# Exact Recovery of Sparsely Used Overcomplete Dictionaries

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#### Abstract

We consider the problem of learning overcomplete dictionaries in the context of sparse coding, where each sample selects a sparse subset of dictionary elements. Our method consists of two stages, viz., initial estimation of the dictionary, and a clean-up phase involving estimation of the coefficient matrix, and re-estimation of the dictionary. We prove that our method exactly recovers both the dictionary and the coefficient matrix under a set of sufficient conditions.

**Keywords:** Dictionary learning, sparse coding, overcomplete dictionaries, incoherence, lasso.

## 1 Introduction

The dictionary learning problem is as follows: given observations Y, the task is to factorize it as

$$Y = AX, \quad Y \in \mathbb{R}^{d \times n}, \ A \in \mathbb{R}^{d \times r}, \ X \in \mathbb{R}^{r \times n},$$
 (1)

where X is referred to as the *coefficient* matrix and the columns of A are referred to as the *dictionary* elements. There are indeed infinite factorizations for (1) unless further constraints are imposed. A natural assumption is that the coefficient matrix X is sparse, and in fact, that each sample  $y_i$  selects a sparse subset of dictionary elements from A. This instance of dictionary learning is popularly known as the *sparse coding* problem [29, 24]. It has been argued that sparse coding can provide a succinct representation of the observed data, given only unlabeled samples [24]. Through this lens of unsupervised learning, dictionary learning has received an increased attention from the machine learning community in the last few years; see Section 1.2 for a brief survey.

Although the above problem has been extensively studied, most of the methods are heuristic and lack guarantees. Spielman et. al [30] provide exact recovery results for this problem, when the coefficient matrix has Bernoulli-Gaussian entries and the dictionary matrix  $A \in \mathbb{R}^{r \times d}$  has full column rank. This condition entails that the dictionary is undercomplete, i.e., the observed dimensionality needs to be greater than the number of dictionary elements  $(r \leq d)$ . However, for most practical settings, it has been argued that overcomplete representations, where  $r \gg d$ , are far more relevant, and can provide greater flexibility in modeling as well as greater robustness to

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noise [25, 11]. Moreover, in the context of blind source separation (BSS) of audio, image or video signals, the dictionary learning problem is typically overcomplete, since there are more sources than observations [14]. In this work, we provide guaranteed methods for exact recovery of overcomplete dictionaries.

## 1.1 Summary of Results

In this paper we present two algorithms. The first algorithm is employed for an initial estimation of the dictionary. The second algorithm takes the output of the first one, and then provides a final estimate for both the dictionary as well as the coefficient matrix. The first algorithm can be seen as a clustering style method followed by a singular value decomposition (SVD) within each cluster resulting in an estimate for each dictionary element. The clusters are formed based on the magnitudes of the correlation between pairs of samples. Our second algorithm takes the output of the first algorithm as an initial dictionary estimate, and provides estimates for the coefficients X by solving a sparse linear system through Lasso, followed by thresholding. The dictionary is then re-estimated by solving the linear system Y = AX with these estimated coefficients.

We consider a random coefficient matrix, where each column of X has s non-zero entries which are randomly chosen, i.e., each sample  $y_i$  selects s dictionary elements uniformly at random. We additionally assume that the dictionary elements are pairwise incoherent and that the dictionary matrix has a bounded spectral norm. Under these conditions, we establish that our first algorithm estimates the dictionary elements with bounded error when the number of samples scales as  $n = \mathcal{O}(r(\log r + \log d))$ , and when the sparsity  $s = \mathcal{O}(d^{1/4}, r^{1/4})$ . Combining the first and the second algorithms, we establish success in exactly recovering both the true dictionary A and the coefficient matrix X. The sample complexity is again  $\mathcal{O}(r(\log r + \log d))$ , but we require more stringent sparsity constraints:  $s = \mathcal{O}(d^{1/5}, r^{1/6})$ . We outline our method as well as our analysis techniques in Section 1.3. This is the first work to provide a tractable method for exact recovery of overcomplete dictionaries, and we discuss the previous results below. Finally, concurrently with our work, an approximate recovery result with a similar procedure as our initialization step was recently announced by Arora et al. [7]. A detailed discussion comparing our and their results is presented in Section 1.2.

### 1.2 Related Works

This work overlaps with and relates to prior works in many different communities and we discuss them below in turn.

**Dictionary Learning:** Hillar and Sommer [19] consider conditions for identifiability of sparse coding and establish that when the dictionary succeeds in reconstructing a certain set of sparse vectors, there exists a unique sparse coding, up to permutation and scaling. However, the number of samples required to establish identifiability is exponential in r for the general case. In contrast, we show that efficient recovery is possible using  $\mathcal{O}(r(\log r + \log d))$  samples, albeit under additional conditions such as *incoherence* among the dictionary elements.

Spielman et. al [30] provide exact recovery results for a  $\ell_1$  based method in the undercomplete setting, where  $r \leq d$ . In contrast, we allow for the overcomplete setting where r > d. There exist a plethora of heuristics for dictionary learning, which work well in practice in many contexts, but lack theoretical guarantees. For instance, Lee et. al. propose an iterative  $\ell_1$  and  $\ell_2$  optimization

procedures [24]. This is similar to the the method of optimal directions (MOD) proposed in [15]. Another popular method is the so-called K-SVD, which iterates between estimation of X and given an estimate of X, updates the dictionary estimate using a spectral procedure on the residual. Other works consider more sophisticated methods from an optimization viewpoint while still alternating between dictionary and coefficient updates [23, 17]. While we also employ the  $\ell_1$  procedure to estimate the coefficient matrix, our overall approach is non-iterative and we employ a combinatorial method for obtaining an initial estimate of the dictionary elements. Moreover, while Geng et al. [17] and Jenatton et al. [22] study the local optimality properties of an alternating minimization procedure, we focus on establishing exact recovery.

Recent works [31, 28, 26] provide generalization bounds for predictive sparse coding, where the goal of the learned sparse representation is to obtain good performance on some predictive task. This differs from our framework since we do not consider predictive tasks here, but the task of recovering the underlying dictionary elements.

Finally, part of our results are closely related to the very recent work of Arora et al. [7], carried out independently and concurrently with our work. There are however some important distinctions: while the work of [7] does not result in exact recovery of the dictionary elements, we can guarantee exact recovery. This is because we follow up the dictionary estimation procedure, which bears similarities to the one employed in Arora et al. [7], with a cleanup process using Lasso. Moreover, our method for initial estimation of dictionary elements is simpler since it only involves pairwise relationships among the samples, but at the cost of a more stringent sparsity requirement. We note that our exact recovery result requires even stronger sparsity constraints, since we require a certain error guarantee for the first step of estimating the dictionary elements.

Blind Source Separation/ICA/Topic Models: The problem of dictionary learning is applicable to blind source separation (BSS), where the rows of X are signals from the sources and A represents the linear mixing matrix. The term blind implies that the dictionary matrix A is unknown and needs to be jointly estimated with the coefficient matrix X, given samples Y. This problem has been extensively studied and the most popular setting is the independent component analysis (ICA), where the sources are assumed to be independent. In contrast, for the sparse component analysis problem, no assumptions are made on the statistics of the sources. Many works provide guarantees for ICA in the undercomplete setting, where there are fewer sources than observations [20, 8, 3] and some works provide guarantees in the overcomplete setting [13, 18]. However, the techniques are very different since they rely on the independence among the sources. The problem of learning topic models can be cast as a similar factorization problem, where A now corresponds to the topic-word matrix and X corresponds to the proportions of topics in various documents. There are various recent works providing guaranteed methods for learning topic models, e.g [1, 6, 5, 4]. However, these works make different assumptions on either A or X or both to guarantee recovery. For instance, the work [6] assumes that the topic-word matrix A has rows such that for each column, only the entry corresponding to that column is non-zero. The work [5] assumes expansion conditions on A and provides recovery through  $\ell_1$ -based optimization. We note that the techniques of [5] are related to those employed by Spielman et. al [30] for dictionary learning, but make different assumptions. All these works only deal with the undercomplete setting. The recent work [4] considers topic models in the overcomplete setting, and provides guarantees when A satisfies certain higher order expansion conditions. The techniques are very different from the ones employed here since they involve higher order moments and tensor forms.

Connection to Learning Overlapping Communities: Our initial step for estimating the dictionary elements involves finding large cliques in the sample correlation graph, where the nodes are the samples and the edges represent sufficiently large correlations among the endpoints. The clique finding problem is a special instance of the overlapping community detection problem, which has been studied in various contexts, e.g. [2, 10, 9, 21, 27]. However, the correlation graph here has different kinds of constraints than the ones studied before as follows. In our setting involving noise-free dictionary learning, each community corresponds to a clique and there are no edges across two different communities. In contrast, many works on community detection are concerned about handling noise efficiently, where each community is not a full clique, and there are edges across different communities. Here, we need to learn overlapping communities, while many community detection methods limit to learning non-overlapping ones. In our setting, we argue that the overlap across different communities is small under a random coefficient matrix, and thus, we can find the communities efficiently through simple random sampling and neighborhood testing procedures.

## 1.3 Overview of Techniques

Our dictionary learning procedure consists broadly of two steps: (1) the initial step for obtaining an estimate of the dictionary matrix A, and (2) the clean-up step involving estimation of the coefficient matrix X, and subsequent re-estimation of A. We provide an overview of the two steps below.

Initial step involving estimation of dictionary elements: This step first involves construction of the sample correlation graph  $G_{\text{corr}(\rho)}$ , where the nodes are samples  $\{y_1, y_2, \dots y_n\}$  and an edge  $(y_i, y_j) \in G_{\text{corr}(\rho)}$  implies that  $|\langle y_i, y_j \rangle| > \rho$ , for some  $\rho > 0$ . We then employ a *clustering* procedure on the graph to obtain a subset of samples, which are then employed to estimate each dictionary element. Roughly, we search for large cliques in the correlation graph and obtain a spectral estimate of each dictionary element using samples from such sets.

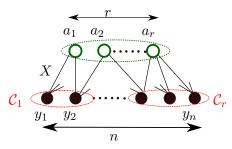
Lasso cleanup step: This is a relatively straightforward procedure. Once an initial estimate of the dictionary matrix is obtained, we estimate the coefficient matrix X through Lasso and then perform thresholding. Now, we re-estimate the dictionary, given this coefficient matrix, by solving another linear system. This provides us with a final estimate of the both the dictionary as well as the coefficient matrix.

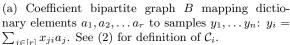
Intuition why initial dictionary estimation step works: The core intuitions can be described in terms of the relationships between the two graphs, viz., the coefficient bipartite graph  $B_{\text{coeff}}$  and the sample correlation graph  $G_{\text{corr}}$ , shown in Figures 1a and 1b. As described earlier, the correlation graph  $G_{\text{corr}}$  consists of edges between well correlated samples. The coefficient bipartite graph  $B_{\text{coeff}}$  consists of dictionary elements  $\{a_i\}$  on one side and the samples  $\{y_i\}$  on the other, and the bipartite graph  $B_{\text{coeff}}$  encodes the sparsity pattern of the coefficient matrix X. In other words, it maps the dictionary elements  $\{a_i\}$  to samples  $\{y_i\}$  on which they are supported on and  $\mathcal{N}_B(y_i)$  denotes the neighborhood of  $y_i$  in the graph  $B_{\text{coeff}}$ .

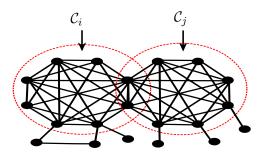
Now given this bipartite graph  $B_{\text{coeff}}$ , for each dictionary element  $a_i$ , consider a set of samples which (pairwise) have only one dictionary element  $a_i$  in common, and denote such a set by  $C_i$  i.e.

$$C_i := \{ y_k, k \in S : \mathcal{N}_B(y_k) \cap \mathcal{N}_B(y_l) = a_i, \ \forall k, l \in S \}.$$
 (2)

<sup>&</sup>lt;sup>1</sup>Note that such a set need not be unique.







