Undulator Equation and Radiated Power

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Undulator Radiated Power in the Central Cone

$$\lambda_x = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

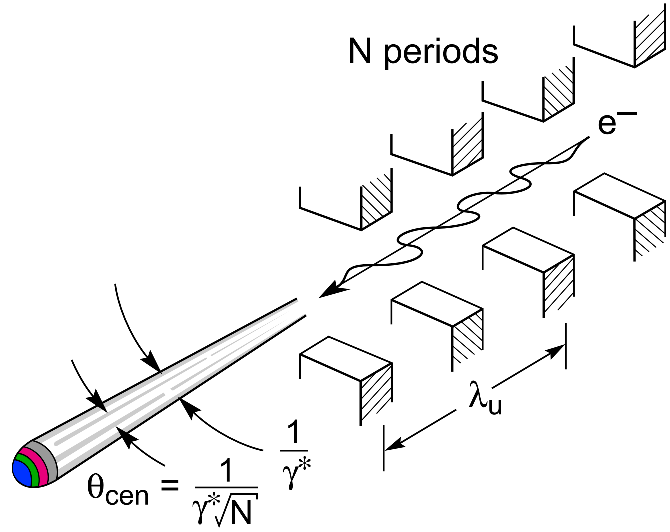
$$\bar{P}_{cen} = \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{\left(1 + \frac{K^2}{2}\right)^2} f(K)$$

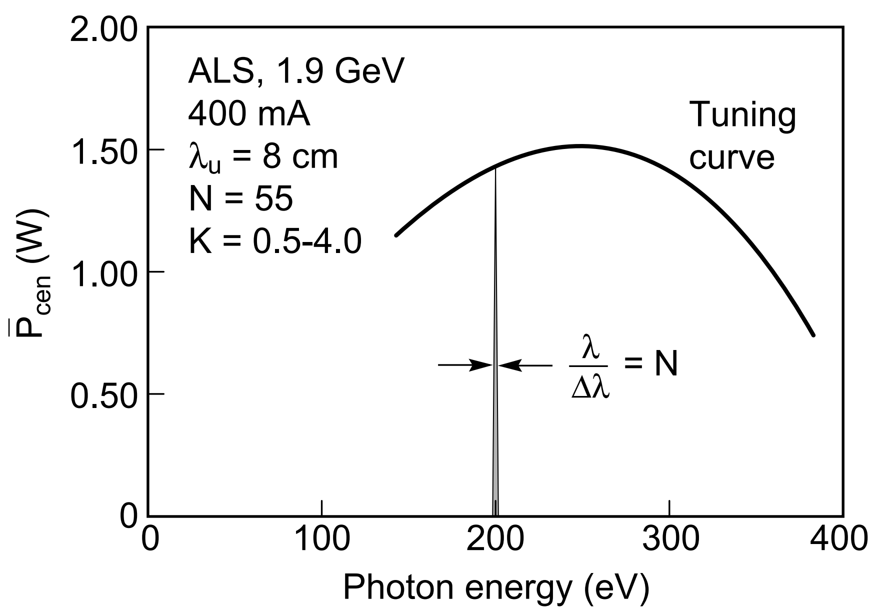
$$\theta_{cen} = \frac{1}{\gamma^* \sqrt{N}}$$

