# CS294-204 Phase Transitions (Fall 2021)
# Homework #4
# Due Fr. 10/22

## 1 Counting Connected Subgraphs [5 Points]

Let $G = (V, E)$ be a finite or countably infinite graph of maximal degree $D$. Consider the "generating function" $H_x(\lambda)$ defined by

$$H_x(\lambda) = \sum_{G' \subset G}{}' \lambda^{|E(G')|} \tag{1}$$

where the sum goes over finite, connected subgraphs $G' \subset G$ such that the vertex set $V(G')$ of $G'$ contains the vertex $x$ and the edge set $E(G')$ contains at least one edge. Our goal is a proof of the estimate

$$H_x(\lambda) \le a \qquad \text{if} \qquad 0 \le D\lambda \le \frac{\log(1+a)}{1+a}. \tag{2}$$

To this end, it will be convenient to consider the modified function

$$\tilde{H}_{n,x}(\lambda) = \sum_{G' \subset G}{}'' \lambda^{|E(G')|}$$

where the sum now goes over finite, connected subgraphs $G' \subset G$ such that $V(G')$ contains the vertex $x$ and $E(G')$ contains between 0 and $n$ edges (observe that here we included the case $E(G') = \emptyset$; this will be convenient for the following, inductive proof).

(a) Show that

$$\tilde{H}_{n,x}(\lambda) \le \sum_{\mathscr{N} \subset \mathscr{N}(x)} \prod_{y \in \mathscr{N}} \left(\lambda \tilde{H}_{n-1,y}(\lambda)\right) = \prod_{y \in \mathscr{N}(x)} \left(1 + \lambda \tilde{H}_{n-1,y}(\lambda)\right)$$

where $\mathscr{N}(x) = \{y \in V \mid xy \in E\}$ is the neighborhood of $x$ in $G$.

(b) Conclude by induction that $\tilde{H}_{n,x}(\lambda) \le 1 + a$ provided $0 \le \lambda D \le \frac{\log(1+a)}{1+a}$. At this point, you have actually proved (2). Why?

(c) Use (2) to show that the number of finite, connected subgraphs $G' \subset G$ such that $V(G')$contains a given vertex $x$ and $E(G')$ contains $n$ edges is bounded by $(2D/log2)^n \le (3D)^n$.

## 2 Cluster Expansion for General Spin Systems [8 Points]

In Lecture 14, we considered a spin system on a finite graph $G = (V, E)$ with partition function

$$Z_G = \sum_{\sigma_V \in [q]^V} \prod_{v \in V} \mu_0(\sigma_v) \prod_{xy \in E} g(\sigma_x, \sigma_y),$$

where $\mu_0$ is a probability measure on $[q]$ and $g(\sigma, \sigma') \ge 0$ (in the lecture, and later in this exercise, we will assume $g \le 1$, but for now we don't). We derived a polymer system, where polymers are subsets $\gamma \subset V$ of size at least two such that the induced subgraph $G[\gamma] = (\gamma, E[\gamma])$ is connected. Two polymers $\gamma$, $\gamma'$ are adjacent if $\gamma \cap \gamma' \ne 0$ and the activity of a polymer $\gamma$ is

$$z(\gamma) = \sum_{\substack{E' \subset E[\gamma] \\ (\gamma, E') \text{ is connected}}} \mathbb{E}_0[\prod_{xy \in E'} \rho_{xy}] \tag{3}$$

where $\mathbb{E}_0$ denotes expectations with respect to $\prod_{v \in V} \mu_0(\sigma_v)$ and $\rho_{xy}$ is the function $\rho_{xy} = \rho_{xy}(\sigma_x, \sigma_y) = g(\sigma_x, \sigma_y) - 1$.

(a) Let $\epsilon_\infty = \|\rho_{xy}\|_\infty = \max_{\sigma,\sigma'} |1 - g(\sigma,\sigma')|$. Observe that $|\gamma| \le |E'| + 1$ whenever $E'$ contributes to $z(\gamma)$ in (3). Combine this observation with the bound (2) to prove that the Dobrushin convergence condition holds if

$$\epsilon_\infty \le \frac{1}{DK(a)} \quad \text{where} \quad K(a) = \frac{1 + ae^{-a}}{\log(1 + ae^{-a})} e^a. \tag{4}$$