(b) Sample correlation graph  $G_{\text{corr}}$  with nodes  $\{y_k\}$  and edge  $(y_i, y_j)$  s.t.  $|\langle y_i, y_j \rangle| > \rho$ .  $C_i$ , defined in (2), is a clique in the correlation graph. See Lemma 3.1.

Figure 1: Coefficient bipartite graph and the sample correlation graph.

For a random coefficient matrix (resulting in a random bipartite graph), we argue that there exists (large) sets  $C_i$ , for each  $i \in [r]$ , which consists of a large fraction of  $\mathcal{N}_B(a_i)$ , and no two elements  $a_i$  and  $a_j$  have a large fraction of samples in common. In other words, for random coefficient matrices, we see a diversity in the dictionary elements among the samples, and this can be viewed as an *expansion* property from the dictionary elements to the set of samples. We exploit this property to establish success for our method.

Our subsequent analysis is broadly divided into two parts, viz., establishing that (large) sets  $\{C_i\}$  can be found efficiently, and that the dictionary elements can be estimated accurately once such sets  $\{C_i\}$  are found. We establish that the sets  $\{C_i\}$  are cliques in the correlation graph when the dictionary elements are incoherent, as shown in Figure 1b. Combined with the previous argument that the different sets  $C_i$ 's have only a small amount of overlap for random coefficient matrices, we argue that these sets can be found efficiently through simple random sampling and neighborhood testing on the correlation graph. Once a large enough set  $C_i$  is found, we argue that under incoherence, the dictionary element  $a_i$  can be estimated accurately through SVD over the samples in  $C_i$ .

Analysis of Lasso cleanup step: Given an approximate estimate of the dictionary, we now analyze the performance of Lasso for estimating the coefficient matrix. Since we only have a noisy estimate of the dictionary, our analysis here is different from the usual analysis for a sparse linear system. The noise in our system is dependent on the approximate dictionary employed, which differs from the typical statistical setting, where noise is assumed to be independent. We exploit the known guarantees available for Lasso under deterministic noise [12] for our setting. Combining Lasso with a simple thresholding procedure, we guarantee exact recovery of the coefficient matrix, albeit under a more stringent condition on the sparsity. The dictionary is then re-estimated by solving another linear system. Combining the above analysis with the guarantees for the earlier step on dictionary estimation, we obtain the main result of this paper on exact recovery of the dictionary as well as the coefficient matrix.

## 2 Method and Guarantees

**Notation:** Let  $[n] := \{1, 2, ..., n\}$  and for a vector w, let  $\mathrm{Supp}(w)$  denote the support of w, i.e. the set of indices where w is non-zero. Let ||w|| denote the  $\ell_2$  norm of vector w, and similarly for a matrix W, ||W|| denotes its spectral norm. Let  $A = [a_1|a_2|\dots|a_r]$ , where  $a_i$  denotes the  $i^{\mathrm{th}}$  column, and similarly for  $Y = [y_1|y_2|\dots y_n]$  and  $X = [x_1|\dots|x_n]$ . For a graph G = (V, E), let  $\mathcal{N}_G(i)$  denote set of neighbors for node i in G.

### 2.1 Method

## 2.1.1 Initialization Step: Estimation of Dictionary Elements

The first step is to obtain an initial estimate  $\widehat{A}$  of the dictionary elements, and is given in Algorithm 1. The estimate  $\widehat{A}$  is then employed to obtain an estimate of the coefficient matrix X through Lasso.

Given samples Y, we first construct the correlation graph  $G_{\operatorname{corr}(\rho)}$ , where the nodes are samples  $\{y_1,y_2,\ldots y_n\}$  and an edge  $(y_i,y_j)\in G_{\operatorname{corr}(\rho)}$  implies that  $|\langle y_i,y_j\rangle|>\rho$ , for some threshold  $\rho>0$ . We then determine a good subset of samples via a clustering procedure on the graph as follows: we first randomly sample an edge  $(y_{i^*},y_{j^*})\in G_{\operatorname{corr}(\rho)}$  and then consider the intersection of their neighborhoods, denoted by  $\widehat{S}$ . We then employ UniqueIntersection routine in Procedure 1 to determine if  $\widehat{S}$  is a "good set" for estimating a dictionary element, and this is done by ensuring that the set  $\widehat{S}$  has sufficient number of mutual neighbors<sup>2</sup> in the correlation graph. Once  $\widehat{S}$  is determined to be a good set, we then proceed by estimating the matrix  $\widehat{M}$  using samples in  $\widehat{S}$  and output its top singular vector as the estimate of a dictionary element. The method is repeated over all edges in the correlation graph to ensure that all the dictionary elements get estimated with high probability.

#### 2.1.2 Exact recovery step: Lasso

Once we obtain an estimate of the dictionary elements, we proceed to estimate the coefficient matrix. The main observation at this step is that the coefficient vector  $x_i$  for each sample  $y_i$  is a s-sparse vector in r-dimensions. Hence, recovering the coefficients would be a standard sparse linear problem if we knew the dictionary A exactly. Our analysis will show that even an approximately correct dictionary  $\bar{A}$  from Algorithm 1 suffices to provide guarantees for this recovery. Once the coefficients are estimated, the dictionary can be re-estimated by solving another linear system. The procedure is formally described in Algorithm 2. We do not prescribe any particular choice of computational procedure to solve the optimization problem (3), but there are many algorithms available in standard literature. As a concrete example, the GraDeS algorithm of Garg and Khandekar [16] works in our setting.

#### 2.2 Guarantees

**Assumptions:** We now provide guarantees for the proposed method under the following assumptions on A and X.

<sup>&</sup>lt;sup>2</sup>For convenience to avoid dependency issues, in Procedure 1, we partition  $\hat{S}$  into sets consisting of node pairs and determine if there are sufficient number of node pairs which are neighbors.

**Algorithm 1** InitDictionaryLearn $(Y, \epsilon_{\text{dict}}, \rho)$ : Initial step for estimating dictionary elements.

Input: Samples  $Y = [y_1| \dots |y_n]$ . Correlation threshold  $\rho$ . Desired separation parameter  $\epsilon$  between recovered dictionary elements.

Output: Initial Dictionary Estimate  $\bar{A}$ .

```
Construct correlation graph G_{\operatorname{corr}(\rho)} s.t. (y_i,y_j) \in G_{\operatorname{corr}(\rho)} when |\langle y_i,y_j \rangle| > \rho. Set \bar{A} \leftarrow \emptyset.

for each edge (y_{i^*},y_{j^*}) \in G_{\operatorname{corr}(\rho)} do
\widehat{S} \leftarrow \mathcal{N}_{G_{\operatorname{corr}(\rho)}}(y_{i^*}) \cap \mathcal{N}_{G_{\operatorname{corr}(\rho)}}(y_{j^*}).
if UniqueIntersection(\widehat{S},G_{\operatorname{corr}(\rho)}) then
\widehat{M} \leftarrow \sum_{y \in \widehat{S}} yy^{\top} \text{ and } \bar{a} \leftarrow u_1, \text{ where } u_1 \text{ is top singular vector of } \widehat{M}.
if \min_{b \in \bar{A}} ||\bar{a} - b|| > 2\epsilon_{\operatorname{dict}} then
\bar{A} \leftarrow \bar{A} \cup \bar{a}
end if
end if
```

**Procedure 1** UniqueIntersection(S,G): Determine if samples in S have a unique intersection.

**Input:** Set S with 2m vectors  $y_1, \ldots, y_{2m}$  and graph G with  $y_1, \ldots, y_{2m}$  as nodes.

Output: Indicator variable UNIQUE\_INT

Return  $\bar{A}$ 

```
Partition S into sets S_1, \ldots, S_m such that each |S_i| = 2.

if Number of S_i which are edges in G is greater than \frac{61m}{64} then UNIQUE_INT \leftarrow 1

else

UNIQUE_INT \leftarrow 0

end if

Return UNIQUE_INT
```

- (A1) Unit-norm Dictionary Elements: All the elements are normalized:  $||a_i|| = 1$ , for  $i \in [r]$ .
- (A2) **Incoherent Dictionary Elements:** We assume pairwise incoherence condition on the dictionary elements, for some constant  $\mu_0 > 0$ ,

$$|\langle a_i, a_j \rangle| < \frac{\mu_0}{\sqrt{d}}.\tag{4}$$

(A3) Spectral Condition on Dictionary Elements: The dictionary matrix has bounded spectral norm, for some constant  $\mu_1 > 0$ ,

$$||A|| < \mu_1 \sqrt{\frac{r}{d}}.\tag{5}$$

(A4) Sparse Coefficient Matrix: The columns of coefficient matrix have bounded number of non-zero entries s which are selected randomly, i.e.

$$|\operatorname{Supp}(x_i)| = s, \quad \forall i \in [n].$$
 (6)

## **Algorithm 2** RecoverCoeff $(Y, A, \epsilon_{coeff})$ : Exact Recovery through lasso

input Samples Y, approximate dictionary  $\bar{A}$  and accuracy parameter  $\epsilon_{\text{coeff}}$ . sign(X) returns a matrix with signs of the entries of X.

- for samples  $i = 1, 2, \dots, n$  do
- Estimate

$$\widehat{x}_i = \arg\min_{x \in \mathbb{R}^r} ||x||_1, \quad \text{subject to} \quad ||y_i - \bar{A}x_i||_2 \le \epsilon_{\text{coeff}}.$$
 (3)

- Threshold:  $\widehat{X} \leftarrow \operatorname{sign}(\widehat{X})$ .
- 4: end for
- 5: Estimate  $\widehat{A} = Y \widehat{X}^T (\widehat{X} \widehat{X}^T)^{-1}$ 6: Normalize:  $\widehat{a}_i = \frac{\widehat{a}_i}{\|\widehat{a}_i\|_2}$

## output $\widehat{A}$

We require s to be

$$s < \min\left(\sqrt{\frac{\sqrt{d}}{2\mu_0}}, \sqrt[3]{\frac{r}{1536}}\right). \tag{7}$$

- (A5) Entries in Coefficient Matrix: We assume that the non-zero entries of X are drawn from Bernoulli $\{-1,1\}$  with equal probability.
- (A6) Sample Complexity: Given a parameter  $\alpha \in (0, 1/20)$  (which is related to the error in recovery of dictionary, see Theorem 2.1), and a universal constant c > 0, choose  $\delta > 0$  and the number of samples n such that

$$n := n(d, r, s, \delta, \alpha) = \max\left(\frac{r}{c\alpha^2 s} \log \frac{d}{\delta}, \frac{32768r}{s} \log \frac{4}{\delta}\right), \quad n^2 \delta < 1.$$

(A7) Choice of Threshold for Correlation Graph: The correlation graph  $G_{\text{corr}(\rho)}$  is constructed using threshold  $\rho$  such that

$$\rho = \frac{1}{2} - \frac{s^2 \mu_0}{\sqrt{d}} > 0. \tag{8}$$

(A8) Choice of Separation Parameter  $\epsilon_{\text{dict}}$  between Estimated Dictionary Elements: This is the desired accuracy of the estimated dictionary elements to the true dictionary elements using just the initialization step. It can be chosen to be:

$$32s\left(\frac{\mu_1}{\sqrt{ds}} + \frac{\mu_1^2}{d} + \frac{s^3}{r} + \alpha^2 + \frac{\alpha}{\sqrt{s}}\right) < \epsilon_{\text{dict}}^2 < \frac{1}{4}.\tag{9}$$

The assumption (A1) on normalization is without loss of generality since we can always rescale the dictionary elements and the corresponding coefficients and obtain the same observations. The assumption (A2) on incoherence is crucial to our analysis. In particular, incoherence also leads to a bound on the RIP constant; see Lemma A.5 in Appendix A.6. The assumption (A3) provides a bound on the spectral norm of A.

