CS294-179 Network Structure and Epidemics
Fall 2020
Homework #1 (added bonus problem on 9/4)
Due Fr. 9/11

This and the next assignment will review some of the statistics/probability tools needed to follow the lectures. Most of the material needed to do these assignments can be found in Section 2.1 - 2.5 (pp. 55 - 76) of the course of Remco van der Hofstad on Complex Networks and Random Graphs., `https://www.win.tue.nl/~rhofstad/NotesRGCN.pdf`. If you are not familiar with that material, you might want to consider reading these 20 pages as part of this assignment. You can use the material from this course to solve the problems below, but please make sure you understand the proofs you might find there, and reformulate them in your own words.

1. Birth process [6 Points]

1. Calculate the generating function $G(x)$ for a Poisson random variable $Poi(c)$; write out the implicit equation for the extinction probability η of a birth process with offspring distribution $Poi(c)$ to show that the survival probability, $\theta = 1 - \eta$ obeys the equation $\theta + e^{-c\theta} = 1$. [2 Points]

2. Calculate the generating function for $Bin(n, p)$, and express it in the form $f(x, p)^n$. Show that if $n \to \infty$ and $np \to c \in (0, \infty)$, it converges to that of a Poisson random variable with mean c. Use this to conclude that the extinction probability of a branching process with offspring distribution $Bin(n, p)$ converges to that of a Poisson branching process, i.e., a branching process with Poisson offspring distribution. [2 Points]

3. Recall that conditioned on extinction, a supercritical birth process with offspring distribution p_k and extinction probability η becomes subcritical, with offspring distribution $\tilde{p}_k = \eta^{k-1} p_k$. Show that if $X \sim Poi(c)$ with $c > 1$, this distribution is again Poisson, with parameter $\tilde{c} = c\eta$, and show that $ce^{-c} = \tilde{c}e^{-\tilde{c}}$. [2 Points]

2. Concentration Bounds [6 Points]

Let X be a random variable with $\mathbb{E}[X] = c$ and let $X_1, X_2, \dots$ be i.i.d. with the same distribution as X. In this exercise we will show that

$$\Pr(\sum_{i=1}^{n} X_i \geq nx) \leq e^{-nI(x)} \quad \text{if } x > c; \tag{1}$$

$$\Pr(\sum_{i=1}^{n} X_i \leq nx) \leq e^{-nI(x)} \quad \text{if } x < c, \tag{2}$$

where $I(x)$ is the "rate function"

$$I(x) = \sup_{t \in \mathbb{R}} (tx - \log \mathbb{E}[e^{tX}]).$$

We will use this bound to derive the standard Chernoff bound for binomial random variables.

1. Use the standard trick that for any random variable Z and any $t \geq 0$, $\Pr(Z \geq z) = \Pr(e^{tZ} \geq e^{tz})$ to show that for $x \geq c$ and $t \geq 0$

   $$\Pr(\sum_{i=1}^{n} X_i \geq nx) \leq e^{-n\phi_t(x)}$$

   where $\phi_t(x) = tx - \log \mathbb{E}[e^{tX}]$. Deduce the bound (1).

   Hint: to extend the range of t from $t \geq 0$ to $t \in \mathbb{R}$, use Jensen's inequality ~~Hölder~~ to show that $\phi_t(x) \leq t(x - c) \leq 0 = \phi_0(x)$ if $x \geq c$ and $t \leq 0$). [2 Points]

2. Consider the variables $Y_i = -X_i$ to prove (2). [1 Point]

3. Let $Be(p)$ be the Bernoulli distribution. Show that for $X \sim Be(p)$ and $x \in (0, 1)$, $I(x)$ is the relative entropy (also called Kullback–Leibler divergence)

   $$I_p(x) = x \log\left(\frac{x}{p}\right) + (1 - x) \log\left(\frac{1 - x}{1 - p}\right).$$

   Taylor expand $I_p(x)$ to second order around $x = p$ and bound the second derivative from below by $\min\{1/x, 1/p\}$ to show that

   $$I_p(x) \geq \frac{(x - p)^2}{2} \min\{\frac{1}{x}, \frac{1}{p}\}.$$

   Use this to derive the following version of the Chernoff bound for the binomial distribution

   $$\Pr\Big(Bin(n, p) \geq (1 + \epsilon)np\Big) \leq e^{-np\frac{\epsilon^2}{2(1+\epsilon)}}$$

   $$\Pr\Big(Bin(n, p) \leq (1 - \epsilon)np\Big) \leq e^{-np\frac{\epsilon^2}{2}}.$$