$$\left(\frac{\Delta\lambda}{\lambda}\right)_{cen} = \frac{1}{N}$$

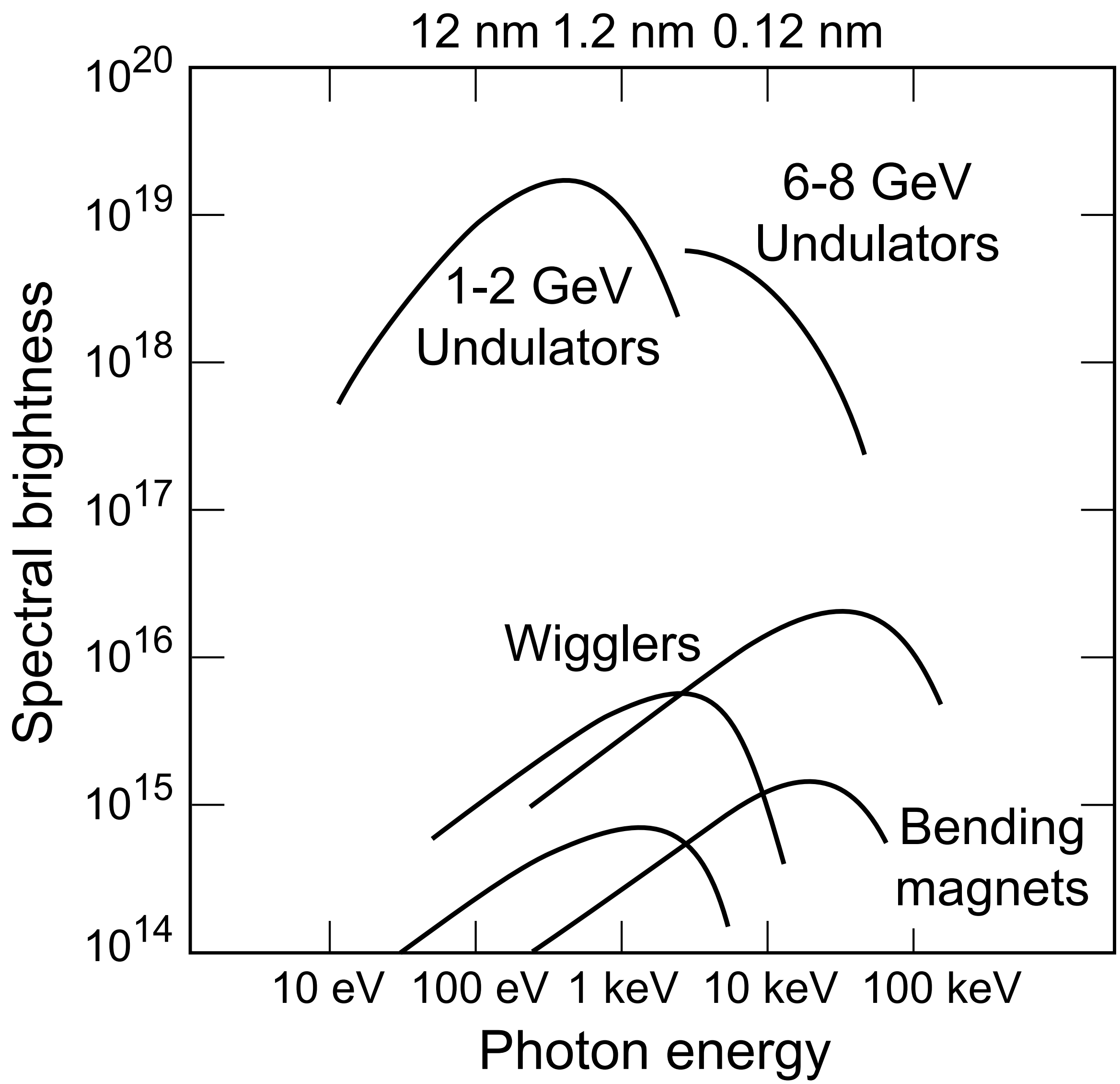
$$K = \frac{eB_0\lambda_u}{2\pi m_0 c}$$