The assumption (A4) on sparsity in the coefficient matrix is crucial for identifiability of dictionary learning problem. We require for the sparsity to be not too large for recovery. The assumption (A5) assumes that the non-zero entries of X have a fixed magnitude with arbitrary signs. Our results for the initial dictionary learning part does not require this assumption and we can handle general values for X which are bounded away from zero. Note that a similar assumption is made in the work of Arora et.al [7]. However, our subsequent cleanup step after performing Lasso requires the assumption that the non-zero entries of X be discrete with enough separation between them, and for simplicity, we assume them to be Bernoulli $\{-1,1\}$ .

The assumption (A6) provides a bound on sample complexity. We subsequently establish that in order to have decaying error for recovery of dictionary elements, we require  $n = \omega(r)$  samples for recovery. Thus, we obtain a nearly linear sample complexity for our method.

Assumption (A7) specifies the threshold for the construction of the correlation graph. Intuitively, we require a threshold such that we can distinguish pairs of samples which share a dictionary element from those which do not.

**Theorem 2.1** (Approximate recovery of dictionary). Suppose the output of Algorithm 1 is A. Then with probability greater than  $1 - 2n^2\delta$ , there exists a permutation matrix P such that:

$$\epsilon_A := \|a_i - (P\bar{A})_i\|_2^2 < 32s \left(\frac{\mu_1}{\sqrt{ds}} + \frac{\mu_1^2}{d} + \frac{s^3}{r} + \alpha^2 + \frac{\alpha}{\sqrt{s}}\right). \tag{10}$$

**Remark:** The assumption on sparsity in (A4) implies that the first two terms in (10) decay. For the third term in (10) to decay, we require  $s = o(r^{1/4})$  instead of  $s = \mathcal{O}(r^{1/3})$  as in (A4). Moreover, we require that  $\alpha^2 s = o(1)$ . Since the sample complexity in (A7) scales as  $n = \Omega\left(\frac{r}{\alpha^2 s}\right)$ , we require  $n = \omega(r)$  samples for recovery of dictionary with decaying error. Thus, we obtain a near linear sample complexity for our method.

### 2.2.1 Exact recovery through lasso step

Our second main result is that under somewhat stronger conditions than before, it is possible to exactly recover the unknown dictionary A with high probability. This result will be obtained by initializing Algorithm 2 with the output of Algorithm 1. We start with the additional assumptions, putting restrictions on the allowed sparsity level s as a function of r and d.

**Assumption 2.1** (Conditions for exact recovery). The sparsity level s, and the number of dictionary elements r and the observed dimension d satisfy

$$32s\left(\frac{\mu_1}{\sqrt{ds}} + \frac{\mu_1^2}{d}\right) \le \frac{1}{1200s^2}, \quad and \quad \frac{32s^4}{r} \le \frac{1}{1200s^2}.$$

The constant  $\alpha$  in Theorem 2.1 satisfies

$$32s\left(\alpha^2 + \frac{\alpha}{\sqrt{s}}\right) \le \frac{1}{1200s^2}.$$

The number of samples n, in addition to assumption (A6), satisfies

$$n \ge \frac{4r}{c_0} \log \frac{d}{\delta},$$

where  $c_0$  is a universal constant.

The accuracy parameter  $\epsilon_{\text{coeff}}$  in Algorithm 2 is chosen as  $\epsilon_{\text{coeff}} = s\epsilon_A$ , where  $\epsilon_A$  is the error in estimating the dictionary elements in (10).

**Theorem 2.2** (Exact recovery of dictionary matrix). Under the conditions of Theorem 2.1, and suppose, in addition Assumption 2.1 holds, then the output  $\widehat{A}$  of Algorithm 2 initialized with Algorithm 1 satisfies  $\widehat{A} = A$  up to permutation of columns, with probability at least  $1 - 3n^2\delta$ .

**Remark:** Assumption 2.1 for exact recovery places a more stringent condition on the sparsity level s, compared to (A4) for approximate recovery. While for approximate recovery, we require  $s = \mathcal{O}(d^{1/4}, r^{1/4})$ , in Assumption 2.1, we require  $s = \mathcal{O}(d^{1/5}, r^{1/6})$  for exact recovery. Note that the additional constraint on sample complexity n in Assumption 2.1 still has the same scaling, and thus,  $n = \mathcal{O}(r(\log r + \log d))$  suffices both for approximate and exact recovery.

We also observe that the result of Theorem 2.2 as the initialization procedure, but in principle we can also use a different approximate recovery procedure to initialize Algorithm 2. In particular, a different initialization procedure with a better error guarantee would also directly translate to better recovery properties in the second step, in terms of the assumptions relating s to r and d. Understanding these issues appears to be an interesting direction for future research.

## 3 Proofs of main results

In this section we will present the proofs of our main results, Theorems 2.1 and 2.2. We will start by presenting a host of useful lemmas, and sketch out how they fit together to yield the main results before moving on to the proofs.

#### 3.1 Correlation graph properties

In this section we will present some useful properties of the correlation graph  $G_{\operatorname{corr}(\rho)}$  described in Section 1.3. Recall that  $G_{\operatorname{corr}(\rho)}$ , where the nodes are samples  $\{y_1, y_2, \dots y_n\}$  and an edge  $(y_i, y_j) \in G_{\operatorname{corr}(\rho)}$  implies that  $|\langle y_i, y_j \rangle| > \rho$ , for some  $\rho > 0$ . This is employed by Algorithm 1 as a proxy for identifying samples which have commond dictionary elements. We now make this connection concrete in the next few lemmas. For this we also recall our notation  $\mathcal{N}_B(y)$  which is the neighborhood of a sample y in the coefficient bipartite graph (see Figure 1a), that is, the set of dictionary elements that combine to yield y.

**Lemma 3.1** (Correlation graph). Under the incoherence assumption (A2) and the threshold  $\rho$  in assumption (A7), the following is true for the edges in the correlation graph  $G_{\text{corr}(\rho)}$ :

$$|\mathcal{N}_B(y_k) \cap \mathcal{N}_B(y_l)| = 1 \Rightarrow (y_k, y_l) \in G_{corr(\rho)}, \quad \forall i \in [r],$$
(11)

$$(y_k, y_l) \in G_{corr(\rho)} \Rightarrow |\mathcal{N}_B(y_k) \cap \mathcal{N}_B(y_l)| \ge 1,$$
 (12)

for all  $k, l \in \{1, 2, \dots, n\}, k \neq l$ .

Lemma 3.1 suggests that nodes which intersect in *exactly one* dictionary element are special, in that they are guaranteed to have an edge between them in  $G_{\text{corr}(\rho)}$ . Our next lemma works towards establishing something even stronger. We will next establish that there are large cliques in

the correlation graph where any two samples in the clique intersect in the same unique dictionary element. In order to state the lemma, we need some additional notation.

For each dictionary element  $a_i$ , consider a set of samples  $\{y_k, k \in S\}$ , for some  $S \subset \{1, 2, ..., n\}$ , such that they only have  $a_i$  in common, and denote such a set by  $C_i$  i.e.

$$C_i := \{ y_k, k \in S : \mathcal{N}_B(y_k) \cap \mathcal{N}_B(y_l) = \{ a_i, \} \ \forall \, k, l \in S \}.$$
 (13)

Lemma 3.1 implies that in the correlation graph, the set of nodes in  $C_i$  form a clique (not necessarily maximal), for each  $i \in \{1, 2, ..., r\}$ , as shown in Figure 1b. The above implication can be exploited for recovery of dictionary elements: if we find the set  $C_i$ , then we can hope to recover the element  $a_i$ , since that is the only element in common to the samples in  $C_i$ .

For ease of stating the next lemma, we further define two shorthand notations.

Uniq-intersect
$$(y_i, y_j) := \{(y_i, y_j) \in G_{corr(\rho)} \text{ and } |\mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_j)| = 1\},$$
 (14)

Intuitively, the samples satisfying Uniq-intersect $(y_i, y_j)$  are guaranteed to have an edge between them by Lemma 3.1. In order to guarantee large cliques, we will also need to measure the number of triangles in  $G_{\text{corr}(\rho)}$ .

In order to do this, given anchor samples  $y_{i^*}$  and  $y_{j^*}$  have a unique intersection, we now bound the probability that a randomly chosen sample  $y_i$ , among the neighborhood set of  $y_{i^*}$  and  $y_{j^*}$  in the correlation graph also has a unique intersection. Now define unique intersection event for a new sample  $y_i$  with respect to anchor samples  $y_{i^*}$  and  $y_{j^*}$  as follows

Uniq-intersect
$$(y_i; y_{i^*}, y_{i^*}) := \{ \mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_{i^*}) = \mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_{i^*}) = \{a_k\} \},$$
 (15)

where  $a_k = \mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*})$  is the unique intersection of the anchor samples  $y_{i^*}$  and  $y_{j^*}$ . In other words, Uniq-intersect $(y_i; y_{i^*}, y_{j^*})$  indicates the event that the pairwise intersections of the new sample  $y_i$  with each of the anchors  $y_{i^*}$  and  $y_{j^*}$  is unique and equal to the unique intersection of  $y_{i^*}$  and  $y_{j^*}$ .

**Lemma 3.2** (Formation of clique under good anchor samples).

$$\mathbb{P}\left[\text{Uniq-intersect}(y_i; y_{i^*}, y_{j^*}) \mid \text{Uniq-intersect}(y_{i^*}, y_{j^*}), \text{ and } (y_i, y_{i^*}), (y_i, y_{j^*}) \in G_{\text{corr}(\rho)}\right]$$

$$\geq 1 - \frac{s^3}{r}.$$

Lemma 3.2 is crucial for our algorithm. It guarantees that given a pair of good anchor elements—one satisfying unique intersection property—a large fraction of their neighbors also contain this common dictionary element. Some further arguments can then be made to establish that a large fraction of the neighbors of  $y_{i^*}$  and  $y_{j^*}$  also have edges amongst themselves and hence form cliques as defined in Equation 13.

<sup>&</sup>lt;sup>3</sup>Note that such a set need not be unique.

### 3.2 Correctness of Procedure 1

A key component in our analysis is the correctness of Procedure 1. As we saw in the previous lemmas, it is crucial for a chosen pair of anchor elements to have a unique intersection in order to use them for identifying large cliques  $C_i$  in  $G_{\text{corr}(\rho)}$ . Procedure 1 plays a crucial role by providing a verifiable test for whether a pair of anchor elements have a unique intersection or not. Our next two lemmas help us establish that this test is sound with high probability. We first show that two neighbors of a bad anchor pair do not have an edge amongst them with high probability.

Denote the event

$$\Delta(y_i, y_j, y_k) := \{(y_i, y_j), (y_j, y_k), (y_i, y_k) \in G_{corr(\rho)}\},\$$

i.e., the samples  $y_i, y_j, y_k$  form a triangle in the correlation graph.

**Lemma 3.3** (Detection of bad anchor samples). For randomly chosen samples  $y_i, y_j$ 

$$\mathbb{P}\left[(y_i, y_j) \notin G_{\operatorname{corr}(\rho)} \mid \Delta(y_i, y_{i^*}, y_{j^*}), \Delta(y_j, y_{i^*}, y_{j^*}), \neg \operatorname{Uniq-intersect}(y_{i^*}, y_{j^*})\right] > \frac{1}{16}.$$

Intuitively, this means that the number of sets  $S_i$  which will be edges in  $G_{\text{corr}(\rho)}$  is rather small for an anchor pair with multiple dictionary elements in common. In order for correctness of the procedure, we will in fact need this number to be substantially smaller than that for a good anchor pair. This is indeed the case as we next establish.