   Hint: As an intermediate step, prove that the distribution of $Bin(n, p)$ is the same as the sum of n i.i.d. Bernoulli random variables. [3 Points]

3. Couplings and Stochastic Domination [5 Points]

Given n random variables $X_1, \ldots, X_n$ not necessarily defined over the same probability space, a coupling of $X_1, \ldots, X_n$ is a joint distribution $\mathbb{P}$ for these variables such that for each i, the marginal probability distribution of X_i is the correct one. I.e., if A_i is an event defined for the random variable X_i, then $\Pr(X_i \in A_i) = \mathbb{P}(X_i \in A_i)$ for all i. The simplest example is an independent, or product, coupling:

$$\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \prod_i \Pr(X_i \in A_i).$$

An another example is the identity coupling: if X and Y have the same distribution, we can couple them by setting $\mathbb{P}(X = Y, X \in A) = \Pr(X \in A)$ and $\mathbb{P}(X \neq Y) = 0$. Couplings will be explicitly or implicitly used throughout this course.

The first use we will examine are stochastic orderings. Given two random variables X, Y with values in $\mathbb{R}$, we say that X stochastically dominates Y if X and Y can be coupled in such a way that $\mathbb{P}(X \geq Y) = 1$.

An example are two Poisson random variables $X \sim Poi(c)$ and $Y \sim Poi(d)$ with $c > d$. To couple them in such a way that $X \geq Y$ with probability 1, we define a third random variable $Z \sim Poi(c - d)$, and couple Y and Z so that they are independent. As an easy calculation shows (part of this exercise, see below), the distribution of $Y + Z$ is then that of X, i.e., $Y + Z \sim Poi(c)$. Setting $X = Y + Z$, this gives a coupling of X and Y, and

$$\mathbb{P}(X \geq Y) = \mathbb{P}(Z \geq 0) = 1.$$

1. Complete the above proof by showing that if Y and Z are independent Poisson random variables with mean d and $c - d$, then $Y + Z \sim Poi(c)$, i.e., show that

   $$\sum_{i,j \geq 0 : i+j=k} \Pr(Y = i)\Pr(Z = j) = \frac{c^k}{k!} e^{-c}.$$

   [1 Point]

2. Let $X \sim Bin(n, p)$ and $Y \sim Bin(m, p)$ with $n \geq m$. Use a similar strategy as the one above to show that X stochastically dominates Y. Hint: Use the fact that the distribution of $Bin(n, p)$ is the same as the sum of n i.i.d. Bernoulli random variables, to produce such a coupling. [2 Points]

3. Prove that if X stochastically dominates Y, then for all $x \in \mathbb{R}$, $\Pr(X \geq x) \leq \Pr(Y \geq x)$. Or is this a typo, and the correct statement should be $\Pr(X \geq x) \geq \Pr(Y \geq x)$? Whichever you decide it is, please prove that one. [2 Points]

4^*. Unified Bernoulli, Binomial and Poisson Concentration Bound

Here we will show that if X is a Bernoulli, Binomial or Poisson random variable with expectation c, and $X_1, \ldots, X_k$ are i.i.d. with the same distribution as X, then then

$$\Pr(\sum_{i=1}^{k} X_i \leq kx) \leq e^{-k\frac{(x-c)^2}{2c}} \quad \text{if} \quad x \leq c$$

$$\Pr(\sum_{i=1}^{k} X_i \geq kx) \leq e^{-k\frac{(x-c)^2}{2x}} \quad \text{if} \quad x \geq c.$$

1. For the Bernoulli distribution this follows from Exercise 2.3 of this set.
2. Prove that if $\tilde{X} = \sum_{i=1}^{n} X_i$, where the X_i are i.i.d with rate function $I(\cdot)$, then $\tilde{X}$ has rate function $nI(x/n)$. This reduces the above bound to that for the Bernoulli distribution, since a $Bin(n,p)$ random variable is the sum of n i.i.d. $Be(p)$ random variables. [2 Bonus Points]
3. Calculate the rate function for $Poi(c)$ and show that the second derivative at a point y is equal to $1/y$. Bound this from below by the minimum of $1/c$ and $1/x$ to obtain the above concentration bound for $Poi(c)$. [2 Bonus Points]