# CS294-204 Phase Transitions (Fall 2021)
# Homework #6
# Due Fr. 11/19

## 1 Variation Distance [2 Points]

Consider a finite set $S$, and for $f : S \to \mathbb{R}$, define $\delta f = \max_{\sigma,\sigma' \in S} |f(\sigma) - f(\sigma')|$. Prove that if $\mu, \tilde{\mu}$ are probability measures on $S$, then

$$d_{TV}(\mu, \tilde{\mu}) = \max_{f \,:\, \delta f \neq 0} \frac{|E_\mu(f) - E_{\tilde{\mu}}(f)|}{\delta f}.$$

*Hint: If you restrict the maximum to the maximum over functions of the form $f_A(x) = 1_{x \in A}$, you get one bound. The other bound is actually a lemma in the lectures (see Scribes). It is formulated slightly differently, in matrix multiplication form, rather than with expectations, so all you need to do is cite it and give the translation when applying it.*

## 2 Dobrushin Uniqueness for the Ising Model [6 Points]

**(a)** Consider measures of the form $\mu_h(\sigma) = \frac{e^{\sigma h}}{e^h + e^{-h}}$ on the space $S_0 = \{-1, +1\}$. Show that

$$d_{TV}(\mu_h, \mu_{\tilde{h}}) = \frac{1}{2} |\tanh h - \tanh \tilde{h}|$$

**(b)** Prove that the right hand side is bounded by $\tanh\left(\frac{|h-\tilde{h}|}{2}\right)$. To this end, you may assume w.l.o.g. that $h \geq \tilde{h}$, which means you can write $h = a + b$ and $\tilde{h} = a - b$ for some $b > 0$. So all you need to show is that the right hand side is maximized when $a = 0$, which reduces the problem to a calculus problem.

**(c)** Let $G$ be a graph whose degrees are bounded by $D$. Prove that the condition of the Dobrushin Uniqueness Theorem 19.1 from lecture 19 is satisfied when $D \tanh \beta < 1$.

## 3 Dobrushin Uniqueness for general spin models [8 Points]

Let $G$ be a graph of maximal degree $D$. Consider a spin model with spin space $S_0 = [q]$, a priory measure $\mu_0$, and interaction $g : S \times S \to [0, 1]$, and let

$$\epsilon = \max_{s \in [q]} \sum_{s' \in [q]} \mu_0(s') |g(s, s') - 1| = \max_{s \in [q]} \sum_{s' \in [q]} \mu_0(s')(1 - g(s, s'))$$

The goal of this exercise is to prove Dobrushing Uniqueness for $2D\epsilon < 1$. To this end, we will show that for any $i \in V(G)$ with neighborhood $N = N(i)$, and any $k \in N$,

$$C_{ik} = d_{TV}(K_{i,\sigma_N}, K_{i,\tilde{\sigma}_N}) \leq 2\epsilon \tag{1}$$

where $\sigma_N, \tilde{\sigma}_N \in [q]^N$ differ only at $k$, and $K_{i,\sigma_N}$ is the single spin measure

$$K_{i,\sigma_N}(\sigma_j) = \frac{1}{Z(\sigma_N)} \mu_0(\sigma_i) \prod_{j \in N} g(\sigma_j, \sigma_i) \quad \text{where} \quad Z(\sigma_N) = \sum_{\sigma_i \in S} \mu_0(\sigma_i) \prod_{j \in N} g(\sigma_j, \sigma_i).$$

To prove (1), we first observe that the measures $K = K_{i,\sigma_N}$ and $\tilde{K} = K_{i,\tilde{\sigma}_N}$ have a common factor

$$F_0(\cdot) = \mu_0(\cdot) \prod_{j \in N : j \neq k} g(\sigma_j, \cdot).$$

and can be written as

$$K(s) = \frac{F(s)}{Z} \quad \text{and} \quad \tilde{K}(s) = \frac{\tilde{F}(s)}{\tilde{Z}}$$

where

$$F(s) = g(\sigma_k, s)F_0(s), \quad \tilde{F} = g(\tilde{\sigma}_k, s)F_0(s), \quad Z = \sum_s F(s) \quad \text{and} \quad \tilde{Z} = \sum_s \tilde{F}(s).$$

**(a)** Let $Z_0 = \sum_s F_0(s)$, and let $\delta = \max\{|Z - Z_0|, |\tilde{Z} - Z_0|\}$. In a first step, we will prove that

$$d_{TV}(K, \tilde{K}) = \frac{1}{2} \sum_{s \in [q]} |K(s) - \tilde{K}(s)| \leq \frac{\delta}{Z_0 - \delta} \quad \text{if} \quad Z_0 \geq \delta$$

- **(i)** Assume without loss of generality that $Z \leq \tilde{Z}$, and use the triangle inequality twice to prove that

$$\sum_{s \in [q]} |K(s) - \tilde{K}(s)| \leq \left| \frac{1}{Z} - \frac{1}{\tilde{Z}} \right| \sum_s F(s) + \sum_{s \in [q]} \frac{|F(s) - \tilde{F}(s)|}{\tilde{Z}}$$
$$\leq 1 - \frac{Z}{\tilde{Z}} + \sum_{s \in [q]} \frac{|F_0(s) - F(s)| + |F_0(s) - \tilde{F}(s)|}{\tilde{Z}}$$

- **(ii)** Use that $0 \leq g \leq 1$ and the definition of $F$ and $\tilde{F}$ to remove the absolute values on the right hand side and simplify the resulting expression to

$$2\frac{Z_0 - Z}{\tilde{Z}}$$

- **(iii)** Argue that $Z, \tilde{Z} \leq Z_0$ to conclude that $Z_0 - Z \leq \delta$ and $\tilde{Z} \geq Z_0 - \delta$ and complete the proof of part (a).