**Lemma 3.4** (Detection of good anchor samples). For randomly chosen samples  $y_i, y_j$ 

$$\mathbb{P}\left[(y_i, y_j) \notin G_{\operatorname{corr}(\rho)} \mid \Delta(y_i, y_{i^*}, y_{j^*}), \Delta(y_j, y_{i^*}, y_{j^*}), \operatorname{Uniq-intersect}(y_{i^*}, y_{j^*})\right] \leq \frac{24s^3}{r}.$$

Combining the above two lemmas, the correctness of Procedure 1 naturally follows.

**Proposition 3.1** (Correctness of Procedure 1). Suppose  $(y_{i^*}, y_{j^*}) \in G_{\operatorname{corr}(\rho)}$ . Suppose that  $s^3 \leq r/1536$  and  $\gamma \leq 1/64$ . Then Algorithm 1 returns the value of Uniq-intersect $(y_{i^*}, y_{j^*})$  correctly with probability greater than  $1 - 2 \exp(-\gamma^2 m)$ .

#### 3.3 Estimation of the Dictionary Elements via SVD

In this section we will put all the pieces together and establish Theorem 2.1. We start by establishing that given a pair of good anchor elements, the SVD step in Algorithm 1 approximately recovers the unique dictionary element in the intersection of the two anchors.

**Proposition 3.2** (Accuracy of SVD). Consider anchor samples  $y_{i^*}$  and  $y_{j^*}$  such that Uniq-intersect $(y_{i^*}, y_{j^*})$  is satisfied, and wlog, let  $\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*}) = \{a_1\}$ . Recall the definition of  $\widehat{S}$  (25), and further define  $\widehat{M} := \sum_{i \in \widehat{S}} y_i y_i^{\top}$  and  $|\widehat{S}| = m$ . If  $\widehat{a}$  is the top singular vector of  $\widehat{M}$ , then there exists a universal constant c such that we have:

$$\|\widehat{a} - a_1\|_2^2 < 32s \left( \frac{\mu_1}{\sqrt{ds}} + \frac{\mu_1^2}{d} + \frac{s^3}{r} + \alpha^2 + \frac{\alpha}{\sqrt{s}} \right),$$

with probability greater than  $1 - d \exp(-c\alpha^2 m)$  for  $\alpha < 1/20$ .

Given the above proposition, the proof of Theorem 2.1 is relatively straightforward. Indeed, the key missing piece is the dependence on the random quantity  $|\hat{S}|$  in the error probability in Proposition 3.2. We now present the proof.

#### Proof of Theorem 2.1:

Consider a particular iteration of Algorithm 1. Procedure 1 returns Uniq-intersect  $(y_{i^*}, y_{j^*})$  with probability greater than  $1-2\exp(-\gamma^2|\widehat{S}|/2)$ . If  $\neg$  Uniq-intersect  $(y_{i^*}, y_{j^*})$ , then Algorithm 1 proceeds to the next iteration. Consider the case of Uniq-intersect  $(y_{i^*}, y_{j^*})$  and suppose  $\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*}) = \{a_l\}$ . Using Proposition 3.2, with probability greater than  $1 - d\exp\left(-c\alpha^2|\widehat{S}|\right)$ , we have:

$$||a_l - \widehat{a}||_2^2 < 32s \left( \frac{\mu_1}{\sqrt{ds}} + \frac{\mu_1^2}{d} + \frac{s^3}{r} + \alpha^2 + \frac{\alpha}{\sqrt{s}} \right).$$

Using Lemma A.4 and Lemma 3.1, we see that  $|\widehat{S}| \geq \frac{ns}{4r}$  with probability greater than  $1 - \exp\left(\frac{-ns}{16r}\right)$ . Using a union bound over all the iterations (which are at most  $n^2$ ), the above claims hold for all iterations with probability greater than  $1 - n^2 d \exp\left(\frac{-c\alpha^2 ns}{r}\right) - 2n^2 \exp\left(\frac{-\gamma^2 ns}{8r}\right) - n^2 \exp\left(\frac{-ns}{16r}\right)$ .

Using Lemma A.4 and Lemma 3.1, with probability greater than  $1-r\exp\left(\frac{-ns}{64r}\right)$ , for every  $l \in [r]$ , there are at least  $\frac{ns}{8r}$  pairs  $(i^*, j^*)$  such that  $\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*}) = \{a_l\}$  and  $(i^*, j^*) \in G_{\operatorname{corr}(\rho)}$ . Lines 9-11 of the algorithm then ensure that there is a unique copy of the approximation to  $a_l$  dictionary element. Using a union bound now gives the result.

## 3.4 Exact recovery through lasso

In this section, we will show how to clean up the approximate recovery of the previous section and obtain exact recovery of the dictionary under our assumptions on the coefficients. We start by setting up the problem as that of sparse estimation with deterministic noise and describing some guarantees in a general setup. We then specialize these to the assumptions of our problem and present the proof of Theorem 2.2.

### 3.4.1 Lasso with determinstic noise

Recalling the model (1), we see that each observation  $y_i$  is generated according to the linear model

$$y_i = Ax_i$$
, for  $i = 1, 2, ..., n$ ,

where  $x_i$  is a s-sparse vector in r dimensions. If we knew the dictionary A, then this is the usual sparse linear system. Given the knowledge of an approximate dictionary  $\bar{A}$  however, we can rewrite the system as

$$y_i = \bar{A}x_i + \underbrace{(A - \bar{A})x_i}_{w_i},\tag{16}$$

where  $W \in \mathbb{R}^{d \times n}$  is the error matrix. Note that the errors in W are not zero mean, or even independent of  $\bar{A}$  unlike typical statistical settings. Under our initialization, however, they are bounded, which we establish subsequently. For the remainder of this section, we assume the following facts about  $\bar{A}$ :

**Assumption 3.1** (Approximate initialization). Assume that  $\bar{A}$  is an approximately correct initialization for A, meaning the following hold:

**RIP:** The 2s-RIP constant of the matrix  $\bar{A}$ ,  $\delta_{2s} < \frac{1}{7}$ . That is, for every  $S \subseteq \{1, 2, ..., r\}$  with  $|S| \leq 2s$ , the smallest and largest singular values,  $\sigma_{\min}$  and  $\sigma_{\max}$  respectively of the  $d \times |S|$  matrix  $\bar{A}_S$  satisfy:

$$\frac{6}{7} < \sigma_{\min} < \sigma_{\max} < \frac{8}{7}.$$

**Bounded error:**  $\|\bar{a}_i - a_i\|_2 \le \epsilon_A$  for all  $i = 1, 2, \dots, r$ .

Under these general assumptions, we can provide a guarantee on the error incurred in step (3) of Algorithm 2. While this result has been obtained in many contexts by various authors, we use the following precise form from Candes [12].

**Theorem 3.1** (Theorem 1.2 from Candes [12]). Suppose  $y_i$  is generated according to the linear model (16), where  $x_i$  is s-sparse and assume that  $\delta_{2s} \leq \sqrt{2} - 1$ . Then the solution to Equation 3 obeys the following, for a universal constant  $C_1$ ,

$$\|\widehat{x}_i - x_i\|_2 \le C_1 \|w_i\|_2.$$

In particular,  $C_1 = 8.5$  suffices for  $\delta_{2s} \leq 0.2$ .

#### 3.4.2 Proof of Theorem 2.2

In order to prove Theorem 2.2, we first establish that under our assumptions, the coefficients  $x_i$  are exactly recovered in Equation (3). Once this is established, Theorem 2.2 follows in a straightforward manner. We start with a useful proposition.

**Proposition 3.3.** Under conditions of Theorem 2.1, assume further that  $\epsilon_A \leq 1/(20s)$  for the dictionary returned in Algorithm 1. Then Algorithm 2 guarantees that  $\hat{x}_i = x_i$  for all i = 1, 2, ..., n.

**Proof:** We would like to use Theorem 3.1 that we recover the coefficients  $x_i$  correctly in the lasso step (3) of Algorithm 2. In order to do this, we first need to verify Assumption 3.1 for the dictionary returned by Algorithm 1, and then obtain bounds on the quantity  $||w_1||_2$ . We start with the former.

Consider any 2s-sparse subset S of [r]. We have:

$$\sigma_{\min}(\bar{A}_S) \ge \sigma_{\min}(A_S) - \|A_S - \bar{A}_S\|_2 \stackrel{(\zeta_1)}{\ge} 1 - \frac{2\mu_0 s}{\sqrt{d}} - \|A_S - \bar{A}_S\|_F \quad \text{and},$$

$$\sigma_{\max}(\bar{A}_S) \le \sigma_{\max}(A_S) + \|A_S - \bar{A}_S\|_2 \stackrel{(\zeta_2)}{\le} 1 + \frac{2\mu_0 s}{\sqrt{d}} + \|A_S - \bar{A}_S\|_F,$$

where  $\zeta_1$  and  $\zeta_2$  follow from Lemma A.5 in Appendix A.6. Since  $A_S$  is a  $d \times 2s$  matrix, it satisfies that  $||A_S - \bar{A}_S||_F \leq \sqrt{2s\epsilon_A}$ . Given the assumption  $\epsilon_A \leq 1/(20s)$ , it immediately follows that the minimum and maximum singular values of  $\bar{A}_s$  are at least 6/7 and 8/7 respectively, so that we obtain  $\delta_{2s} = 1/7 < 0.2$ .

This shows that  $\bar{A}$  satisfies Assumption 3.1. Next we bound the  $\ell_2$  norm of the noise vector  $w_i$ . Again bounding the frobenius norm of the error in the dictionary in the same way as above, we obtain

$$||w_i||_2 \le ||(A - \bar{A})_{S_i}||_2 ||x_i||_2 \le ||(A - \bar{A})_{S_i}||_F \sqrt{s} \le s\epsilon_A,$$

where  $S_i$  is the support of  $x_i$ . Consequently, we obtain from Theorem 3.1 that the output  $\hat{x}_i$  of Equation 3 satisfies

$$\|\widehat{x}_i - x_i\|_2 \le C_1 \, s\epsilon_A \le 9s\epsilon_A \le 9/20. \tag{17}$$

We now observe that an  $\ell_2$  error guarantee is also an  $\ell_{\infty}$  error guarantee. Recall that by the model assumption, each non-zero coefficient of X has an absolute value of 1. Since Equation (17) guarantees that the  $\ell_2$  error guarantee is no larger than 1/2, all the coefficients will be uniquely recovered and hence  $\hat{x}_i = x_i$ .

#### Proof of Theorem 2.2:

We are now ready to provide our proof of exact recovery. Based on Proposition 3.3, we only need to verify two things. First is that the initialization  $\bar{A}$  satisfies  $\epsilon_A < 1/(20s)$  and the second is that the linear system Y = AX is well-posed when we solve for A. In order to verify the former, we observe that our additional conditions in Assumption 2.1 guarantee that

$$32s\left(\frac{\mu_1}{\sqrt{ds}} + \frac{\mu_1^2}{d}\right) \le \frac{1}{1200s^2},$$

$$\frac{32s^4}{r} \le \frac{1}{1200s^2}, \quad \text{and}$$

$$32s\left(\alpha^2 + \frac{\alpha}{\sqrt{s}}\right) \le \frac{1}{1200s^2}.$$

Hence we obtain from Theorem 2.1 that with probability at least  $1 - n^2 d \exp\left(\frac{-c\alpha^2 ns}{r}\right) - 4 \max(n^2, r) \exp\left(\frac{-ns}{32768r}\right)$ ,  $\epsilon_A < 1/(20s)$ . Hence, it only remains to verify that the linear system is well-posed.