$$\gamma^* = \gamma / \sqrt{1 + \frac{K^2}{2}}$$





Spectral Brightness of Synchrotron Radiation



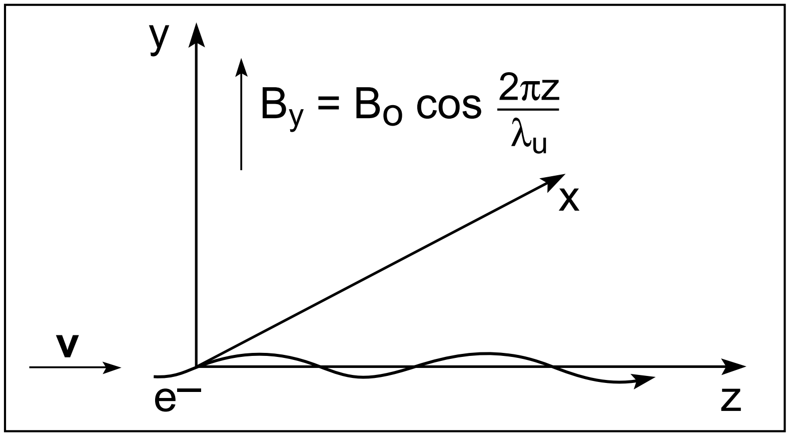
The Equation of Motion in an Undulator

Magnetic fields in the periodic undulator cause the electrons to oscillate and thus radiate. These magnetic fields also slow the electrons axial (z) velocity somewhat, reducing both the Lorentz contraction and the Doppler shift, so that the observed radiation wavelength is not quite so short. The force equation for an electron is

$$\frac{d\mathbf{p}}{dt} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (5.16)$$

where $\mathbf{p} = \gamma m \mathbf{v}$ is the momentum. The radiated fields are relatively weak so that

$$\frac{d\mathbf{p}}{dt} \simeq -e(\mathbf{v} \times \mathbf{B})$$



Taking to first order $v \simeq v_z$, motion in the x-direction is

$$m\gamma \frac{dv_x}{dt} = +ev_z B_y$$

$$m\gamma \frac{dv_x}{dt} = e\frac{dz}{dt} \cdot B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right) \qquad (0 \le z \le N\lambda_u)$$

$$m\gamma\, dv_x = e\, dz\, B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right)$$

The Equation of Motion in an Undulator (cont.)

$$m\gamma\, d\mathrm{v}_x = e\, dz\, B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right)$$

integrating both sides

$$m\gamma \mathrm{v}_x = e B_0 \frac{\lambda_u}{2\pi} \int \cos\left(\frac{2\pi z}{\lambda_u}\right) \cdot d\left(\frac{2\pi z}{\lambda_u}\right)$$

$$m\gamma \mathrm{v}_x = \frac{e B_0 \lambda_u}{2\pi} \sin\left(\frac{2\pi z}{\lambda_u}\right) \qquad (5.17)$$

$$\boxed{\mathrm{v}_x = \frac{Kc}{\gamma} \sin\left(\frac{2\pi z}{\lambda_u}\right)} \qquad (5.19)$$

$$\boxed{K \equiv \frac{e B_0 \lambda_u}{2\pi mc} = 0.9337\, B_0(\mathrm{T}) \lambda_u(\mathrm{cm})} \qquad (5.18)$$

is the non-dimensional "magnetic deflection parameter."
The "deflection angle", θ, is

$$\theta = \frac{\mathrm{v}_x}{\mathrm{v}_z} \simeq \frac{\mathrm{v}_x}{c} = \frac{K}{\gamma}\, \sin k_u z$$

The Axial Velocity Depends on K

In a magnetic field γ is a constant; to first order the electron neither gains nor looses energy.

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v_x^2+v_z^2}{c^2}}}$$

thus

$$\frac{v_z^2}{c^2} = 1 - \frac{1}{\gamma^2} - \frac{v_x^2}{c^2} \qquad (5.22)$$

$$\frac{v_z^2}{c^2} = 1 - \frac{1}{\gamma^2} - \frac{K^2}{\gamma^2}\sin^2\left(\frac{2\pi z}{\lambda_u}\right)$$

Taking the square root, to first order in the small parameter K/γ

$$\frac{v_z}{c} = 1 - \frac{1}{2\gamma^2} - \frac{K^2}{2\gamma^2}\sin^2\left(\frac{2\pi z}{\lambda_u}\right) \qquad (5.23a)$$

Using the double angle formula $\sin^2 k_u z = (1 - \cos 2k_u z)/2$, where $k_u = 2\pi/\lambda_u$,

$$\frac{v_z}{c} = 1 - \frac{1+K^2/2}{2\gamma^2} + \frac{K^2}{4\gamma^2}\cos\left(2\cdot\frac{2\pi z}{\lambda_u}\right)$$

Reduced axial velocity — A double frequency component of the motion

The first two terms show the reduced axial velocity due to the finite magnetic field (K). The last term indicates the presence of harmonic motion, and thus harmonic frequencies of radiation.