**(b)** Prove that

$$Z_0 \geq 1 - (D - 1)\epsilon \quad \text{and} \quad \max\{Z_0 - Z, Z_0 - \tilde{Z}\} \leq \epsilon$$

- **(i)** Establish that

$$Z_0 = \sum_s \mu_0(s) \prod_{\substack{j \in N \\ j \neq k}} g(\sigma_j, s) \geq \sum_s \mu_0(s)\Big(1 - \sum_{\substack{j \in N \\ j \neq k}} (1 - g(\sigma_j, s)\Big) \geq 1 - (D - 1)\epsilon.$$

- **(ii)** Establish that

$$Z = Z_0 - \sum_s F_0(s)(1 - g(\sigma_k, s)) \geq Z_0 - \sum_s \mu_0(s)(1 - g(\sigma_k, s)) \geq Z_0 - \epsilon$$

  What bound do you get if you bound $\tilde{Z}$ from below?

**(c)** Use **(a)** and **(b)** to prove that $C_{ik} \leq 2\epsilon$ if $2D\epsilon \leq 1$.

# 4 Isoperimetric Inequality for Low Temperature Potts Contours [4 Points + 4 Bonus Points]

Let $G = (V, E)$, where $V = \{1, \dots, L\}^d$ and $E = \{xy\colon x, y \in V,\ \|x - y\|_1 = 1\}$, with $\|x - y\|_1 = \sum_{i=1}^{d} |x_i - y_i|$. Consider a coloring of the vertices of $G$ by $[q]$, and let $\pi = \{W_1, \dots, W_q\}$ be the resulting partition of $V$ into groups of vertices of equal color. Define the edge-boundary $\partial_e \pi$ as the set of edges $xy$ such that $x \in W_i$ and $y \in W_j$ for $i \neq j$. We will prove the following

**Claim 4.1** *There exists a constant $c \in (0, 1]$ such that the following holds: If $|\partial_e \pi| < cL^{1-d}$, then one of the $W_i$'s has a component of size larger than $\frac{3}{4}L^d$.*

We will prove the claim for a constant $c$ that depends on $q$ and $d$, in the main part, and improve it to $c = 1$ in the bonus part.

We are interested in the case where the set of facets $\gamma$ intersecting the edges in $\partial_e \pi$ is connected, in which case $\gamma$ is a contour, and the claim proves that for low-temperature Potts contours with free boundary conditions, contours smaller than $cL^{d-1}$ have well defined exteriors, which means we can analyze the large deviations for these contours as in the case of the Ising model. This in turn will eventually again lead to a lower bound on the mixing time of the order $e^{const\, L^{d-1}}$. But the claim holds in general, even if the set of facets $\gamma$ intersecting the edges in $\partial_e \pi$ is not connected.

Throughout, we will assume that $|\partial_e \pi| < L^{d-1}$ (we make the stronger assumption $|\partial_e \pi| < cL^{d-1}$ only at the end).

**(a)** Show that $|\partial_e W_i| < L^{d-1}$ for all $i$, and use the results from the last homework to show that if there exists a set $W_i$ such that $|W_i| \geq L^d/2$, then $W_i$ must have a component $W_{ij}$ of size larger than $\frac{3}{4}L^d$, giving us a well defined exterior of size larger than $\frac{3}{4}L^d$.

**(b)** Applying the isoperimetric inequality to $W_i$ with $|W_i| < L^d/2$, use the properties of the function $f$ from the isoperimetric inequality to conclude that in fact, $|W_i| < L^d/4$.

**(c)** Analyze the function $f(x)$ for $0 \leq x < 1/4$, to show that there exists a constant $c$ (depending on $q$ and possibly the dimension $d$) such that for $|\partial_e W_i| < cL^{d-1}$, $|W_i| < \frac{1}{q}L^d$.

**(d)** Use that fact that the inequality $|W_i| < \frac{1}{q}L^d$ can't hold for all $i$ (why?), to conclude that if $|\partial_e \pi| < cL^{d-1}$, one of the color groups must have a component of size larger than $\frac{3}{4}L^d$.

**(e)** **[Bonus]** Prove the stronger statement that if $|\partial_e \pi| < L^{d-1}$, then one of the $W_i$ must have a component of size at least $\frac{3}{4}L^d$. You will proceed by contradiction, so you may assume that $|W_i| < \frac{1}{4}L^d$ for all $i$ by parts **(a)** and **(b)**.

- **(i)** Use that
$$\sum_i |\partial_e W_i| = 2|\partial_e \pi|$$
to prove that it must be possible to dived $[q]$ into three sets $I_1, I_2, I_2$ such that for $k = 1, 2, 3$, $\sum_{i \in I_k} |\partial_e W_i| < L^{d-1}$.
- **(ii)** Use the strategy from the last homework set to prove that this implies that $\sum_{i \in I_k} |W_i| < \frac{1}{4}L^d$.
- **(iii)** This is a contradiction. Why?