CS294-204 Phase Transitions (Fall 2021)
Homework #2
Due Fr. 9/24

1 SIR Birth Process

Let $T^{n,\lambda}_{SIR}$ be the birthprocess with off-spring distribution X, where

$$\Pr(X = k) = \int_0^\infty dT e^{-T} \binom{n}{k} p_T^k (1 - p_T)^{n-k}, \qquad p_T = 1 - e^{-\lambda T}. \tag{1}$$

In the above expression

- T represents the duration of an infection (assumed to have distribution exp(1)),
- p_T represents the probability that an individual is infected, if it is exposed for time T and the infections spreads at rate λ,
- n is the number of people that could get infected,
- $\binom{n}{k} p_T^k (1-p_T)^{n-k}$ is the probability that of n independent trials to pass an infection, k are "successful".

Note that in this branching process approximation of an epidemic, we neglect the fact that as more and more people get infected, the infection runs out of people to infect. In particular, while in the above branching process approximation, an outbreak corresponds to infinitely many infected people, in an actual infection, an outbreak will correspond to a positive fraction of all n people being infected.

But the branching process approximation turns out to be a good approximation for the early stages of an infection, where the effect of "running out of people to infect" is negligible. In addition, the bounds you prove in this exercise will provide the technical estimates needed to analyze the homogeneous mixing SIR model in which the "running out of people to infect" *is* accounted for.

In this homework set, you will in particular see that if $\lambda = \lambda_n$ is chosen appropriately, the survival probability takes a very simple form in the limit of large n. Indeed, you will prove the following lemma, which lies at the core of the intuition behind the results for homogeneous mixing we derived in class..

Lemma 1 *Assume $\lambda = \lambda_n$ is chosen in such a way that*

$$n\lambda \to \beta \qquad as \qquad n \to \infty.$$

Then the survival probability converges to

$$\Pr(|T^{n,\lambda}_{SIR}| = \infty) \to \begin{cases} 0 & \text{if } \beta \le 1 \\ P = 1 - \frac{1}{\beta} & \text{if } \beta > 1. \end{cases}$$

In addition, you will derive exponential tail bounds for the size of an outbreak below the epidemic threshold, as well as for the size of an outbreak above the threshold, if you consider the case where $T^{n,\lambda}_{SIR}$ does not survive.

1.1 Generating Function and Survival Probability [6 Points + 2 Bonus Points]

1. First, consider the case where $\lambda = \beta/n$ for some fixed β. Observe that conditioned on T, X is a binomial random variable $Bin(n, p_T)$. Calculate $\mathbb{E}[X]$ and conclude that for $\beta < 1$, the process $T^{n,\lambda}_{SIR}$ dies out with probability one, while for $\beta > 1$ and n large enough, it survives with non-zero probability.

2. Assume that $\lambda = \lambda_n$ is chosen in such a way that $n\lambda \to \beta$ as $n \to \infty$. Consider the generating function for the random variable X defined in (1). Denoting this generating function by G_n, prove that $G_n(\eta)$ converges to $G^\beta(\eta)$, where

$$G^\beta(\eta) = \frac{1}{1 + \beta(1-\eta)}.$$

   To this end

   (a) Use that conditioned on T, X is a binomial random variable $Bin(n, p_T)$ to write G_n as an integral over the generating function of a Binomial random variable.

   (b) Prove convergence of the integrand

   (c) Use dominated convergence to get convergence of the integral.

   (d) Evaluate the integral to finish the proof

3. Solve the implicit equation $G^\beta(\eta) = \eta$, identify the smallest solution, and show that $P = 1 - \eta$ is equal to the expression given in Lemma 1.

4. **[Bonus]** Prove Lemma 1. To this end, you will need to establish uniform convergence of $G_n(\eta)$ to $G^\beta(\eta)$ for $\eta \in [0, 1]$. *Hint: First choose T_0 such that $\int_{T_0}^\infty e^{-T} \leq \epsilon/2$. Then control all errors you make when approximating the integrand uniformly in η and $T \leq T_0$.* [2 Bonus Points].