K-Dependent Axial Velocity Affects the Undulator Equation

Averaging the z-component of velocity over a full cycle (or N full cycles) gives

$$\frac{\bar{v}_z}{c} = 1 - \frac{1 + K^2/2}{2\gamma^2} \qquad (5.25)$$

We can use this to define an effective Lorentz factor γ^* in the axial direction

$$\gamma^* \equiv \frac{\gamma}{\sqrt{1 + K^2/2}} \qquad (5.26)$$

As a consequence, the observed wavelength in the laboratory frame of reference is modified from Eq. (5.12), taking the form

$$\lambda = \frac{\lambda_u}{2\gamma^{*2}}(1 + \gamma^{*2}\theta^2)$$

that is, the Lorentz contraction and relativistic Doppler shift now involve γ^* rather than γ

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\left(1 + \frac{\gamma^2}{1 + K^2/2}\theta^2\right)$$

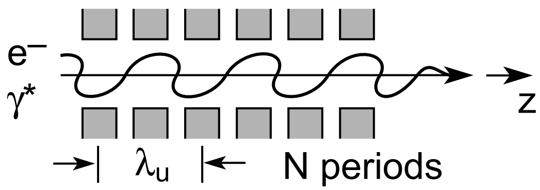
$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)} \qquad (5.28)$$

where $K \equiv e B_o \lambda_u / 2\pi mc$. This is the *undulator equation*, which describes the generation of short (x-ray) wavelength radiation by relativistic electrons traversing a periodic magnet structure, accounting for magnetic tuning (K) and off-axis ($\gamma\theta$) radiation. In practical units

$$\lambda(\text{nm}) = \frac{1.306\lambda_u(\text{cm})\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)}{E_e^2(\text{GeV})} \qquad (5.29a)$$

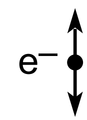
Calculating Power in the Central Radiation Cone: Using the well known "dipole radiation" formula by transforming to the frame of reference moving with the electrons

x, z, t laboratory frame of reference



Lorentz transformation

x′, z′, t′ frame of reference moving with the average velocity of the electron



$\lambda'_u = \dfrac{\lambda_u}{\gamma^*}$

x', z', t' motion

$a'(t')$ acceleration

Determine *x, z, t* motion:

$$\frac{d\mathbf{p}}{dt} = -e\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$m\gamma \frac{dv_x}{dt} = e\,\frac{dz}{dt} B_0 \cos \frac{2\pi z}{\lambda_u}$$

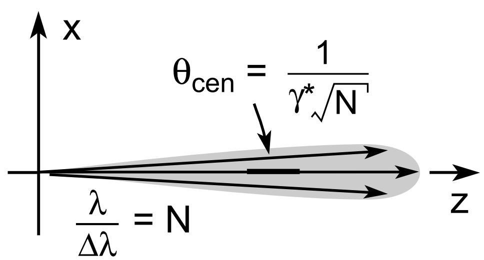
$$v_x(t);\ a_x(t) = \cdots$$

$$v_z(t);\ a_z(t) = \cdots$$

Dipole radiation:

$$\frac{dP'}{d\Omega'} = \frac{e^2 a'^2 \sin^2 \Theta'}{16\pi^2 \epsilon_0 c^3}$$

