, \( v_i \sim u[0,1] \) independent. VCG on \( \phi_i(b_i) \) is just the second price auction, except we don’t want to pay the winner any money. Thus we only consider bidders with virtual bids at least 0, i.e. \( v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \geq 0 \Rightarrow v_i \geq \frac{1}{2} \). Essentially Myerson’s auction in this case is Vickrey with reserve price \( \frac{1}{2} \).

Corollary 1. When bidders are i.i.d according to any distribution, the optimal auction is Vickrey (with reserve price).

The reason is that \( \phi_i \)’s are all the same, and the \( \phi \) in step 2 cancels with \( \phi^{-1} \) in step 3.

Is Myerson’s auction truthful? We know that VCG is IC, and we want Myerson to be IC. What will make that work is for \( \phi_i \) to be increasing. \( \phi_i \) is indeed increasing for regular distributions.

Even when \( \phi_i \) is not increasing, we can further reduce it to another virtual space.

Ironing \( \phi_i \) (not increasing) \( \rightarrow \hat{\phi}_i \) (increasing)

Some high-level intuition. Write the expected revenue with price \( z \) as

\[
R_i(z) = z \int_{z}^{1} f_i(t) dt
\]

Then it’s easy to check

\[
\phi_i(v_i) = \left[ -\frac{\partial}{\partial z} R_i(z) \right]_{z=v_i} \frac{f_i(v_i)}{F_i(v_i)}
\]

We can rewrite \( R_i() \) as a function of \( 1 - F_i(z) \), since \( \frac{\partial}{\partial z} \left( 1 - F_i(z) \right) = -f_i(z) \), when we take the derivative w.r.t \( z \), the chain rule will kill the \( f_i(v_i) \) in the denominator. Thus

\[
\phi_i = R'_i \text{ when we write } R \text{ as a function of } F_i(z)
\]

\( \phi_i \) is increasing when \( R_i \) is convex, thus we can let \( \hat{R}_i \) be the convex hull of \( R_i \), and \( \hat{\phi}_i = \hat{R}' \).

Ironed SW \( \geq \) Revenue. Equality holds if \( x_i \) constant across ironed region.
10.3 Approximation

10.3.1 Many items?

Don’t know optimal solution. For 1 bidder, \( n \) items with unit-demand, [Chawla, Hartline, Kleinberg EC07] gives 2-approximation, and [Cai Daskalakis FOCS11] gives PTAS for some distributions.

10.3.2 Simple versus Optimal auctions

[Hartline, Roughgarden EC09] What if you run Vickrey \(_m\) (when the distributions of bidders’ values not i.i.d)

\[
m_i = \phi_i^{-1}(0)
\]

1. Reject all bidders with value \( v_i < m_i \)
2. Run Vickrey on remaining bidders.

**Theorem 2.** Vickrey \(_m\) is 2-approximation if users’ valuations are regular

10.3.3 Optimal auction with correlated valuations?

\( n \) bidders, \((v_1, \ldots, v_n) \sim f\), 1 item

[Ronen, Saberi FOCS02] gives \( \frac{1}{2} \) of optimal revenue for any \( n \) bidders

[Papadimitriou Pierrakos STOC11] Optimally for \( n = 2 \) bidders, NP-hard to get a PTAS for \( n \geq 3 \)

10.3.4 Prior-freeness?

Define benchmark for revenue [Hartline, Roughgarden STOC08]