CS294-204 Phase Transitions (Fall 2021)
Homework #1
Due Fr. 9/10

1. Poisson, Geometric, and Binomial Birth process [7 Points]

1. Calculate the generating function $G(\eta)$ for a Poisson random variable $Poi(c)$; write out the implicit equation for the extinction probability η of a birth process with offspring distribution $Poi(c)$ to show that the survival probability, $\theta = 1-\eta$ obeys the equation $\theta + e^{-c\theta} = 1$ [2 Points].

2. Calculate the generating function $G(\eta)$ of a geometric random variable X with $\mathbb{P}(X=k) = q^k(1-q)$, $k \geq 0$. Calculate the expectation, $c = \mathbb{E}[X]$, and determine the threshold for the survival of a birth process with offspring distribution X. Solve the implicit equation for the survival probability $\theta = 1-\eta$ and express it in terms of c [2 Points].

3. Calculate the generating function $G_{n,p}(x)$ for $Bin(n,p)$, and express it in the form $f(x,p)^n$. Show that if $n \to \infty$ and $np \to c \in (0,\infty)$, $G_{n,p}(x)$ converges to the generating function of a Poisson random variable with mean c. Prove that the convergence is uniform for $x \in [0,1]$, and use this to conclude that the extinction probability of a branching process with offspring distribution $Bin(n,p)$ converges to that of a branching process with offspring distribution $Poi(c)$ [3 Points].

2. Concentration for General Random Variables [4 Points]

Let X be a random variable with $\mathbb{E}[X] = c$ and let $X_1, X_2, \ldots$ be i.i.d. with the same distribution as X. In this exercise we will show that

$$\Pr(\sum_{i=1}^{k} X_i \geq kx) \leq e^{-kI(x)} \quad \text{if } x > c; \tag{1}$$

$$\Pr(\sum_{i=1}^{k} X_i \leq kx) \leq e^{-kI(x)} \quad \text{if } x < c, \tag{2}$$

where $I(x)$ is the "rate function"

$$I(x) = \sup_{t\in\mathbb{R}}(tx - \log \mathbb{E}[e^{tX}]). \tag{3}$$

1. Use the standard trick that for any random variable Z and any $t > 0$, $\Pr(Z \geq z) = \Pr(e^{tZ} \geq e^{tz})$ to show that for $x > c$ and $t > 0$

$$\Pr(\sum_{i=1}^{k} X_i \geq kx) \leq e^{-k\phi_t(x)}$$

   where $\phi_t(x) = tx - \log \mathbb{E}[e^{tX}]$ [1 Point].

2. Use Jensen's inequality to prove that $\phi_t(x) \leq (x-c)t$, and use this to conclude that the supremum in (3) can be taken over t such that $(x-c)t \geq 0$ (Hint: use that $\phi_t(x) = 0$ if $t = 0$.) Infer the bound (1) [2 Points].

3. Consider the variables $Y_i = -X_i$ to prove (2) [1 Point].

3. Concentration for Poisson and Binomial Random Variables [7 Points]

1. Calculate $\phi_t(x) = tx - \log \mathbb{E}[e^{tX}]$ for a Poisson random variable $X \sim Poi(c)$. Optimize over t to show that

$$I(x) = c - x + x \ln \frac{x}{c}.$$

   Prove that $I(x) = \int_c^x dy \int_c^y \frac{1}{z} dz$ if $x > c$. Use this, and the analogous formula for $x < c$ to show that $I(x) \geq \frac{(x-c)^2}{2\max\{x,c\}}$. Conclude that

$$\begin{aligned} \Pr\Big(\sum_{i=1}^k X_i \geq kx\Big) &\leq e^{-k\frac{(x-c)^2}{2x}} \quad \text{if} \quad x > c \\ \Pr\Big(\sum_{i=1}^k X_i \leq kx\Big) &\leq e^{-k\frac{(c-x)^2}{2c}} \quad \text{if} \quad x < c. \end{aligned} \tag{4}$$

   [2 Points]

2. Let $Be(p)$ be the Bernoulli distribution. Optimize over t in the expression for $I(x)$ and prove that for $X \sim Be(p)$ and $x \in (0,1)$, $I(x)$ is the relative entropy (also called Kullback–Leibler divergence)

$$I_p(x) = x \log\left(\frac{x}{p}\right) + (1-x)\log\left(\frac{1-x}{1-p}\right).$$

   Taylor expand $I_p(x)$ to second order around $x = p$ and bound the second derivative from below by $\min\{1/x, 1/p\}$ to show that $I_p(x) \geq \dfrac{(x-p)^2}{2\max\{x,p\}}$. [3 Points]

3. Consider now k i.i.d. random variables $X_i \sim Bin(n,p)$ with $np = c$. Use the last result to show that the bounds (4) hold for $X_i \sim Bin(n,p)$ with $np = c$. Hint: As an intermediate step, prove that the distribution of $Bin(n,p)$ is the same as the sum of n i.i.d. Bernoulli random variables and use this to express $I(x)$ as $nI_p(x/n)$. [2 Points]

4. Branching process with arbitrary off-spring distribution [4 Bonus Points]

In class, we showed that for a Branching Process with off-spring $X \sim Poi(c)$ with $c > 1$, there exists some constant $D > 0$ such that

$$\Pr(k \leq |T_X| < \infty) \leq \frac{e^{-kD}}{1 - e^{-D}}, \tag{5}$$

i.e., we proved that a Poisson branching process above criticality that dies out is not very likely to reach a large size. Here you will prove this result for an arbitrary off-spring distributions, as long as $\mathbb{E}[X] > 1$.

1. Let X be a random variable taking values in $\mathbb{R}_+ = [0, \infty)$ with $0 < \mathbb{E}[X] < \infty$. Prove that for $x < \mathbb{E}[X]$, the rate function $I(x)$ is strictly positive. *Hint: Use that $\phi_0(x) = 0$ and consider the derivative of $\phi_t(x)$ with respect to t - but be careful, $\mathbb{E}[e^{tX}]$ may not be defined for all $t \in \mathbb{R}$.*

2. Show the same statement in the case where $\mathbb{E}[X] = \infty$.

3. Apply this result for $x = 1$ and $c = \mathbb{E}[X] > 1$, and proceed as in class to prove (5) for general X. Express D in terms of the rate function $I(\cdot)$.