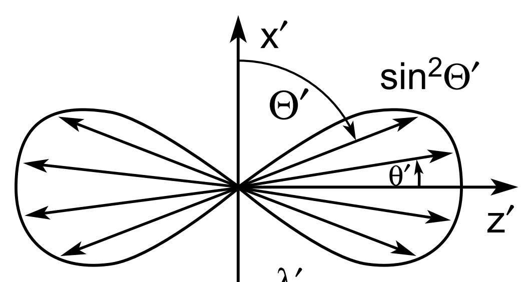
$$\frac{dP'}{d\Omega'} = \frac{e^2 c \gamma^2}{4\epsilon_0 \lambda_u^2} \frac{K^2}{(1 + K^2/2)^2} (1 - \sin^2 \theta' \cos^2 \phi') \cos^2 \omega'_u t'$$



$\theta_{cen} = \dfrac{1}{\gamma^* \sqrt{N}}$

$\dfrac{\lambda}{\Delta\lambda} = N$

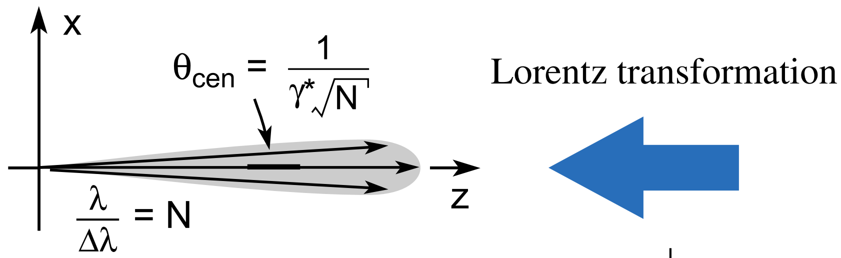
Lorentz transformation

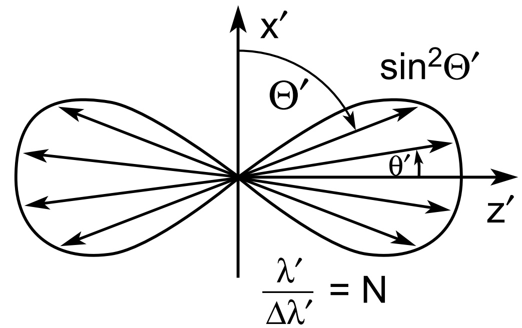


$\sin^2\Theta'$, Θ', θ'

$\dfrac{\lambda'}{\Delta\lambda'} = N$

Calculating Power in the Central Radiation Cone: Using the well known "dipole radiation" formula by transforming to the frame of reference moving with the electrons (cont.)





$$\frac{dP}{d\Omega} = 8\gamma^{*2}\frac{dP'}{d\Omega'}$$

$$\frac{d\bar{P}}{d\Omega} = \frac{e^2 c\gamma^4}{\epsilon_0\lambda_u^2}\frac{K^2}{(1+K^2/2)^3}\begin{cases} K \le 1 \\ \theta \le \theta_{cen}\end{cases}$$

$$\Delta\Omega_{cen} = \pi\theta_{cen}^2 = \pi/\gamma^{*2} N$$

$$\bar{P}_{cen} = \frac{\pi e^2 c\gamma^2}{\epsilon_0\lambda_u^2 N}\frac{K^2}{(1+K^2/2)^2}$$

N_e uncorrelated electrons:

$$N_e = IL/ec,\ L = N\lambda_u$$

$$\bar{P}_{cen} = \frac{\pi e\gamma^2 I}{\epsilon_0\lambda_u}\frac{K^2}{(1+K^2/2)^2}$$

$$\frac{dP'}{d\Omega'} = \frac{e^2 c\gamma^2}{4\epsilon_0\lambda_u^2}\frac{K^2}{(1+K^2/2)^2}(1-\sin^2\theta'\cos^2\phi')\cos^2\omega'_u t'$$

