CS294-179 Network Structure and Epidemics
Fall 2020
Homework #6
Due Fr, Nov. 20

Each of the following exercises has 9 points. Choose 2 (if you do 3, I'll count the one with the least points as bonus).

1 Differential Equations for SIR on $CM(\tilde{d})$

In this question we rewrite Volz equations to eliminate dependency of variables p_I and p_S.

1. By the definition of θ show that $\tau p_I = -\frac{\dot{\theta}}{\theta}$ and use this to eliminate p_I from the differential equation for p_S,
$$\frac{dp_S}{dt} = \tau p_S p_I (1 - \frac{\theta G''(\theta)}{G'(\theta)})$$

2. Solve the differential equation from 1) and conclude that there is a constant C that
$$p_S = \frac{G'(\theta)}{C\theta}.$$
Also, show that $C = G'(1)$.
Hint: As an intermediate step, rewrite the differential equation obtained in 1) in the form
$$\frac{d \log p_S}{dt} = F'(\theta)\dot{\theta}$$
for some function F and use this to concluded that p_S is equal to $e^{F(\theta)}$ times some integration constant.

3. Show that
$$\frac{dp_I}{dt} = \frac{1}{\tau}(\frac{\dot{\theta}^2}{\theta^2} - \frac{\ddot{\theta}}{\theta}).$$

4. Recall the following equation
$$\frac{dp_I}{dt} = -p_I(\tau + \gamma) + \tau p_I^2 + \tau p_S p_I \frac{\theta G''(\theta)}{G'(\theta)}.$$
Use parts 1) - 3) to show that
$$\ddot{\theta} = -(\gamma + \tau)\dot{\theta} + \tau \frac{G''(\theta)}{G'(1)}\dot{\theta}.$$

5. Show that
$$\dot{\theta} = -\tau\theta + \gamma(1 - \theta) + \tau \frac{G'(\theta)}{G'(1)}.$$
Hint: Use the initial conditions $\theta(0) = 1$, $p_I(0) = 0$ to eliminate the integration constant you get.

2 Generating Function for Final Outbreak Size

Recall that the generating function for the in-degree of a vertex of degree d in the underlying graph is

$$G_d^- = \Big(1 - p(1-x)\Big)^d \qquad \text{where} \qquad p = \frac{\tau}{\gamma + \tau};$$

Recall further that the generating functions for the degree distribution in the configuration model is $G(x) = \sum_k p_k x^k$, while the generating function for the forward degrees is

$$\tilde{G}(x) = \sum_k p_k^* x^k \qquad \text{where} \qquad p_k^* = \frac{(k+1)p_{k+1}}{\bar{d}},$$

with $\bar{d} = \sum_k p_k$ denoting the average degree.

1) Calculate the generating functions

$$G^-(x) = \mathbb{E}_d[G_d^-(x)] \qquad \text{and} \qquad \tilde{G}^-(x) = \mathbb{E}_{\tilde{d}}[G_{\tilde{d}}^-(x)]$$

   where the expectations are with respect to p_k and p_k^*. Express the results in terms of G and $\tilde{G}$. *Hint, you should get something of the form $G^-(x) = G(A + Bx)$, and similarly for $\tilde{G}^-$.*

2) Recall that in the birth process approach, the final fraction of recovered vertices is equal to

$$R_\infty = 1 - G^-(\tilde{\eta}^-)$$

   where $\tilde{\eta}^-$ is the smallest solution of $\tilde{\eta}^- = \tilde{G}^-(\tilde{\eta}^-)$. Rewrite these equations in terms of G and $\tilde{G}$.

In the previous exercises, we showed that the Volz-variable θ governing the time evolutions of the fraction of susceptible nodes of degree, $S = G(\theta)$, obeys a differential equation which can be written in the form

$$\frac{d\theta}{dt} = (\gamma + \tau)\Big(1 - \theta + p\Big(\frac{G'(\theta)}{G'(1)} - 1\Big)\Big).$$

3) Calculate the derivative $G'(x)$ of $G(x)$, write $\tilde{G}(x)$ in terms of $G'(x)$ and $G'(1)$, and use the result to express the above differential equation in terms of $\tilde{G}$.

The equation $d\theta/dt = 0$ clearly has the solution $\theta = 1$. As discussed in class, for $R_0 > 1$, the second solution of $d\theta/dt = 0$ is the asymptotic value of θ as $t \to \infty$, $\theta_\infty = \lim_{t\to\infty} \theta(t)$.

4) Write the implicit equation for θ_∞ using the generating function $\tilde{G}$. Bonus: show that for $R_0 > 1$ it has exactly two solutions in $[0, 1]$: the solution $\theta = 1$ and a unique solution $\theta_\infty \in (0, 1)$.

5) Write R_∞ in terms of $S_\infty = G(\theta_\infty)$ and guess how θ_∞ and $\tilde{\eta}^-$ are related. *Hint use the expression for R_∞ you derived in part 2); you should get a linear relationship between the two.*

6) Show that θ_∞ satisfies the implicit equation derived in part 4) if and only if $\tilde{\eta}^-$ satisfies the implicit equation obtained in part 2).

3 The effect of degrees fluctuations

Consider a configuration model with degree distribution (p_k) and generating function $G(\theta) = \sum_k \theta^k p_k$. Recall that in the notation of Volz, at any given time, θ^d is the fraction of degree d vertices which are still susceptible, and $S = G(\theta) = \sum_k p_k \theta^d$ is the ratio of the number of susceptible vertices to the total number of vertices.

In the previous exercises, we showed that the Volz-variable θ obeys the differential equation

$$\frac{d\theta}{dt} = (\gamma + \tau)\Big(1 - \theta + p\Big(\frac{G'(\theta)}{G'(1)} - 1\Big)\Big), \qquad p = \frac{\tau}{\tau + \gamma},$$

and we showed that the asymptotic value of θ as $t \to \infty$ is the unique solution $\theta_\infty \in (0, 1)$ of the equation obtained by setting $\frac{d\theta}{dt} = 0$. In this exercise, we will study this solution for two particular graphs.

1) Let G be a 4-regular random graph (drawn from configuration model). Set $\tau = 2$ and $\gamma = 1$, so that $p = \tau/(\tau + \gamma) = 2/3$. Calculate R_0, and the final fraction of nodes which were infected (and then removed) at the end of the infection, $R_\infty = 1 - S(\theta_\infty)$.

2) Consider the configuration model where all nodes have either degree 2 or degree 10, with the fraction of degree 2 nodes equal to $p_2 = \frac{3}{4}$ (and the fraction of degree 10 nodes equal to $p_{10} = \frac{1}{4}$), so that the average degree is the same as in part 1, i.e., $\bar{d} = 4$. Again choose $\tau = 2$ and $\gamma = 1$. Compute R_0.

3) What fraction of degree 2 nodes are infected (and then removed) by the end of the process? How about degree 10 nodes?

4) Adjust τ so that R_0 becomes the same as in part 1. What fraction of degree 2 nodes are infected (and then removed) by the end of the process? How about degree 10 nodes?

Remark: You can use WolframAlpha to solve the polynomial equations for θ_∞ you will get.