According to Lemma A.7 in Appendix A.6, the matrix  $\mathbb{E}\left[XX^T\right] = \frac{s}{r}I_{r\times r}$  so that all of its singular values are equal to s/r. We now appeal to Theorem A.1 with W=X, d=r and  $u=\sqrt{s}$ . Then we obtain for any t>0 with probability at least  $1-r\exp(-ct^2)$ 

$$\sigma_{\min}(XX^T) \ge \frac{ns}{r} - n \max\left\{\sqrt{\frac{s}{r}}\delta, \delta^2\right\},$$

where  $\delta = t\sqrt{s/n}$ . Substituting the value of  $\delta$ , we obtain the lower bound

$$\sigma_{\min}(XX^T) \ge \frac{ns}{r} - n \max\left\{\sqrt{\frac{s}{r}}t\sqrt{\frac{s}{n}}, t^2 \frac{s}{n}\right\}$$
$$\ge \frac{ns}{r} \left(1 - t\sqrt{\frac{r}{n}} - \frac{t^2r}{n}\right)$$
$$= \frac{ns}{4r},$$

for  $t = \sqrt{n/(4r)}$ . This means that the linear system is well-posed with probability at least  $1 - r \exp(-cn/(4r))$ . Choosing  $c_0$  to now be  $\min(c, 1/32768)$  finishes the proof.

## 4 Discussion and Conclusion

In this paper, we proposed simple and tractable methods for dictionary learning. We present an initialization step that uses simple clustering-based approach in order to approximately recover the dictionary. We also analyzed a simple denoising strategy based on sparse recovery algorithms for reconstructing the dictionary exactly under some assumptions. In particular, the second step is not tied to the first step in any critical way, and a different approximate recovery procedure can also be further cleaned with the same algorithm.

In the analysis of the clustering step, we provide guarantees when the coefficient matrix is sparse and randomly drawn. In principle, our analysis can be extended to general sparse coefficient matrices and can be cast as a higher-order expansion condition on the coefficient bipartite graph. Similar (and yet not the same) expansion conditions have appeared in other contexts involving learning of overcomplete models. For instance, in [4], Anandkumar et. al. establish that under an expansion condition on the topic-word matrix, unsupervised learning of the model is possible. Here, the hidden topics correspond to dictionary elements, and the observed words correspond to the samples in the dictionary setting.

Finally, our work suggests some natural and interesting directions for future research. While both the steps of our algorithm seem inherently robust to noise, it remains important to quantify the recovery properties when the observations are noisy in future work. Another natural question is raised by the fact that we use only one step of lasso and least squares for exact recovery. It seems plausible that we can recover the dictionary under more general conditions by doing further alternating steps of dictionary and coefficient recovery. Analyzing such a procedure is an interesting question for future work. Since our study was motivated by natural applications of dictionary learning in signal processing and machine learning, it would also be interesting to investigate how our provably correct procedures perform compared to the popular heuristic methods.

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## A Proofs for approximate initialization

In this section we will provide the proofs of many of the Lemmas along with some auxilliary results in Sections 3.1- 3.3. Some of the more technical results that are required will be deferred to Appendix A.6.

## A.1 Proofs of correlation graph properties

We start by proving Lemmas 3.1 and 3.2 in Section 3.1.

### Proof of Lemma 3.1:

We first prove (12) via contradiction. Suppose  $\mathcal{N}_B(y_k) \cap \mathcal{N}_B(y_l) = \emptyset$ , we then have

$$\begin{aligned} |\langle y_k, y_l \rangle| &= |\sum_{i,j} x_{ik} x_{jl} \langle a_i, a_j \rangle| \le \sum_{i,j} |x_{ik} x_{jl} \langle a_i, a_j \rangle| \\ &\le |\mathcal{N}_B(y_k)| \cdot |\mathcal{N}_B(y_l)| \cdot \max_{i,j,k,l} |x_{ik} x_{jl}| \cdot \max_{i \neq j} |\langle a_i, a_j \rangle| \le \frac{s^2 \mu_0}{\sqrt{d}} \end{aligned}$$

For (11), let  $\{a_{i^*}\} = \mathcal{N}_B(y_k) \cap \mathcal{N}_B(y_l)$ 

$$\begin{aligned} |\langle y_k, y_l \rangle| &= |\sum_{i,j} x_{ik} x_{jl} \langle a_i, a_j \rangle| \ge |x_{i^*k} x_{i^*l}| \langle a_{i^*}, a_{i^*} \rangle - \sum_{i \neq j} |x_{ik} x_{jl} \langle a_i, a_j \rangle| \\ &\ge 1 - \frac{s^2 \mu_0}{\sqrt{d}}, \end{aligned}$$

using the above analysis. The claims now follow from the setting of  $\rho$ .

We next establish Lemma 3.2.

Proof of Lemma 3.2: Define the event

$$\mathcal{A} := \{ |\mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_{i^*})| \ge 1 \} \cap \{ |\mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_{i^*})| \ge 1 \}.$$

From Lemma 3.1, we have that

$$\mathbb{P}\left[\text{Uniq-intersect}(y_i; y_{i^*}, y_{j^*}) \mid \text{Uniq-intersect}(y_{i^*}, y_{j^*}), \text{ and } (y_i, y_{i^*}), (y_i, y_{j^*}) \in G_{\text{corr}(\rho)}\right] \\ \geq \mathbb{P}\left[\text{Uniq-intersect}(y_i; y_{i^*}, y_{j^*}) \mid \text{Uniq-intersect}(y_{i^*}, y_{j^*}), \mathcal{A}\right]$$

In order to lower bound  $\mathbb{P}\left[\text{Uniq-intersect}(y_i; y_{i^*}, y_{j^*}) \mid \text{Uniq-intersect}(y_{i^*}, y_{j^*}), \mathcal{A}\right]$ , we instead upper bound the probability of the complementary event  $\mathbb{P}\left[\neg \text{Uniq-intersect}(y_i; y_{i^*}, y_{j^*}) \mid \text{Uniq-intersect}(y_{i^*}, y_{j^*}), \mathcal{A}\right]$ In order to do so, we first bound the following

$$\mathbb{P}\left[\mathcal{A} \mid \text{Uniq-intersect}(y_{i^*}, y_{j^*})\right] \ge \frac{s}{r},\tag{18}$$

since  $\mathcal{A}$  holds when the unique element in  $\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*})$  is chosen and its probability is s/r. We also have

$$\mathbb{P}\left[\neg \text{Uniq-intersect}(y_i; y_{i^*}, y_{j^*}) \cap \mathcal{A} \mid \text{Uniq-intersect}(y_{i^*}, y_{j^*})\right] \leq \frac{(s-1)^2 \binom{r-3}{s-2}}{\binom{r}{s}},$$

since for  $\neg$  Uniq-intersect $(y_i; y_{i^*}, y_{j^*})$  to hold, we need to choose at least one of the s-1 elements in  $\mathcal{N}_B(y_{i^*})/\mathcal{N}_B(y_{j^*})$ , and similarly one from the s-1 elements of  $\mathcal{N}_B(y_{j^*})/\mathcal{N}_B(y_{i^*})$ . The rest of the s-2 elements can be picked arbitrarily from the r-3 dictionary atoms that remain after excluding the two already picked and the unique intersection  $\mathcal{N}_B(y_{j^*}) \cap \mathcal{N}_B(y_{i^*})$ .

It is easy to check that

$$\frac{(s-1)^2 \binom{r-3}{s-2}}{\binom{r}{s}} = \frac{(s-1)^2 (r-s)s(s-1)}{r(r-1)(r-2)} \le \frac{s^4}{r^2}.$$
(19)

Taking the ratio of the two bounds in (18) and (19) completes the proof.

### A.2 Proofs of Lemmas 3.3 and 3.4

We now prove the two lemmas that are crucial to establishing the correctness of Proedure 1. **Proof of Lemma 3.3:** Let  $A_1$  and  $A_2$  denote the following events:

$$\mathcal{A}_{1} := \{ |\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{i^{*}})| \geq 1 \} \cap \{ |\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{j^{*}})| \geq 1 \} \\
\cap \{ |\mathcal{N}_{B}(y_{j}) \cap \mathcal{N}_{B}(y_{i^{*}})| \geq 1 \} \cap \{ |\mathcal{N}_{B}(y_{j}) \cap \mathcal{N}_{B}(y_{j^{*}})| \geq 1 \} \\
\mathcal{A}_{2} := \{ |\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{i^{*}})| = 1 \} \cap \{ |\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{j^{*}})| = 1 \} \\
\cap \{ |\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{i^{*}})| = 1 \} \cap \{ |\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{j^{*}})| = 1 \} \tag{20}$$

In words, both  $y_i$  and  $y_j$  have at least dictionary element in common with each of  $y_{i^*}$  and  $y_{j^*}$  under the event  $\mathcal{A}_1$ , while the number of common elements is exactly one under the event  $\mathcal{A}_2$ . We have

$$\mathbb{P}\left[(y_{i},y_{j}) \notin G_{\text{corr}(\rho)} \mid \Delta(y_{i},y_{i^{*}},y_{j^{*}}), \Delta(y_{j},y_{i^{*}},y_{j^{*}}), \neg \text{Uniq-intersect}(y_{i^{*}},y_{j^{*}})\right] \\
\stackrel{(a)}{=} \mathbb{P}\left[(y_{i},y_{j}) \notin G_{\text{corr}(\rho)} \mid \mathcal{A}_{1}, \Delta(y_{i},y_{i^{*}},y_{j^{*}}), \Delta(y_{j},y_{i^{*}},y_{j^{*}}), \neg \text{Uniq-intersect}(y_{i^{*}},y_{j^{*}})\right] \\
= \mathbb{P}\left[(y_{i},y_{j}) \notin G_{\text{corr}(\rho)}, \Delta(y_{j},y_{i^{*}},y_{j^{*}}) \mid \mathcal{A}_{1}, \Delta(y_{i},y_{i^{*}},y_{j^{*}}), \Delta(y_{j},y_{i^{*}},y_{j^{*}}), \neg \text{Uniq-intersect}(y_{i^{*}},y_{j^{*}})\right] \\
\geq \mathbb{P}\left[(y_{i},y_{j}) \notin G_{\text{corr}(\rho)}, \Delta(y_{i},y_{i^{*}},y_{j^{*}}), \Delta(y_{j},y_{i^{*}},y_{j^{*}}) \mid \mathcal{A}_{1}, \neg \text{Uniq-intersect}(y_{i^{*}},y_{j^{*}}), (y_{i^{*}},y_{j^{*}}), (y_{i^{*}},y_{j^{*}}) \in G_{\text{corr}(\rho)}\right] \\
\stackrel{(b)}{\geq} \mathbb{P}\left[(y_{i},y_{j}) \notin G_{\text{corr}(\rho)}, \mathcal{A}_{2} \mid \mathcal{A}_{1}, \neg \text{Uniq-intersect}(y_{i^{*}},y_{j^{*}}), (y_{i^{*}},y_{j^{*}}) \in G_{\text{corr}(\rho)}\right] \\
\stackrel{(c)}{\geq} \mathbb{P}\left[\{\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{j}) = \emptyset\} \cap \mathcal{A}_{2} \mid \mathcal{A}_{1}, \neg \text{Uniq-intersect}(y_{i^{*}},y_{j^{*}}), (y_{i^{*}},y_{j^{*}}), (y_{i^{*}},y_{j^{*}}) \in G_{\text{corr}(\rho)}\right], \tag{21}$$

where the inequalities (a), (b) and (c) follow from Lemma 3.1. We will now work on lower bounding this resulting probability.