Power Radiated in the Central Radiation Cone

To use the "dipole radiation" formula

$$\frac{dP'}{d\Omega'} = \frac{e^2\, a'^2 \sin^2 \Theta'}{16\pi^2 \epsilon_0 c^3}$$

we need the acceleration a′(t′) in the frame of reference moving with the electron.
We already know the velocity in the laboratory frame of reference

$$\mathrm{v}_x = \frac{Kc}{\gamma} \sin k_u z \,,\; k_u = 2\pi/\lambda_\mathrm{u}$$

Take $z \simeq ct$, and $\omega_\mathrm{u} = k_\mathrm{u} c$

$$\mathrm{v}_x = \frac{Kc}{\gamma} \sin \omega_u t$$

Integrating once

$$x \simeq -\frac{K}{k_u \gamma} \cos \omega_u t$$

We can use the Lorentz transformation to the primed frame of reference

$$t = \gamma^* \left(t' + \frac{z'}{c} \right)$$

$$x = x'$$

Thus in the primed frame of reference

$$\mathrm{x}' = -\frac{K}{k_u \gamma} \cos \underbrace{\omega_\mathrm{u} \gamma^*}_{\omega_\mathrm{u}'} \left(t' + \frac{z'}{c}\right)$$

small: z′ is the small axial motion about the mean

Power Radiated in the Central Radiation Cone (cont.)

$$x' = -\frac{K}{k_u\gamma}\cos\underbrace{\omega_u\gamma^*}_{\omega_u'}\left(t' + \frac{z'}{c}\right)$$

small: z' is the small axial motion about the mean

Taking the second derivative

$$a_x' \simeq \frac{K\omega_u'^2}{k_u\gamma}\cos\omega_u' t'$$

since $\omega_u' = \gamma^* k_u c = \gamma k_u c/(1 + K^2/2)^{1/2}$

$$a_x' \simeq \frac{2\pi c^2\gamma}{\lambda_u}\frac{K}{(1 + K^2/2)}\cos\omega_u' t' \qquad (5.33)$$

Then

$$\frac{d\bar{P}'}{d\Omega'} = \frac{e^2 c\gamma^2}{8\epsilon_0\lambda_u^2}\frac{K^2}{(1 + K^2/2)^2}\sin^2\Theta' \qquad (5.34)$$

Power in the Central Radiation Cone (continued)

The central radiation cone, $\theta_{\text{cen}} = \dfrac{1}{\gamma^* \sqrt{N}}$, corresponds to only a small part of the $\sin^2 \Theta'$ radiation pattern, near $\Theta' \simeq \pi/2$, where

$$\sin^2 \Theta' \simeq 1$$

Within this small angular cone, a Lorentz transformation back to the laboratory frame of reference gives

$$\frac{dP}{d\Omega} \simeq 8\gamma^{*2} \frac{dP'}{d\Omega'}$$

so that

$$\left.\frac{d\bar{P}}{d\Omega}\right|_{e^-} \simeq \frac{e^2 c \gamma^4}{\epsilon_0 \lambda_u^2} \frac{K^2}{(1 + K^2/2)^3} \qquad (K \leq 1,\ \theta \leq \theta_{\text{cen}}) \qquad (5.37)$$

For the central radiation cone (1/N relative spectral bandwidth)

$$\Delta\Omega_{\text{cen}} = \pi\theta_{\text{cen}}^2 = \pi/(\gamma^* \sqrt{N})^2$$

So that for a single electron, the power radiated into the central cone is

$$\left.\bar{P}_{\text{cen}}\right|_{e^-} \simeq \frac{\pi e^2 c \gamma^2}{\epsilon_0 \lambda_u^2 N} \frac{K^2}{(1 + K^2/2)^2} \qquad (5.38)$$

Power in the Central Radiation Cone (continued)

$$\bar{P}_{cen}\Big|_{e^-} \simeq \frac{\pi e^2 c \gamma^2}{\epsilon_0 \lambda_u^2 N} \frac{K^2}{(1+K^2/2)^2} \tag{5.38}$$