(*Recall that it is enough to verify that* $\sum_{\gamma: x \in \gamma} |z(\gamma)| e^{a|\gamma|} \le a$). Conclude that, in particular, the cluster expansion converges if $\epsilon_\infty \le 1/8D$. [2 Points]

(b) Next consider the case where $0 \le g \le 1$, in which case we will consider the parameter $\epsilon = \|\rho_{xy}\|_x = \max_{\sigma'} \sum_\sigma \mu_0(\sigma) |g(\sigma,\sigma') - 1|$ where $\|\cdot\|_x$ is the mixed $\ell_\infty - \ell_1$ norm

$$\|f\|_x = \max_\sigma \mathbb{E}_0[|f| \mid \sigma_x = \sigma] = \max_{\sigma_x} \sum_{\{\sigma_v\}_{v \ne x}} |f(\sigma)| \prod_{v \ne x} \mu_0(\sigma_v). \tag{5}$$

We will prove that if

$$\epsilon \le \frac{1}{DK(a)} \quad \text{where} \quad K(a) = \frac{a + e^a}{\log(1 + ae^{-a})}, \tag{6}$$

then the polymer system obeys the Dobrushin convergence condition. [6 Points]

The following notation will be useful: for any non-empty set $Y \subset V$, we will set

$$\rho(Y) = \sum_{\substack{E' \subset E[\gamma] \\ (Y,E') \text{ is connected}}} \prod_{xy \in E'} \rho_{xy},$$

with the convention that $\rho(Y) = 1$ if $|Y| = 1$. Note that with this notation, $z(\gamma) = \mathbb{E}_0[\rho(\gamma)]$ for all polymers $\gamma$.

(i) Given $x \in \gamma$ and a term $E'$ contribution to expression for $\rho(\gamma)$, consider the graph $G' = (\gamma, E')$, and the induced subgraph of $G'$ on $\gamma \setminus \{x\}$. Let $G_1 = (Y_1, E_1), \ldots, G_k = (Y_k, E_k)$ be the connected components of this induced subgraph. If we fix the sets $Y_1, \ldots, Y_k$ and sum over all $E'$ leading to a given collection $\{Y_1, \ldots, Y_k\}$, we will need to sum over $E_1, \ldots, E_k$, as well as over sets of edges between $x$ and the various $Y_i$'s (note that $E'$ can't contain edges between the different $Y_i$'s since $G_1, \ldots, G_k$ were different connected components of the induced subgraph on $\gamma \setminus \{x\}$. ) Show that summing over all sets $E'$ leading to the same set of $Y_i$'s gives the following identity

$$\rho(\gamma) = \sum_{\pi \text{ of } \gamma \setminus \{x\}} \prod_{Y \in \pi} \left( \rho(Y) \sum_{\substack{\mathscr{N} \subset \mathscr{N}(x) \cap Y \\ \mathscr{N} \ne \emptyset}} \prod_{z \in \mathscr{N}} \rho_{xz} \right), \tag{7}$$

where the sum goes over partitions of $\gamma \setminus \{x\}$. Explain why you get a sum over partitions, why you get the factor $\rho(Y)$, and why you get the sum over $\mathscr{N}$.

(ii) Given (7), prove that

$$|\rho(\gamma)| \le \sum_{\pi \text{ of } \gamma \setminus \{x\}} \prod_{Y \in \pi} |\rho(Y)| \sum_{z \in \mathscr{N}(x) \cap Y} |\rho_{xz}| \tag{8}$$

*Hint: As an intermediate step, argue that the sum over* $\mathscr{N}$ *in* (7) *can be rewritten as* $\prod_{z \in \mathscr{N}(x) \cap Y} (\rho_{xz} + 1) - 1$.

(iii) Use the bound (8) to prove that

$$\|\rho(\gamma)\|_x \le \sum_{\pi \text{ of } \gamma \setminus \{x\}} \prod_{Y \in \pi} \left( \sum_{z \in \mathscr{N}(x) \cap Y} \epsilon \|\rho(Y)\|_z \right), \tag{9}$$

with the norm $\|\cdot\|_x$ defined in (5).

(iv) Consider the generating function $H_\Lambda(x) = \sum_{\gamma: x\in\gamma\subset\Lambda} \|\rho(\gamma)\|_x e^{a(|\gamma|-1)}$, where again it will be convenient to include the term $\|\rho(\{x\})\|_x = 1$, and prove by induction that $H_\Lambda(x) \le 1 + ae^{-a}$ for all finite $\Lambda \subset V$. This will be enough to prove convergence. Why?