We first lower bound the numerator in writing the above conditional probability as the ratio of a joint to marginal probability. We begin by noting that

$$\mathbb{P}\left[\left\{\mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_j) = \emptyset\right\} \cap \mathcal{A}_2 \cap \mathcal{A}_1 \mid \neg \text{Uniq-intersect}(y_{i^*}, y_{j^*}, (y_{i^*}, y_{j^*}) \in G_{\text{corr}(\rho)})\right] \\ = \mathbb{P}\left[\left\{\mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_j) = \emptyset\right\} \cap \mathcal{A}_2 \mid \neg \text{Uniq-intersect}(y_{i^*}, y_{j^*}), (y_{i^*}, y_{j^*}) \in G_{\text{corr}(\rho)}\right]$$

Let us define  $m = |\mathcal{N}_B(y_{i^*}) \cup \mathcal{N}_B(y_{j^*})| \in [s, 2s]$  and  $l = |\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*})| \ge 2^4$ . The event in the probability above, that is  $\mathcal{A}_2$  holds while  $y_i$  and  $y_j$  do not share a dictionary element, can be arranged by choosing two of the l elements, and assigning a unique element to each  $y_i$  and  $y_j$ . Similarly the remaining elements can be chosen outside  $\mathcal{N}_B(y_{i^*}) \cup \mathcal{N}_B(y_{j^*})$  in a non-overlapping manner: for  $y_i$  assign s-1 elements among r-m elements, and then for  $y_j$  assign from remaining r-m-s+1 elements. This logic yields the following lower bound on the probability

$$\mathbb{P}\left[\left\{\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{j}) = \emptyset\right\} \cap \mathcal{A}_{2} \mid \neg \text{Uniq-intersect}(y_{i^{*}}, y_{j^{*}})\right]$$

$$\geq \frac{2\binom{l}{2}\binom{r-m}{s-1}\binom{r-m-s+1}{s-1}}{\binom{r}{s}^{2}} \geq \frac{2\binom{l}{2}\binom{r-2s}{s-1}\binom{r-3s+1}{s-1}}{\binom{r}{s}^{2}},$$

where the second inequality uses  $m \leq 2s$ . Now with some straightforward algebra, we can further lower bound this expression as

$$\mathbb{P}\left[\left\{\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{j}) = \emptyset\right\} \cap \mathcal{A}_{2} \mid \neg \text{Uniq-intersect}(y_{i^{*}}, y_{j^{*}})\right] \\
\geq \frac{s^{2}(l-1)^{2}}{r^{2}} \left(1 - \frac{3s-3}{r-s}\right)^{s-1} \left(1 - \frac{2s-1}{r-s}\right)^{s-1} \\
\geq \frac{s^{2}(l-1)^{2}}{r^{2}} \left(1 - \frac{3s}{r-s}\right)^{s} \left(1 - \frac{2s}{r-s}\right)^{s}.$$

Now we invoke Lemma A.6 to further lower bound the RHS and obtain

$$\mathbb{P}\left[\left\{\mathcal{N}_{B}(y_{i}) \cap \mathcal{N}_{B}(y_{j}) = \emptyset\right\} \cap \mathcal{A}_{2} \mid \neg \text{Uniq-intersect}(y_{i^{*}}, y_{j^{*}})\right] \\
\geq \frac{s^{2}(l-1)^{2}}{r^{2}} \exp\left(-\frac{3s^{2}}{r-s}\right) \exp\left(-\frac{2s^{2}}{r-s}\right) \geq \frac{s^{2}(l-1)^{2}}{r^{2}} \left(1 - \frac{10s^{2}}{r-s}\right) \\
\geq \frac{s^{2}(l-1)^{2}}{2r^{2}},$$

where the final inequality holds since  $s^2 \leq r/40$ .

In order to lower bound the conditional probability in Equation 21, we need to further upper bound the marginal probability in the denominator. To this end, we observe that we have to upper bound  $\mathbb{P}\left[\mathcal{A}_1|\neg \text{Uniq-intersect}(y_{i^*},y_{j^*})\right]$ . Now conditioned on  $\neg \text{Uniq-intersect}(y_{i^*},y_{j^*})$ , for each  $y_i$  and  $y_j$ ,  $\mathcal{A}_1$  can be satisfied in two ways: choose at least one element from l elements in  $\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*})$  or choose at least two elements from m-l elements in  $\mathcal{N}_B(y_{i^*}) \cup \mathcal{N}_B(y_{j^*})$ . Making this precise, we obtain

<sup>&</sup>lt;sup>4</sup>the intersection is at least 1 by Lemma 3.1

$$\mathbb{P}\left[\mathcal{A}_{1} \mid \neg \text{Uniq-intersect}(y_{i^{*}}, y_{j^{*}})\right] \leq \left(\frac{ls}{r} + \frac{(m-l)^{2}\binom{r-2}{s-2}}{\binom{r}{s}}\right)^{2} \\
\leq \left(\frac{ls}{r} + \frac{s^{2}(m-l)^{2}}{(r-1)^{2}}\right)^{2} \\
\leq \left(\frac{ls}{r} + \frac{s^{2}(2s-2)^{2}}{(r-1)^{2}}\right)^{2} \\
\leq \frac{2l^{2}s^{2}}{r^{2}}, \text{ (since } 4s^{3} < r - 1)$$

The result follows by using the fact that  $l \geq 2$ .

The proof of Lemma 3.4 is similar, but involves controlling slightly different events.

### Proof of Lemma 3.4:

We will establish the lemma by lower bounding the probability of the complementary event. We recall the events  $A_1$  and  $A_2$  defined in Equation 20 in the proof of Lemma 3.3. We can mimick the initial arguments in the proof of Lemma 3.3 to conclude that

$$\mathbb{P}\left[(y_i, y_j) \in G_{\text{corr}(\rho)} \mid \Delta(y_i, y_{i^*}, y_{j^*}), \Delta(y_j, y_{i^*}, y_{j^*}), \text{Uniq-intersect}(y_{i^*}, y_{j^*})\right] \\ \geq \mathbb{P}\left[\text{Uniq-intersect}(y_i, y_j) \cap \mathcal{A}_2 \mid \mathcal{A}_1, \text{Uniq-intersect}(y_{i^*}, y_{j^*})\right],$$

and we provide a lower bound for this. Once again, we express the conditional probability as the ratio of a joint to a marginal and then lower bound the numerator and upper bound the denominator. In the numerator, we have the event

We have

$$\mathbb{P}\left[\text{Uniq-intersect}(y_i, y_j) \cap \mathcal{A}_2 \cap \mathcal{A}_1 \mid \text{Uniq-intersect}(y_{i^*}, y_{j^*})\right] \\ = \mathbb{P}\left[\text{Uniq-intersect}(y_i, y_i) \cap \mathcal{A}_2 \mid \text{Uniq-intersect}(y_{i^*}, y_{i^*})\right]$$

The event Uniq-intersect $(y_i, y_j) \cap A_2$  is guaranteed to occur if we choose  $y_i$  and  $y_j$  so that they have the only element in  $\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*})$  in common. This yields the lower bound

$$\mathbb{P}\left[\text{Uniq-intersect}(y_i, y_j) \cap \mathcal{A}_2 \cap \mathcal{A}_1 \mid \text{Uniq-intersect}(y_{i^*}, y_{j^*})\right] \\ \geq \frac{\binom{r-2s+1}{s-1}\binom{r-3s+2}{s-1}}{\binom{r}{s}^2}.$$

It is easy to further conclude that

$$\mathbb{P}\left[\text{Uniq-intersect}(y_{i}, y_{j}) \cap \mathcal{A}_{2} \cap \mathcal{A}_{1} \mid \text{Uniq-intersect}(y_{i^{*}}, y_{j^{*}})\right] \\
\geq \frac{s^{2}}{r^{2}} \left(1 - \frac{3s - 3}{r - s + 1}\right)^{(s - 1)} \left(1 - \frac{2s - 2}{r - s + 1}\right)^{s - 1} \\
\geq \frac{s^{2}}{r^{2}} \exp(-5(s - 1)^{2}/(r - s + 1)) \\
\geq \frac{s^{2}}{r^{2}} \left(1 - \frac{10s^{2}}{r - s}\right) \geq \frac{s^{2}}{r^{2}} \left(1 - \frac{20s^{2}}{r}\right),$$

where we again invoked Lemma A.6 as well as the fact that  $s \leq r/2$ . As for the marginal probability in the denominator, we need to upper bound

$$\mathbb{P}\left[\mathcal{A}_{1} \mid \text{Uniq-intersect}(y_{i^{*}}, y_{j^{*}})\right] \leq \left(\frac{s}{r} + \frac{(2s-1)^{2}\binom{r-2}{s-2}}{\binom{r}{s}}\right)^{2} \\
\leq \left(\frac{s}{r} + \frac{(2s-1)^{2}(s-1)^{2}}{(r-1)^{2}}\right)^{2} \leq \frac{s^{2}}{r^{2}}\left(1 + \frac{4s^{3}}{r}\right)^{2},$$

since for each  $y_i$  and  $y_j$ ,  $\mathcal{A}_1$  can be satisfied in two ways: choose the unique element from  $\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*})$  or choose at least two elements from 2s-1 elements in  $\mathcal{N}_B(y_{i^*}) \cup \mathcal{N}_B(y_{j^*})$ .

Using the above two inequalities, we have:

$$\begin{split} & \mathbb{P}\left[(y_i, y_j) \in G_{\operatorname{corr}(\rho)} \mid \Delta(y_i, y_{i^*}, y_{j^*}), \Delta(y_j, y_{i^*}, y_{j^*}), \operatorname{Uniq-intersect}(y_{i^*}, y_{j^*})\right] \\ & \geq \frac{1 - \frac{20s^2}{r}}{\left(1 + \frac{4s^3}{r}\right)^2}. \end{split}$$

It is easy to verify that  $1/(1+x)^2 \le 1-x$  for  $0 \le x \le (\sqrt{2}-1)/2$ . Since  $s^3 \le r/5$ , we obtain

$$\begin{split} & \mathbb{P}\left[(y_i, y_j) \in G_{\text{corr}(\rho)} \mid \Delta(y_i, y_{i^*}, y_{j^*}), \Delta(y_j, y_{i^*}, y_{j^*}), \text{Uniq-intersect}(y_{i^*}, y_{j^*})\right] \\ & \geq \left(1 - \frac{20s^2}{r}\right) \left(1 - \frac{4s^3}{r}\right) \\ & \geq 1 - \frac{24s^3}{r}. \end{split}$$

## A.3 Proof of Proposition 3.1

Let us start with the case when Uniq-intersect $(y_{i^*}, y_{j^*}) = 1$ . For any pair  $(y_i, y_j)$  where  $y_i$  and  $y_j$  are taken from  $\mathcal{N}_{G_{\operatorname{corr}(\rho)}}(y_{i^*}) \cap \mathcal{N}_{G_{\operatorname{corr}(\rho)}}(y_{j^*})$ , let  $E_{ij}$  be the random variable which is 1 if  $(y_i, y_j) \in G_{\operatorname{corr}(\rho)}$ . Then Lemma 3.4 guarantees  $\mathbb{P}(E_{ij} = 1) \geq 1 - 24s^3/r$ . Let  $S = \{(i, j) : (y_i, y_j) \in G_{\operatorname{corr}(\rho)}$ , and  $y_i, y_j \in \mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*})\}$ . The size of the set constructed in Algorithm 1 is equal

to  $\sum_{(i,j)\in S} E_{ij}$ . Recalling that |S|=m, Hoeffding's inequality guarantees that with probability at least  $1-2\exp(-2m\gamma^2)$ 

$$\left| \frac{1}{m} \sum_{(i,j) \in S} (E_{ij} - \mathbb{P}(E_{ij} = 1)) \right| \le \gamma.$$

Combining with the lower bound on  $\mathbb{P}(E_{ij}=1)$ , we obtain that with probability at least  $1-2\exp(-2m\gamma^2)$ ,

$$\sum_{(i,j)\in S} E_{ij} \ge m\left(1 - 24\frac{s^3}{r}\right) - m\gamma. \tag{22}$$

Using  $\gamma \leq 1/64$ , we see that this quantity is at least 62m/64 under the conditions of the lemma, which means that Algorithm 1 returns 1.