For N_e electrons radiating independently within the undulator

$$N_e = IL/ec, \text{ where } L = N\lambda_u$$

The power radiated into the central cone is then

$$\bar{P}_{cen} \simeq \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{(1+K^2/2)^2} \qquad (K \leq 1) \tag{5.39}$$

or

$$\bar{P}_{cen} = (5.69 \times 10^{-6}\ \text{W}) \frac{\gamma^2 I(\text{A})}{\lambda_u(\text{cm})} \frac{K^2}{(1+K^2/2)^2} \tag{5.41b}$$

Corrections to $\bar{P}_{cen}$ for Finite K

Our formula for calculated power in the central radiation cone ($\theta_{cen} = 1/\gamma^*\sqrt{N}$, $\Delta\lambda/\lambda = 1/N$)

$$\bar{P}_{cen} \simeq \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{(1 + K^2/2)^2} \qquad (5.39)$$

is strictly valid for $K << 1$. This restriction is due to our neglect of K^2 terms in the axial velocity v_z. The $\bar{P}_{cen}$ formula, however, indicates a peak power at $K = \sqrt{2}$, suggesting that we explore extension of this very useful analytic result to somewhat higher K values. Kim* has studied undulator radiation for arbitrary K and finds an additional multiplicative factor, $f(K)$, which accounts for energy transfer to higher harmonics:

$$\bar{P}_{cen} = \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{(1 + K^2/2)^2} f(K) \qquad (5.41a)$$

where

$$f(K) = [J_0(x) - J_1(x)]^2 \qquad (5.40a)$$

and

$$x = K^2/4(1 + K^2/2)$$

$$f(K) = 1 - x - \frac{x^2}{4} + \frac{3x^3}{8} + \cdots \qquad (5.40b)$$

K	x	$f(K)$
0	0	1.000
0.5	0.0556	0.944
1.0	0.1667	0.828
$\sqrt{2}$	0.2500	0.740
1.5	0.2647	0.725
2.0	0.3333	0.653
2.5	0.3788	0.606

* K.-J. Kim, "Characteristics of Synchrotron Radiation", pp. 565-632 in *Physics of Particle Accelerators* (AIP, New York, 1989), M. Month and M. Dienes, Editors.

Also see: P.J. Duke, *Synchrotron Radiation* (Oxford Univ. Press, UK, 2000).
A. Hofmann, "The Physics of Synchrotron Radiation" (Cambridge Univ. Press, 2004).

$\bar{P}_{cen}$ in Terms of Photon Energy

From the undulator equation

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

On axis, $\theta = 0$, and with $f\lambda = c$

$$f = 2\gamma^2 c\,/\lambda_u\left(1 + \frac{K^2}{2}\right)$$

In terms of photon energy (on-axis)

$$\hbar\omega = 4\pi\hbar\gamma^2 c\,/\lambda_u\left(1 + \frac{K^2}{2}\right)$$

We can now replace $K^2/(1 + K^2/2)^2$ in $\bar{P}_{cen}$ by an expression involving $\hbar\omega$. Introducing the limiting photon energy $\hbar\omega_o$, corresponding to $K = 0$,

$$\hbar\omega_o = 4\pi\hbar\gamma^2 c\,/\lambda_u$$

then

$$\bar{P}_{cen} = \frac{2\pi e\gamma^2 I}{\epsilon_0\lambda_u}\cdot\frac{\hbar\omega}{\hbar\omega_0}\left(1 - \frac{\hbar\omega}{\hbar\omega_0}\right) f(\hbar\omega/\hbar\omega_0) \quad (5.41c)$$