1.2 Concentration Below Criticality

1. Below we will prove the following concentration bound for a sum of k i.i.d. random variables $X_1, \dots, X_k$ with the same distribution as X

$$\Pr(\sum_{i=1}^{k} X_i \geq kx) \leq \exp\Big(-\frac{(\lambda n - x)^2}{4}k\Big) \quad \text{if} \quad \lambda n \leq x \leq 1 \tag{2}$$

   Assuming this bound holds, prove that for $\lambda n < 1$,

$$\Pr(|T^{n,\lambda}_{SIR}| > k) \leq e^{-k(\lambda n - 1)^2/4}.$$

   Hint: Use the same reasoning as the one for the subcritical random graph in the lectures. [2 Points]

2. To prove (2), first use that conditioned on T, X is a random variable with distribution $Bin(n, p_T)$ to prove that conditioned on T,

$$\mathbb{E}[e^{tX}|T] = (1 + p_T(e^t - 1))^n$$

   Use that $p_T \leq \lambda T$ to bound

$$\mathbb{E}[e^{tX}] \leq \int_0^\infty dT e^{-T} e^{(e^t-1)n\lambda T}$$

Evaluate the integral, and conclude that for $x \geq \lambda n$, the rate function (defined in Homework 1, Exercise 2) obeys the bound

$$I(x) \geq \sup_{t \geq 0}(tx + \log(1 - \lambda n(e^t - 1))) \tag{3}$$

Use (3) to prove (2). To this end, set $\beta = \lambda n$, and first optimize over t to get the bound

$$I(x) \geq x \log \frac{x}{\beta} + (1+x) \log \frac{1+\beta}{1+x}.$$

Then Tayler-expand the right hand side around $x = \beta$ to prove that

$$x \log \frac{x}{\beta} + (1+x) \log \frac{1+\beta}{1+x} = \int_\beta^x dy \int_\beta^y dz \frac{1}{z(1+z)}$$

Show that for $\beta < x \leq 1$ this implies $I(x) \geq \frac{1}{4}(\beta - x)^2$ and hence (2). [4 Points].

1.3 Concentration above Criticality

1. Below, we will prove the following concentration bounds for k i.i.d. random variables $X_1, \ldots, X_k$ with the same distribution as X,

$$\Pr(\sum_{i=1}^k X_i \leq k(1-\delta)\mathbb{E}[X]) \leq \exp\Big(-\frac{\delta^2}{6}k\Big) \quad \text{if} \quad 0 \leq \delta \leq 1 \quad \text{and} \quad \mathbb{E}[X] \geq 1. \tag{4}$$

   Assuming this bound, prove that if $\lambda n \to \beta > 1$, there exists $D > 0$ such that for n sufficiently large

$$\Pr(k \leq |T_{SIR}^{n,\lambda}| < \infty) \leq \frac{1}{1 - e^{-D}} e^{-Dk}.$$

   Hint: Use the same reasoning as the one from the branching process lecture for the Poisson or Binomial birth process and choose n large enough to control the difference between $c = \mathbb{E}[X]$ and β. [2 Points]

2. To prove (4), establish first that $1 - z \leq e^{-z}$ and $e^{-z} \leq 1 - z + \frac{1}{2}z^2$ whenever $z \geq 0$. Use these bounds to prove that

$$\mathbb{E}[e^{-tX}] \leq \exp\Big(-t\mathbb{E}[X] + \frac{1}{2}t^2\mathbb{E}[X^2]\Big).$$

   Insert this into the expression for the rate function $I(x)$ from the first homework set 1, and conclude that

$$\Pr(\sum_{i=1}^k X_i \leq xk) \leq \exp\Big(-\frac{(\mathbb{E}[X] - x)^2}{2\mathbb{E}[X^2]}k\Big) \quad \text{if} \quad x \leq E[X].$$

   To deduce the bound (4), you will want to bound the expectation of X^2. To this end,

   (a) condition first on T, and use that conditioned on T, X is a binomial random variable to calculate $\mathbb{E}[X^2|T]$. You should obtain an expression involving p_T^2 and $p_T(1-p_T)$.

   (b) To bound the expectation of this expression over T, calculate the expectations of p_T and p_T^2. Show that this gives $\mathbb{E}[X^2] \leq \mathbb{E}[X] + 2(\mathbb{E}[X])^2$. Deduce the bound (4) when $\mathbb{E}[X] \geq 1$.

   [4 Points].