Now let us consider the case when Uniq-intersect $(y_{i^*}, y_{j^*}) = 0$ . Defining  $E_{ij}$  the same way as above, we see that by Lemma 3.3,  $\mathbb{P}(E_{ij} = 1) \leq 15/16$ . Then, a similar application of Hoeffding's inequality yields this time

$$\sum_{(i,j)\in S} E_{ij} \le \frac{m}{16} + m\gamma,\tag{23}$$

which is at most 61m/64 for  $\gamma \leq 1/64$ . Hence Algorithm 1 returns 0 in this case.

### A.4 Proof of Proposition 3.2

We now prove Proposition 3.2. We need a couple of auxilliary results for the proof. We first restate a theorem from [32], which we will heavily use in the sequel.

**Theorem A.1** (Restatement of Theorem 5.44 from [32]). Consider a  $d \times n$  matrix W where each column  $w_i$  of W is an independent random vector with covariance matrix  $\Sigma$ . Suppose further that  $\|w_i\|_2 \leq \sqrt{u}$  a.s. for all i. Then for any  $t \geq 0$ , the following inequality holds with probability at least  $1 - d \exp(-ct^2)$ :

$$\left\| \frac{1}{n} W W^T - \Sigma \right\|_2 \le \max \left( \|\Sigma\|_2^{1/2} \, \delta, \delta^2 \right) \text{ where } \delta = t \sqrt{\frac{u}{n}}.$$

Here c > 0 is an absolute numerical constant. In particular, this inequality yields:

$$||W||_2 \le ||\Sigma||_2^{\frac{1}{2}} \sqrt{n} + t\sqrt{u}.$$

In order to bound the errors made in Algorithm 1, we need some additional notation and auxilliary results. For now, let us consider a fixed pair of anchor samples  $y_{i^*}$  and  $y_{j^*}$  such that Uniq-intersect $(y_{i^*}, y_{j^*})$  is satisfied, and wlog, let  $\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*}) = \{a_1\}$ . We define the following sets of interest

$$\widehat{S} = \mathcal{N}_{\text{corr}}(y_{i^*}) \cap \mathcal{N}_{\text{corr}}(y_{j^*}),$$

$$S = \{ y_i \in \widehat{S} : \mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_{i^*}) = \mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_{j^*}) = \{a_1\} \}, \text{ and}$$
(24)

$$\widetilde{S} = \widehat{S} \setminus S. \tag{25}$$

For the purposes of understanding the errors in Algorithm 1, it would be helpful to decompose each vector  $y_i \in S$  as

$$y_i := y_i - x_{1i}a_1,$$

$$(26)$$

and accordingly define  $Y_S$  to be the  $d \times |S|$  matrix of all such vectors in S. Intuitively, if all the vectors y were 0, then Algorithm 1 can recover  $a_1$  via SVD in a relatively straightforward manner. We start by controlling the norm of the vectors  $y_i$  and  $y_i$ .

**Lemma A.1.** Under the model 1 and given assumptions 4, 6 and 7 we have for all i = 1, 2, ..., n

$$||y_i||_2 \le \sqrt{2s} \quad and \quad ||\check{y}_i||_2 \le 2\sqrt{s}.$$

#### **Proof:**

The proof is relatively straightforward consequence of our model and the assumptions. The model allows us to write

$$||y_i||_2^2 = \langle y_i, y_i \rangle = \sum_{a_p, a_q \in \mathcal{N}_B(y_i)} x_{pi} x_{qi} \langle a_p, a_q \rangle$$

$$\leq \sum_{a_p, a_q \in \mathcal{N}_B(y_i)} |x_{pi} x_{qi}| |\langle a_p, a_q \rangle|$$

$$= \sum_{a_p \in \mathcal{N}_B(y_i)} x_{pi}^2 ||a_p||_2^2 + \sum_{a_p \neq a_q \in \mathcal{N}_B(y_i)} |x_{pi} x_{qi}| |\langle a_p, a_q \rangle|$$

$$\leq s + s^2 \frac{\mu_0}{\sqrt{d}}$$

$$\leq s + \frac{1}{2} \leq \frac{3s}{2}.$$

Finally, by triangle inequality we further have that  $\|\breve{y}_i\|_2 \leq \|y_i\|_2 + 1$ .

Given this result, we would next like to control the amount of contribution the  $\check{y}_i$  directions can have in the SVD step of Algorithm 1. Our next result shows that while these vectors are not zero, their random support along with the incoherence of our dictionary elements ensures that these vectors are not strongly aligned with any one direction. We do so by bounding the spectral norm of the matrix  $\check{Y}_S$ .

**Lemma A.2.** With the vectors  $\check{y}_i$  defined in Equation 26, we have the following bound with probability greater than  $1 - d \exp\left(-c\alpha^2|S|\right)$  for any  $\alpha > 0$ 

$$\left\| \breve{Y}_S \right\|_2 \le \sqrt{s|S|} \left( \frac{\mu_1}{\sqrt{d}} + 2\alpha \right),$$

where c is a universal constant.

#### **Proof:**

In order to prove the lemma, we first calculate the spectral norm of the covariance matrix of  $\check{y}_i$  and then use Theorem A.1. Note that from Lemma A.1, we have  $\|\check{y}_i\|_2 \leq 2\sqrt{s}$ . We first bound the spectral norm of the covariance matrix of  $\check{y}_i \in S$  i.e., we bound  $\|\mathbb{E}\left(\check{y}_i\check{y}_i^T\right)\|_2$ . In order to do this, we first fix  $w \in \mathbb{R}^d$  and calculate:

$$w^{T}\mathbb{E}\left[\breve{y}_{i}\breve{y}_{i}^{T}\right]w = \mathbb{E}\left[\left(w^{T}\breve{y}_{i}\right)^{2}\right] = \mathbb{E}\left[\left(w^{T}A\breve{x}_{i}\right)^{2}\right] = \mathbb{E}\left[\left(z^{T}\breve{x}_{i}\right)^{2}\right],$$

where we use the notation  $z := A^T w$  and  $\check{x}_i$  is the same as  $x_i$  but with  $x_{i1}$  set to 0. We further simplify as

$$w^{T}\mathbb{E}\left[\check{y}_{i}\check{y}_{i}^{T}\right]w \leq \mathbb{E}\left[\left(\sum_{p=1}^{r}z_{p}\check{x}_{pi}\right)^{2}\right]$$

$$=\mathbb{E}\left[\sum_{p=1}^{r}z_{p}^{2}\check{x}_{pi}^{2}\right] + \mathbb{E}\left[\sum_{p\neq q=1}^{r}z_{p}z_{q}\check{x}_{pi}\check{x}_{qi}\right]$$

$$\leq \sum_{p=1}^{r}z_{p}^{2}\mathbb{E}\left[\check{x}_{pi}^{2}\right] + \sum_{p\neq q=1}^{r}|z_{p}z_{q}||\mathbb{E}\left[\check{x}_{pi}\check{x}_{qi}\right]|$$

$$\leq \sum_{p=1}^{r}z_{p}^{2}\frac{s}{r} + 0,$$

where the last inequality uses the fact that the values of  $\mathbb{E}[x_{pi}x_{qi}] = 0$ , since either one of them is zero or both are independently  $\pm 1$ .

Then we can further simplify the upper bound to obtain

$$w^T \mathbb{E}\left[\breve{y}_i \breve{y}_i^T\right] w \le \frac{s}{r} \|z\|_2^2 \le \frac{s}{r} \cdot \frac{\mu_1^2 r}{d} = \frac{\mu_1^2 s}{d},$$

where  $(\zeta)$  follows from Assumption (A3), since

$$\|z\|_2 = \|A^T w\|_2 \le \|A^T\|_2 \|w\|_2 = \|AA^T\|_2^{\frac{1}{2}} \|w\|_2 \le \sqrt{\frac{\mu_1^2 r}{d}}.$$

Recalling that w was an arbitrary unit vector, this immediately yields a spectral norm bound on the expected covariance

$$\left\| \mathbb{E} \left[ \breve{y}_i \breve{y}_i^T \right] \right\|_2 \leq \frac{\mu_1^2 s}{d}.$$

We are now in a position to apply Theorem A.1 with the matrix  $W = \check{Y}_S$  of size  $d \times |S|$ , where  $u = 2\sqrt{s}$  and  $t = \alpha\sqrt{|S|}$  for some  $\alpha > 0$ . Doing so yields the inequality

$$\left\| \breve{Y}_S \right\|_2 \le \sqrt{\frac{\mu_1^2 s}{k d}} \cdot \sqrt{|S|} + \alpha \sqrt{|S|} \cdot 2\sqrt{s}$$
$$\le \sqrt{s|S|} \left( \sqrt{\frac{\mu_1^2}{d}} + 2\alpha \right),$$

with probability greater than  $1 - d \exp(-c\alpha^2 |S|)$ .

Finally we are in a position to establish a bound on the accuracy of the SVD step in Algorithm 1. Having bounded the contribution from the directions apart from  $a_1$  in the previous lemma, we will now lower bound the contribution of the  $a_1$  direction, which will ensure that the largest singular vector is close to  $a_1$ .

**Proof of Proposition 3.2:** Recall the definitions of the sets S and  $\widetilde{S}$  (25). In order for a vector  $y_i$  to end up in  $\widetilde{S}$ , the event in Lemma 3.2 has to fail. Hence, if we define  $E_i$  to be the random variable which is 1 if  $y_i \in \widetilde{S}$ , then we have from Hoeffding's inequality

$$\left|\frac{1}{m}\sum_{i=1}^{m}(E_i - \mathbb{P}[E_i = 1])\right| \le \sqrt{\frac{2\log(2/\delta)}{m}},$$

with probability at least  $1 - \delta/2$ . From Lemma 3.2 we further know that  $\mathbb{P}[E_i = 1] \leq s^3/r$  so that

$$|\widetilde{S}| \le \frac{ms^3}{r} + \alpha m,\tag{27}$$

with probability at least  $1 - \exp(-2\alpha^2 m)$ . As a consequence, the size of S is at least

$$|S| \ge m(1 - s^3/r - \alpha) \ge 9m/10$$
 (28)

for  $\alpha < 1/20$  by our assumption that  $s^3 < r/384$ .

In order to understand the singular vector  $\hat{a}$ , we now write the matrix  $\widehat{M}$  as the sum of two matrices M and  $\widetilde{M}$  as follows:

$$\widehat{M} = M + \widetilde{M}, \text{ where,}$$

$$M := \sum_{y_i \in S} y_i y_i^T \text{ and } \widetilde{M} := \sum_{y_i \in \widetilde{S}} y_i y_i^T.$$

Recalling our earlier notation  $\check{y}_i$  (26), we expand M as follows:

$$M = \sum_{y_i \in S} y_i y_i^T = |S| a_1 a_1^T + \sum_{i: y_i \in S} x_{1i} \left( a_1 \breve{y}_i^T + \breve{y}_i a_1^T \right) + \sum_{i: y_i \in S} \breve{y}_i \breve{y}_i^T$$

We wish to show that  $a_1$  is close to the top singular vector of  $\widehat{M}$ . In order to show this, we bound the spectral norms of the following matrices:  $\sum_{i:y_i \in S} x_{1i} \left( a_1 \check{y}_i^T + \check{y}_i a_1^T \right), \sum_{i:y_i \in S} \check{y}_i \check{y}_i^T$  and  $\widetilde{M}$ .

