CS294-179 Network Structure and Epidemics  
Fall 2020  
Homework #6  
Due Fr, Nov. 20

Each of the following exercises has 9 points. Choose 2 (if you do 3, I’ll count the one with the least points as bonus).

1 Differential Equations for SIR on $CM(\tilde{d})$

In this question we rewrite Volz equations to eliminate dependency of variables $p_I$ and $p_S$.

1. By the definition of $\theta$ show that $\tau p_I = -\frac{\dot{\theta}}{\theta}$ and use this to eliminate $p_I$ from the differential equation for $p_S$, 
\[
\frac{dp_S}{dt} = \tau p_S p_I (1 - \frac{\theta G''(\theta)}{G'(\theta)})
\]
2. Solve the differential equation from 1) and conclude that there is a constant $C$ that 
\[p_S = \frac{G'(\theta)}{C\theta}.
\]
Also, show that $C = G'(1)$.

Hint: As an intermediate step, rewrite the differential equation obtained in 1) in the form 
\[
\frac{d\log p_S}{dt} = F'(\theta)\dot{\theta}
\]
for some function $F$ and use this to concluded that $p_S$ is equal to $e^{F(\theta)}$ times some integration constant.

3. Show that 
\[
\frac{dp_I}{dt} = \frac{1}{\tau} \left( \frac{\dot{\theta}^2}{\theta} - \frac{\dot{\theta}}{\theta} \right).
\]
4. Recall the following equation 
\[
\frac{dp_I}{dt} = -p_I(\tau + \gamma) + \tau p_I^2 + \tau p_S p_I \frac{\theta G''(\theta)}{G'(\theta)}.
\]
Use parts 1) - 3) to show that 
\[
\ddot{\theta} = -(\gamma + \tau)\dot{\theta} + \tau \frac{G''(\theta)}{G'(1)}\dot{\theta}.
\]
5. Show that 
\[
\dot{\theta} = -\tau \theta + \gamma(1 - \theta) + \tau \frac{G'(\theta)}{G'(1)}
\]

Hint: Use the initial conditions $\theta(0) = 1$, $p_I(0) = 0$ to eliminate the integration constant you get.
2 Generating Function for Final Outbreak Size

Recall that the generating function for the in-degree of a vertex of degree \( d \) in the underlying graph is

\[
G_d = (1 - p(1 - x))^d \quad \text{where} \quad p = \frac{\tau}{\gamma + \tau};
\]

Recall further that the generating functions for the degree distribution in the configuration model is \( G(x) = \sum_k p_k x^k \), while the generating function for the forward degrees is

\[
\tilde{G}(x) = \sum_k p^*_k x^k \quad \text{where} \quad p^*_k = \frac{(k + 1)p_{k+1}}{\bar{d}},
\]

with \( \bar{d} = \sum_k p_k \) denoting the average degree.

1) Calculate the generating functions \( G^-(x) = \mathbb{E}_d[G_d^-(x)] \) and \( \tilde{G}^-(x) = \mathbb{E}_d[G_\tilde{d}^-(x)] \)

where the expectations are with respect to \( p_k \) and \( p^*_k \). Express the results in terms of \( G \) and \( \tilde{G} \). Hint, you should get something of the form \( G^-(x) = G(A + Bx) \), and similarly for \( \tilde{G}^- \).

2) Recall that in the birth process approach, the final fraction of recovered vertices is equal to

\[
R_\infty = 1 - G^-(\tilde{\eta}^-)
\]

where \( \tilde{\eta}^- \) is the smallest solution of \( \tilde{\eta}^- = \tilde{G}^-(\tilde{\eta}^-) \). Rewrite these equations in terms of \( G \) and \( \tilde{G} \).

In the previous exercises, we showed that the Volz-variable \( \theta \) governing the time evolutions of the fraction of susceptible nodes of degree, \( S = G(\theta) \), obeys a differential equation which can be written in the form

\[
\frac{d\theta}{dt} = (\gamma + \tau) \left( 1 - \theta + p \left( \frac{G'(\theta)}{G'(1)} - 1 \right) \right).
\]

3) Calculate the derivative \( G'(x) \) of \( G(x) \), write \( \tilde{G}(x) \) in terms of \( G'(x) \) and \( G'(1) \), and use the result to express the above differential equation in terms of \( \tilde{G} \).

The equation \( d\theta/dt = 0 \) clearly has the solution \( \theta = 1 \). As discussed in class, for \( R_0 > 1 \), the second solution of \( d\theta/dt = 0 \) is the asymptotic value of \( \theta \) as \( t \to \infty \), \( \theta_\infty = \lim_{t \to \infty} \theta(t) \).

4) Write the implicit equation for \( \theta_\infty \) using the generating function \( \tilde{G} \). Bonus: show that for \( R_0 > 1 \) it has exactly two solutions in \([0, 1]\): the solution \( \theta = 1 \) and a unique solution \( \theta_\infty \in (0, 1) \).

5) Write \( R_\infty \) in terms of \( S_\infty = G(\theta_\infty) \) and guess how \( \theta_\infty \) and \( \tilde{\eta}^- \) are related. Hint use the expression for \( R_\infty \) you derived in part 2): you should get a linear relationship between the two.

6) Show that \( \theta_\infty \) satisfies the implicit equation derived in part 4) if and only if \( \tilde{\eta}^- \) satisfies the implicit equation obtained in part 2).
3 The effect of degrees fluctuations

Consider a configuration model with degree distribution \( p_k \) and generating function \( G(\theta) = \sum_k \theta^k p_k \). Recall that in the notation of Volz, at any given time, \( \theta^d \) is the fraction of degree \( d \) vertices which are still susceptible, and \( S = G(\theta) = \sum_k \theta^d p_k \) is the ratio of the number of susceptible vertices to the total number of vertices.

In the previous exercises, we showed that the Volz-variable \( \theta \) obeys the differential equation

\[
\frac{d\theta}{dt} = (\gamma + \tau) \left( 1 - \theta + p \left( \frac{G'(\theta)}{G'(1)} - 1 \right) \right), \quad p = \frac{\tau}{\tau + \gamma},
\]

and we showed that the asymptotic value of \( \theta \) as \( t \to \infty \) is the unique solution \( \theta_\infty \in (0, 1) \) of the equation obtained by setting \( \frac{d\theta}{dt} = 0 \). In this exercise, we will study this solution for two particular graphs.

1) Let \( G \) be a 4-regular random graph (drawn from configuration model). Set \( \tau = 2 \) and \( \gamma = 1 \), so that \( p = \tau/(\tau + \gamma) = 2/3 \). Calculate \( R_0 \), and the final fraction of nodes which were infected (and then removed) at the end of the infection, \( R_\infty = 1 - S(\theta_\infty) \).

2) Consider the configuration model where all nodes have either degree 2 or degree 10, with the fraction of degree 2 nodes equal to \( p_2 = \frac{3}{4} \) (and the fraction of degree 10 nodes equal to \( p_{10} = \frac{1}{4} \)), so that the average degree is the same as in part 1, i.e., \( \bar{d} = 4 \). Again choose \( \tau = 2 \) and \( \gamma = 1 \). Compute \( R_0 \).

3) What fraction of degree 2 nodes are infected (and then removed) by the end of the process? How about degree 10 nodes?

4) Adjust \( \tau \) so that \( R_0 \) becomes the same as in part 1. What fraction of degree 2 nodes are infected (and then removed) by the end of the process? How about degree 10 nodes?

Remark: You can use WolframAlpha to solve the polynomial equations for \( \theta_\infty \) you will get.