*Hint: the proof is very similar to the one used to prove convergence of the cluster expansion for the chromatic polymomial, as is the reason why this is enough to prove convergence.*

(c) **[Bonus]** Let $b > 0$. Show that a strengthening of the conditions (4) and (6) to the conditions $\epsilon_\infty \le \frac{1}{DK(a)} e^{-b}$ and $\epsilon \le \frac{1}{DK(a)} e^{-b}$, respectively, leads to the improvement

$$\sum_{\gamma: x\in\gamma} |\tilde{z}(\gamma)| e^{a|\gamma|} \le a \qquad \text{where} \qquad \tilde{z}(\gamma) = z(\gamma) e^{b(|\gamma|-1)}$$

a condition which was used in Lectures 14 and 15.

# 3 Analyticity of the free energy for the high temperature Ising Model [5 Points]

Let $G = (\mathbb{Z}^d, \mathbb{E}_d)$. Given a finite subset $\Lambda \subset \mathbb{Z}^d$, let $G[\Lambda] = (\Lambda, E[\Lambda])$ be the induced subgraph, and let

$$Z_\Lambda^h = \sum_{\sigma_\Lambda \in \{-1,+1\}^\Lambda} \prod_{v\in\Lambda} \mu_0^h(\sigma_v) \prod_{xy\in E[\Lambda]} e^{\beta\sigma_x\sigma_y}$$

be the partition function in a magnetic field, where

$$\mu_0^h(\sigma) = \frac{e^{h\sigma}}{e^h + e^{-h}}.$$

If

$$\epsilon_\infty = e^\beta - 1 \le \frac{1}{2d\tilde{K}(a)},$$

we know by (4) that the cluster expansion for $\log Z_\Lambda^h$ is absolutely convergent - in fact, we have shown in Lecture 14 that the limit $f(h) = -\lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_\Lambda^h$ exists if $|\partial\Lambda|/|\Lambda| \to 0$, and can be expressed as

$$f(h) = -\sum_{\Gamma: x\in V(\Gamma)} \frac{W(\Gamma)}{|V(\Gamma)|}$$

where $V(\Gamma) = \bigcup_{\gamma\in\Gamma} \gamma$ and $W(\Gamma)$ is the weight $W(\Gamma) = \frac{z^\Gamma}{\Gamma!} \phi_c(G(\Gamma))$, with $z$ given by (3). We furthermore have that

$$\sum_{\Gamma: x\in V(\Gamma)} |W(\Gamma)| \le a,$$

showing absolute and uniform convergence of the cluster expansion for $f(h)$ for $h \in \mathbb{R}$.

(a) Write down the explicit expression for $z(\gamma)$ as a finite sum. Show that each term is real analytic in $h$. Does this imply that each term $W(\Gamma)$ contributing to $f(h)$ is real analytic? Does this imply that $f(h)$ is real analytic? (not necessarily - which of the two above conclusions is wrong?)

(b) Add a small imaginary part to $h$. Show that this does not destroy convergence, if the imaginary part is small enough, and $\beta$ is chosen appropriately. To this end, write $z(\gamma)$ as

$$z(\gamma) = \sum_{E'\subset E[\gamma]} \tilde{z}(\gamma, E'),$$

and bound

$$|\tilde{z}(\gamma, E')| \le c^{|\gamma|} \epsilon_\infty^{|E'|} \quad \text{where} \quad c = \frac{|e^h| + |e^{-h}|}{|e^h + e^{-h}|}.$$

Show that $c \le \sqrt{2}$ if $|Im(h)| \le \pi/4$. How small does $\beta$ have to be to absorb this extra factor?

(c) Put everything together, to show that for $\beta$ sufficiently small, $f(h)$ is analytic in the region

$$\mathcal{D} = \{h \in \mathbb{C} \colon |Im(h)| < \pi/4\}.$$

*Hint: All you need is that the uniform limit of functions which are analytic in an open disc in the complex plain is analytic.*

(d) [**Bonus**] Clearly, analyticity in $\mathcal{D}$ implies that $f(h)$ is real analytic. Can you achieve real analyticity without given up anything on the convergence region, except for maybe replacing a bound $e^{\beta} - 1 \leq \frac{1}{2d\tilde{K}(a)}$ by the strict bound $e^{\beta} - 1 < \frac{1}{2d\tilde{K}(a)}$? How would you do this?