Using Lemma A.2, we first obtain:

$$\left\| \sum_{i:y_i \in S} x_{1i} a_1 \check{y}_i^T \right\|_2 \le \|a_1\|_2 \|\check{Y}_S\|_2 \|x_{S1}\|_2$$

$$\le \sqrt{s|S|} \left( \sqrt{\frac{\mu_1^2}{d}} + 2\alpha \right) \cdot \sqrt{|S|}$$

$$= s|S| \left( \frac{\mu_1}{\sqrt{ds}} + \frac{2\alpha}{\sqrt{s}} \right) \text{ and,}$$

$$\left\| \sum_{i:y_i \in S} \check{y}_i \check{y}_i^T \right\|_2 \le 2s|S| \left( \frac{\mu_1^2}{d} + 4\alpha^2 \right).$$

$$(30)$$

Finally, we have the following bound on the spectral norm of  $\widetilde{M}$ :

$$\left\|\widetilde{M}\right\|_{2} = \left\|\sum_{y_{i} \in \widetilde{S}} y_{i} y_{i}^{T}\right\|_{2} \leq \left|\widetilde{S}\right| \left\|y_{i}\right\|_{2}^{2} \leq \left|\widetilde{S}\right| 2s.$$

$$(31)$$

Using (29), (30) and (31), we now prove the statement of the lemma. Let  $|\langle a_1, \hat{a} \rangle| = \theta$ . On one hand, we have:

$$\left\|\widehat{a}^{T}\widehat{M}\widehat{a}\right\|_{2} \leq \theta^{2}|S| + 2\left\|\sum_{i:y_{i} \in S} x_{1i}a_{1}\widecheck{y}_{i}^{T}\right\|_{2} + \left\|\sum_{i:y_{i} \in S}\widecheck{y}_{i}\widecheck{y}_{i}^{T}\right\|_{2} + \left\|\widetilde{M}\right\|_{2}$$

$$\leq \theta^{2}|S| + 2s|S|\left(\frac{\mu_{1}}{\sqrt{ds}} + \frac{2\alpha}{\sqrt{s}}\right) + 2s|S|\left(\frac{\mu_{1}^{2}}{d} + 4\alpha^{2}\right) + |\widetilde{S}|2s$$

$$\leq |S|\left[\theta^{2} + 8s\left(\frac{\mu_{1}}{\sqrt{ds}} + \frac{\mu_{1}^{2}}{d} + \alpha^{2} + \frac{\alpha}{\sqrt{s}} + \left(\frac{s^{3}}{r} + \alpha\right)\right)\right],$$

where the last step uses the bounds (27) and (28). On the other hand, we have

$$\begin{split} \left\|\widehat{a}^T \widehat{M} \widehat{a}\right\|_2 &= \left\|\widehat{M}\right\|_2 \ge |S| \cdot \|a_1\|_2^2 - 2 \left\| \sum_{i:y_i \in S} x_{1i} a_1 \widecheck{y}_i^T \right\|_2 - \left\| \sum_{i:y_i \in S} \widecheck{y}_i \widecheck{y}_i^T \right\|_2 - \left\| \widetilde{M} \right\|_2 \\ &\ge |S| - 2s|S| \left( \frac{\mu_1}{\sqrt{ds}} + \frac{\alpha}{\sqrt{s}} \right) - 2s|S| \left( \frac{\mu_1^2}{d} + 4\alpha^2 \right) - |\widetilde{S}| 2s \\ &\ge |S| \left[ 1 - 8s \left( \frac{\mu_1}{\sqrt{ds}} + \frac{\mu_1^2}{d} + \alpha^2 + \frac{\alpha}{\sqrt{s}} + \left( \frac{s^3}{r} + \alpha \right) \right) \right]. \end{split}$$

Using the above two inequalities, we obtain

$$\theta^2 \ge 1 - 16s \left( \frac{\mu_1}{\sqrt{ds}} + \frac{\mu_1^2}{d} + \frac{s^3}{r} \right) - 16s \left( \alpha^2 + \frac{\alpha}{\sqrt{s}} \right).$$

Now we observe that since  $||a_1||_2 = ||\hat{a}||_2 = 1$ , we have

$$\|\hat{a} - a_1\|_2^2 = 2(1 - \theta) \le 2(1 - \theta^2),$$

for  $0 \le \theta \le 1$ , which completes the proof.

## A.5 Approximate recovery guarantee for Algorithm 1

Building on all our work so far, this section presents the main guarantee for Algorithm 1. So far, we have established that the sub-procedure in Algorithm 1 correctly detects good anchor pairs with high probability. Conditioned on this, Proposition 3.2 shows that we can recover the dictionary element in this intersection to a bounded error with high probability. The next theorem, which puts everything together shows that in an appropriate number of iterations, Algorithm 1 will approximately receover all the dictionary elements with high probability.

**Lemma A.3** (Number of good anchor pairs). Suppose we have n examples. Then, we have:

$$\mathbb{P}\left\{\bigcup_{l\in[r]}|\{(i,j):\mathcal{N}_B(y_i)\cap\mathcal{N}_B(y_j)=\{a_l\}\}|>\frac{ns}{8r}\right\}\geq 1-r\exp\left(\frac{-ns}{64r}\right).$$

**Proof:** Fix  $l \in [r]$ . Define the set  $S \subseteq [n]$  as follows:

$$S := \{i : a_l \in \mathcal{N}_B(y_i)\}.$$

Since for every  $i \in [n]$ , the probability of  $i \in S$  is  $\frac{s}{r}$ , using standard Chernoff bounds, we see that:

$$\mathbb{P}\left[|S| < \frac{ns}{2r}\right] < \exp\left(\frac{-ns}{8r}\right). \tag{32}$$

Consider any two examples  $y_i, y_j \in S$ . Then,

$$\mathbb{P}\left[\mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_j) = \{a_l\}\right] \ge 1 - \frac{s^2}{r}.$$

Dividing the set S into  $\frac{|S|}{2}$  disjoint pairs and using Chernoff bounds, we see that

$$\mathbb{P}\left[\left|\left\{(i,j): \mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_j) = \left\{a_l\right\}\right\}\right| < \frac{|S|}{4}\right] \le \exp\left(\frac{-\left(1 - \frac{s^2}{r}\right)|S|}{16}\right) \le \exp\left(\frac{-|S|}{32}\right). \tag{33}$$

Using (32) and (33), we have:

$$\mathbb{P}\left[\left|\left\{(i,j): \mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_j) = \left\{a_l\right\}\right\}\right| > \frac{ns}{8r}\right] \ge 1 - \exp\left(\frac{-ns}{64r}\right).$$

Using a union bound over different dictionary elements, we have:

$$\mathbb{P}\left[|\{(i,j)|\mathcal{N}_B(y_i)\cap\mathcal{N}_B(y_j)=\{a_l\}\}|>\frac{ns}{8r}\ \forall\ l\in[r]\right]\geq 1-r\exp\left(\frac{-ns}{64r}\right).$$

**Lemma A.4.** In each iteration of Algorithm 1, the size of the set  $\widehat{S}$  satisfies:

$$|\widehat{S}| \ge \frac{ns}{4r},$$

with probability greater than  $1 - \exp\left(\frac{-ns}{16r}\right)$ .

**Proof:** Since  $(y_{i^*}, y_{j^*}) \in G_{\operatorname{corr}(\rho)}$ , from Lemma 3.1, we know that  $\mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*}) \neq \emptyset$ . Wlog let  $a_1 \in \mathcal{N}_B(y_{i^*}) \cap \mathcal{N}_B(y_{j^*})$ . Since each sample  $y_i$  has probability of at least

$$\frac{s}{r} \cdot \frac{\binom{r-2s+1}{s-1}}{\binom{r-1}{s-1}} \ge \frac{s}{r} \cdot \left(\frac{r-3s}{r-s}\right)^s \ge \frac{s}{r} \cdot \left(1 - \frac{2s}{r-s}\right)^s \ge \frac{s}{r} \cdot \left(1 - \frac{2s^2}{r-s}\right) \ge \frac{s}{2r},$$

of satisfying  $\mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_{i^*}) = \mathcal{N}_B(y_i) \cap \mathcal{N}_B(y_{j^*}) = \{a_1\}$ , using Chernoff bounds, we have:

$$\mathbb{P}\left[|i: \text{Uniq-intersect } (y_i, y_{i^*}) \& \text{Uniq-intersect } (y_i, y_{j^*})| < \frac{ns}{4r}\right] \le \exp\left(\frac{-ns}{16r}\right).$$

Using Lemma 3.1 now finishes the proof.

### A.6 Auxiliary Results

Below, we establish that the incoherence assumption on the dictionary elements leads to a bound on the RIP constant.

**Lemma A.5.** The 2s-RIP constant of A,  $\delta_{2s}$  satisfies  $\delta_{2s} < \frac{2\mu_0 s}{\sqrt{d}}$ .

**Proof:** Consider a 2s-sparse unit vector  $w \in \mathbb{R}^r$  with Supp(w) = S. We have:

$$||Aw||^{2} = \left(\sum_{j \in S} w_{j} a_{j}\right)^{2} = \sum_{j} w_{j}^{2} ||a_{j}||^{2} + \sum_{j,l \in S, j \neq l} w_{j} w_{l} \langle a_{j}, a_{l} \rangle$$

$$\geq 1 - \sum_{j,l \in S, j \neq l} |w_{j} w_{l}| ||\langle a_{j}, a_{l} \rangle|$$

$$\geq 1 - \sum_{j,l \in S, j \neq l} |w_{j} w_{l}| \frac{\mu_{0}}{\sqrt{d}}$$

$$\geq 1 - \frac{\mu_{0}}{\sqrt{d}} ||w||_{1}^{2}$$

$$\geq 1 - \frac{\mu_{0}}{\sqrt{d}} 2s \cdot ||w||^{2} = 1 - \frac{2\mu_{0}s}{\sqrt{d}}.$$

Similarly, we have:

$$||Aw||^2 \le 1 + \frac{2\mu_0 s}{\sqrt{d}}.$$

This proves the lemma.

**Lemma A.6.** For r > 2, c > 0, let  $0 \le x \le r/(2c+1)$ . Then  $(1-cx/(r-x))^x \ge \exp(-cx^2/(r-x)) \ge 1 - \frac{2x^2}{r-x}$ .

#### **Proof:**

We start by observing that x/(r-x) is an increasing function of x for x < r, so that x < r/(2c+1) implies that cx/(r-x) < 1/2. Additionally, we have the following fact for any  $\theta > 0$ 

$$1 - \theta \le e^{-\theta} \le 1 - \theta + \frac{\theta^2}{2}.$$
 (34)

The first inequality is a consequence of the convexity of  $e^{-\theta}$  while the second one follows since the second derivative of  $e^{-\theta}$  is at most 1 when  $\theta > 0$ . Since we have  $x/(r-x) \le 1/2$ , it is easy to see that

$$1 - \frac{cx}{r - x} \ge 1 - 2\frac{cx}{r - x} + 2\frac{c^2x^2}{(r - x)^2}.$$

Now applying the inequalities (34) with  $\theta = 2cx/(r-x)$ , we obtain

$$\left(1 - \frac{cx}{r - x}\right)^{x} \ge \left(1 - 2\frac{cx}{r - x} + 2\frac{cx^{2}}{(r - x)^{2}}\right)^{x}$$

$$\ge (\exp(-2cx/(r - x)))^{x} = \exp(-2cx^{2}/(r - x))$$

$$\ge 1 - \frac{2cx^{2}}{r - x},$$

where the second inequality follows from again using (34), this time with  $\theta = 2cx^2/(r-x)$ .

Lemma A.7. We have:

$$\mathbb{E}\left[XX^T\right] = \frac{s}{r}I_{r\times r},$$

where  $I_{r \times r}$  is the  $r \times r$  identity matrix.

**Proof:** Let  $\Sigma := \mathbb{E}[XX^T]$ . We will first calculate the diagonal elements of  $\Sigma$ :

$$\Sigma_{jj} = \mathbb{E}\left[x_{ji}^2\right] = \frac{s}{r}.$$

On the other hand, any off diagonal element can be calculated as follows:

$$\Sigma_{jk} = \mathbb{E}\left[x_{ji}x_{ki}\right] = \mathbb{E}\left[x_{ji}\right]\mathbb{E}\left[x_{ki}\right] = 0.$$

This proves the lemma.

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