CS294-179, Spring 2025
Homework #4, Due 3/21

1 Your friends have more friends than you do [6 Points]

Let G be a graph where all vertices have degree at least one. For a $i \in V(G)$, let $N(i)$ be the set of its neighbors, and $d_i^* = \frac{1}{d_i} \sum_{j \in N(i)} d_j$ be the average degree of the neighbors of i. Finally, let $d^*(G)$ to be the average of d_i^* over all vertices i. We will compare it to $\bar{d}(G)$, the average degree of G, and will in particular show that for arbitrary graphs "on average, your friends have at least as many friends than you do", i.e., $d^*(G) \geq \bar{d}(G)$.

1. Let S_n be the star with n leaves, i.e., let the graph where one vertex, the center, is connected to n others, the leaves, and no other edges. Find $d_*(S_n)$ and $\bar{d}(S_n)$ as a function of n.
2. Let H be a graph that is the disjoint union of a triangle and an isolated edge. Compute $\bar{d}(H)$ and $d^*(H)$.
3. Prove that for any graph G, $d^*(G) \geq \bar{d}(G)$. *Hint: as an intermediate step you might want to prove that $\frac{d_i}{d_j} + \frac{d_j}{d_i} \geq 2$ and the equality holds when $d_i = d_j$.*
4. Characterize graphs for which the equality happens in part 3 ($\bar{d}(G) = d^*(G)$).
5. Express the statement "the friends of every node i in the network have exactly as many friends as i has" in mathematical terms, and prove it for the case that $\bar{d}(G) = d^*(G)$. In other words, prove that if the above statement holds on average ($\bar{d}(G) = d^*(G)$) then it is actually true vertex by vertex, not just on average.

2 Polya Urn [5 Points]

Recall that a polya urn with R initial red, and B initial blue balls is a process where R and B are updated according to the following rule:

- Set $R_0 = R$ and $B_0 = B$
- At time t, draw a random ball from all $B_t + R_t$ balls, and raise R_t by one if the ball is red - otherwise raise B_t by one. Denote the new number of red balls by R_{t+1} and the new number of blue balls by B_{t+1}.

Set $X_t = 1$ if the ball drawn at time t is red, and $X_t = 0$ otherwise. Fix a set $I_R \subset [n]$ and it's complement, I_B. We have seen that the probability of drawing a sequence $X_1, \dots, X_n$ such that $X_i = 1$ for $i \in I_R$ and $X_i = 0$ for $i \in I_B$ does only depend on the sizes n_1 and n_2 of I_1 and I_2 and is equal to

$$\Pr\Big(X_i = 1 \text{ for } i \in I_R \text{ and } X_i = 0 \text{ for } i \in I_B\Big) = \frac{R(R+1)\dots(R+n_1-1) \times B(B+1)\dots(B+n_2-1)}{(B+R)(B+R+1)\dots(B+R+n-1)}.$$

In this exercise, you will prove the following theorem

Theorem 1.1 *If $R > 0$ and $B > 0$, the probability of the sequence $X_1, \dots, X_n$ draws from the Poly Urn with initially R red and B blue can equivalently be calculated by first drawing $p \sim \beta(R, B)$ and then choosing X_i iid with distribution $Be(p)$.*

Here $\beta(R, B)$ is the probability distribution on $[0, 1]$ that has the probability density function $\frac{1}{Z} x^{R-1}(1-x)^{B-1}$, where $Z = \int_0^1 x^{R-1}(1-x)^{B-1}$.

1. Let A and B non-negative integers. Use integration by parts to calculate the integral

$$\int_0^1 x^A(1-x)^B dx.$$

2. Let R and B be positive integers, let n_1 and n_2 be non-negative integers, and assume that $X \sim \beta(A, B)$. Calculate first the normalization factor $Z = \int_0^1 x^{R-1}(1-x)^{B-1}$ for the β-distribution, and then the expectation of $X^{n_1}(1-X)^{n_2}$.

3. Use the result from 2 to prove the theorem.

3 Degrees for preferential attachment[7 Points]

Let D_n be the degree of a random vertex in the preferential attachment graph on n nodes (note that it has two sources of randomness - the randomness from choosing a vertex, and that of the preferential attachment graph). For the various version of preferential attachment discussed in the lectures (independent, conditional, and sequential), the random variable D_n converges in distribution to a random variable D whose distribution is given by

$$\Pr(D=k) = \frac{2m(m+1)}{k(k+1)(k+2)}.$$

A different way of obtaining the distribution of D proceeds by proving that Preferential attachment has a weak local limit, and then establishing the degree distribution for the root of the limiting rooted graph. In that approach, the variable D appears naturally as a sum of m and a mixed Poisson random variable q:

$$D = m + q \quad \text{where} \quad q \sim Poi(\gamma \frac{1-x}{x})$$

where $\gamma \sim \Gamma(m, 1)$ and $x = \sqrt{u}$ with u being a uniform random variable in $[0, 1]$. In this exercise, you will prove that the two are indeed equivalent.

1. Let A be a non-negative integer. Use integration by parts to calculate the integral

$$\int_0^\infty x^A e^{-x} dx$$

2. Let A be a positive integer, let k be a non-negative integer, and let $c > -1$. A random variable X has distribution $\Gamma(A, 1)$ if its probability density function is equal to $\frac{1}{Z}x^{A-1}e^{-x}$, where Z is a normalization constant. Calculate first Z, and then the expectation of $X^k e^{-cX}$.

3. Calculate the probability that the random variable q above takes the value k, conditioned on γ and x.

4. By first taking expectation with respect to the random variable γ, and then with respect to $u = x^2$, get the probability that $q = k$.

5. Relate this result to the explicit formula given above and show that the two give the same probability distribution.