Proof of the Bounded Differences Inequality

Peter Bartlett. March 9, 2006.

Theorem Suppose that $X_1, \ldots, X_n \in \mathcal{X}$ are independent, and $f : \mathcal{X}^n \to \mathbb{R}$. Let $c_1, \ldots, c_n$ satisfy

$$\sup_{x_1,\ldots,x_n,x_i'} |f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \\ - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)| \leq c_i,$$

for $i = 1, \ldots, n$. Then

$$P(f - \mathbb{E}f \geq t) \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^n c_i^2}\right).$$

Proof: We define

$$V_i = \mathbb{E}\left[f|X_1, \ldots, X_i\right] - \mathbb{E}\left[f|X_1, \ldots, X_{i-1}\right].$$

These V_is will play the same role as that played by the terms of the sum in the proof of Hoeffding's inequality. In particular, since the sum telescopes, we have

$$f - \mathbb{E}f = \sum_{i=1}^n V_i.$$

Using this, and the Chernoff bounding technique, we see that

$$\begin{aligned}
&P(f - \mathbb{E}f \geq t) \\
&= P\left(\sum_{i=1}^n V_i \geq t\right) \\
&\leq \inf_{s>0} \exp(-st)\mathbb{E}\exp\left(s\sum_{i=1}^n V_i\right) \\
&= \inf_{s>0} \exp(-st)\mathbb{E}\left(\prod_{i=1}^n e^{sV_i}\right).
\end{aligned}$$

So we need to bound the moment generating function of this sum of (dependent) random variables. Now, we have seen that $\mathbb{E}(V_i|X_1,\dots,X_{i-1}) = 0$. If we define

$$
\begin{aligned}
L_i &= \inf_x \mathbb{E}(f|X_1,\dots,X_{i-1},x) - \mathbb{E}(f|X_1,\dots,X_{i-1}), \\
U_i &= \sup_x \mathbb{E}(f|X_1,\dots,X_{i-1},x) - \mathbb{E}(f|X_1,\dots,X_{i-1}),
\end{aligned}
$$

then we have

$$
\begin{aligned}
V_i - L_i &= \mathbb{E}(f|X_1,\dots,X_i) - \mathbb{E}(f|X_1,\dots,X_{i-1}) \\
&\qquad - \inf_x \mathbb{E}(f|X_1,\dots,X_{i-1},x) + \mathbb{E}(f|X_1,\dots,X_{i-1}) \\
&= \mathbb{E}(f|X_1,\dots,X_i) - \inf_x \mathbb{E}(f|X_1,\dots,X_{i-1},x),
\end{aligned}
$$

and so $L_i \le V_i$ almost surely. Similarly, $V_i \le U_i$ a.s. Furthermore, from the independence of the X_i,

$$
\begin{aligned}
U_i - L_i &= \sup_x \mathbb{E}(f|X_1,\dots,X_{i-1},x) - \inf_x \mathbb{E}(f|X_1,\dots,X_{i-1},x) \\
&= \sup_x \int f(X_1,\dots,X_{i-1},x,x_{i+1},\dots,x_n)dP(x_{i+1},\dots,x_n) \\
&\qquad - \inf_x \int f(X_1,\dots,X_{i-1},x,x_{i+1},\dots,x_n)dP(x_{i+1},\dots,x_n) \\
&= \sup_{x,y} \int (f(X_1,\dots,X_{i-1},x,x_{i+1},\dots,x_n) \\
&\qquad\qquad - f(X_1,\dots,X_{i-1},y,x_{i+1},\dots,x_n))dP(x_{i+1},\dots,x_n) \\
&\le c_i,
\end{aligned}
$$

from the bounded differences assumption. Notice that the independence allows us to write the difference of the conditional expectations in terms an integral of a difference, and hence appeal to the bounded differences property. Thus, we may apply the Hoeffding lemma to the bounded, zero-mean random variables V_i conditioned on $X_1, \ldots, X_{i-1}$, as follows.

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{n} e^{sV_i}\right) &= \mathbb{E}\mathbb{E}\left(\prod_{i=1}^{n-1} e^{sV_i} e^{sV_n} | X_1, \ldots, X_{n-1}\right) \\
&= \mathbb{E}\left(\prod_{i=1}^{n-1} e^{sV_i} \mathbb{E}\left(e^{sV_n} | X_1, \ldots, X_{n-1}\right)\right) \\
&\leq \exp(s^2 c_n^2/8) \mathbb{E}\left(\prod_{i=1}^{n-1} e^{sV_i}\right) \\
&= \exp(s^2 c_n^2/8) \mathbb{E}\mathbb{E}\left(\prod_{i=1}^{n-1} e^{sV_i} | X_1, \ldots, X_{n-2}\right) \\
&= \exp(s^2 c_n^2/8) \mathbb{E}\left(\prod_{i=1}^{n-2} e^{sV_i} \mathbb{E}\left(e^{sV_{n-1}} | X_1, \ldots, X_{n-2}\right)\right) \\
&\leq \exp(s^2 (c_{n-1}^2 + c_n^2)/8) \mathbb{E}\left(\prod_{i=1}^{n-2} e^{sV_i}\right) \\
&\vdots \\
&\leq \exp\left(s^2 \sum_{i=1}^{n} c_i^2/8\right).
\end{aligned}$$

Substituting, we have

$$\begin{aligned}
P(f - \mathbb{E}f \geq t) &\leq \inf_{s>0} \exp\left(-st + \frac{s^2 \sum_{i=1}^{n} c_i^2}{8}\right) \\
&= \exp\left(\frac{-2t^2}{\sum_{i=1}^{n} c_i^2}\right),
\end{aligned}$$

where we selected the optimizing value $s = 4t / \sum_i c_i^2$.