CS294-204 Phase Transitions (Fall 2021)
Homework #5
Due Fr. 11/5

1 Isoperimetric Inequality [8 Points]

Let $G = (V, E)$, where $V = \{1, \ldots, L\}^d$ and $E = \{xy\colon x, y \in V,\ \|x - y\|_1 = 1\}$, with $\|x - y\|_1 = \sum_{i=1}^d |x_i - y_i|$. Given $W \subset V$, we will consider the edge boundary $\partial_e W = \{xy \in E\colon x \in W,\ y \in W^c\}$ where $W^c = V \setminus W$. We will study the consequences of the isoperimetric inequality of Bollobás and Leader[1], which says that for any $W \subset V$

$$|\partial_e W| \geq L^{d-1} f(|W| L^{-d}) \quad \text{where} \quad f(x) = \min_{\ell=1,\ldots,d} \ell \left(\min\{x, 1-x\}\right)^{1-1/\ell},\ x \in [0,1]. \tag{1}$$

We will prove the following claim

Claim 4.1 *If $|W| \geq L^d/2$ and $|\partial_e W| < L^{d-1}$, then there exists a component W_i of W such that $|W_i| > \frac{3}{4} L^d$.*

Proof:

(a) Give an example of a set in $d = 2$ such that the naive bound $|\partial_e W| \geq d|W|^{1-1/d}$ does not hold. [1 Point]

(b) Show that $f(x) \leq 1$ with equality if and only if $x \in [1/4, 3/4]$. [1 Point]

(c) Show that the function f and the function $\tilde{f} : [0,1] \to [0,1]$, defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \leq 3/4 \\ 1 & \text{if } x > 3/4. \end{cases}$$

are concave. *(Hint: Express f as $f = \min_\ell f_\ell$ and first show that f_ℓ is concave).* [2 Points]

(d) Assume that $0 \leq x_i \leq 3/4$ for $i = 1, \ldots, k$. Prove that $\sum_i f(x_i) \geq \tilde{f}(\sum_i x_i)$ whenever $\sum_i x_i \leq 1$. [2 Points]

(e) Prove the claim. (*Hint: Let W_i, $i = 1, \ldots, k$, be the components of W, and assume by contradiction that $x_i = L^{-d}|W_i| \leq 3/4$. Apply the isoperimetric inequality to W_i and use that $\sum_i |\partial_e W_i| = |\partial_e W|$ and $|W| = \sum_i |W_i|$ to arrive at a contradiction.)* [2 Points]

2 Ising Contours with free boundary conditions [8 Points]

It is not hard to see that on the graph considered above, the number of simple Ising contours of size n containing a given facet is again bounded by 3^{n-2}, and that the number of simple Ising contours of size n that are connected to a (not necessarily simple) contour γ_0 but don't share a facet with γ_0 is bounded by $(d-1)|\gamma_0| 3^{n-3}$. However, simple contours now can have even or odd sizes, and they can be as small as d (rather than $2d$ for the contours we considered earlier.

(a) Let F be a facet. Prove that

$$\sum_{\gamma \ni F} (3e^{2/5})^{-|\gamma|} \leq \frac{1}{9} \frac{e^{-2d/5}}{1 - e^{-2/5}} \leq \frac{1}{3d},$$

where the sum goes over simple contours containing the facet F (Hint: use that $e^{-x} \leq 1/(xe)$.) [1 Point]

[1] Bollobás and I. Leader: Edge-isoperimetric inequalities in the grid. *Combinatorica* **11** (1991) 299–314.

(b) Fix a not necessarily simple contour γ_0 and show that

$$\sum_{\gamma:\gamma\cap\gamma_0}{}' (3e^{2/5})^{-|\gamma|} \leq \frac{d-1}{27}\frac{e^{-2d/5}}{1-e^{-2/5}}|\gamma_0| \leq \frac{1}{14}|\gamma_0|,$$

where the sum goes over simple contours connected to γ_0 which don't share a facet with γ_0 (Hint: again use that $e^{-x} \leq 1/(xe)$, this time to kill the factor $d-1$.) [1 Point]

(c) Let S be a set of facets, let γ_0 be a simple contour, and let $\epsilon = \frac{1}{14}$. Prove by induction on the size of S that

$$H_S(\gamma_0) = \sum_{\gamma:\gamma_0\subset\gamma\subset S}{}'' \left(\frac{1}{5}\right)^{|\gamma|} \leq \left(\frac{e^{\epsilon}}{5}\right)^{|\gamma_0|},$$

where the sum goes over contours containing the simple contour γ_0. You may want to consult the lecture notes (there is no scribe for that one) for the corresponding bounds in $\mathbb{Z}^d$, but in principle, the below should give enough hints so that you don't need to. [4 Points]

(i) Decompose $\gamma \setminus \gamma_0$ into its connected components $\gamma_1, \ldots, \gamma_k$ (which are now contours in $S \setminus \gamma_0$), and replace the sum over sets of contours $\{\gamma_1, \ldots, \gamma_k\}$ by $\frac{1}{k!}$ times a sum over sequences. Show that this gives

$$\sum_{\gamma:\gamma_0\subset\gamma\subset S}{}'' \left(\frac{1}{5}\right)^{|\gamma|} \leq \left(\frac{1}{5}\right)^{|\gamma_0|} \exp\left(\sum_{\substack{\gamma\subset S\setminus\gamma_0 \\ \gamma\cap\gamma_0}} 5^{-|\gamma|}\right),$$

where the sum in the exponent goes over contours γ that are adjacent to γ_0 (where does this constraint come from?).

(ii) To bound the sum in the exponent, observer that each contour γ that is adjacent to γ_0 must contain a primitive contour that is adjacent to γ_0. Use this to show that the exponent can be bounded by

$$\sum_{\gamma_0':\gamma_0'\cap\gamma_0}{}' H_{S\setminus\gamma_0}(\gamma_0'),$$

where the sum goes over simple contours γ_0' that are adjacent to γ_0 but don't share a facet with it. How do you get to the expression $S \setminus \gamma_0$? Why does γ_0' not share a facet with γ_0?

(iii) Complete the induction (Hint: use first the inductive assumption, and then your results from **(b)**.)

(d) Combined the results from **(c)** with **(a)** and **(b)** to prove that

$$\sum_{\gamma\ni F} 5^{-|\gamma|} \leq \frac{1}{3d}$$

and

$$\sum_{\gamma:\gamma\cap\gamma_0} 5^{-|\gamma|} \leq \frac{1}{4}|\gamma_0|$$

where the sums now go over all possible contours, not just simple ones, and the second sum goes over contours adjacent to γ_0. (Hint: for the second sum, you will need to deal with both contours sharing a facet with γ_0, and with contours adjacent to γ_0 that don't share a facet with it). [2 Points]

3 Detailed Balance [2 Points]

Prove that the Gibbs measure of the Ising model and the Gibbs sampler (aka Glauber Dynamics) of the Ising model are in detailed balance.