or

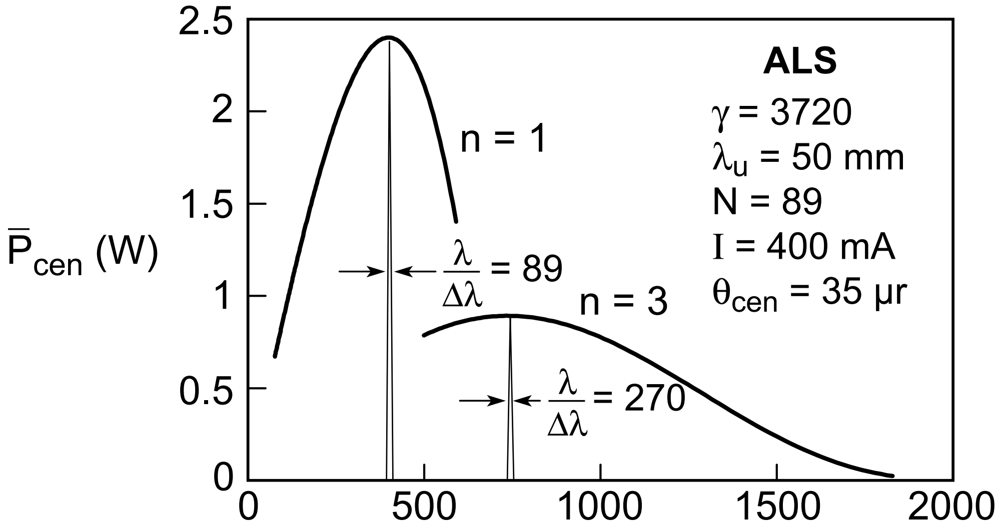
$$\bar{P}_{cen} = (1.14 \times 10^{-5}\ \mathrm{W})\frac{\gamma^2 I(\mathrm{A})}{\lambda_u(\mathrm{cm})}\cdot\frac{\hbar\omega}{\hbar\omega_0}\left(1 - \frac{\hbar\omega}{\hbar\omega_0}\right) f(\hbar\omega/\hbar\omega_0) \quad (5.41e)$$

where

$$f(\hbar\omega/\hbar\omega_0) \simeq \frac{7}{16} + \frac{5}{8}\frac{\hbar\omega}{\hbar\omega_0} - \frac{1}{16}\left(\frac{\hbar\omega}{\hbar\omega_0}\right)^2 + \cdots \quad (5.41d)$$

For $\lambda_u = 5.00$ cm and $\gamma = 3720$, $\hbar\omega_o = 686$ eV. For $\lambda_u = 3.30$ cm and $\gamma = 13{,}700$, $\hbar\omega_o = 14.1$ keV

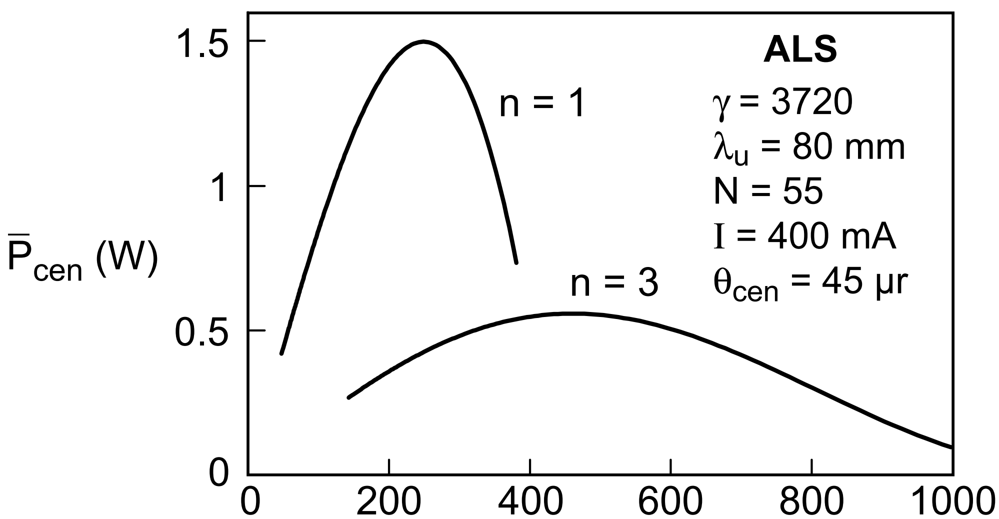
Power in the Central Radiation Cone For Three Soft X-Ray Undulators

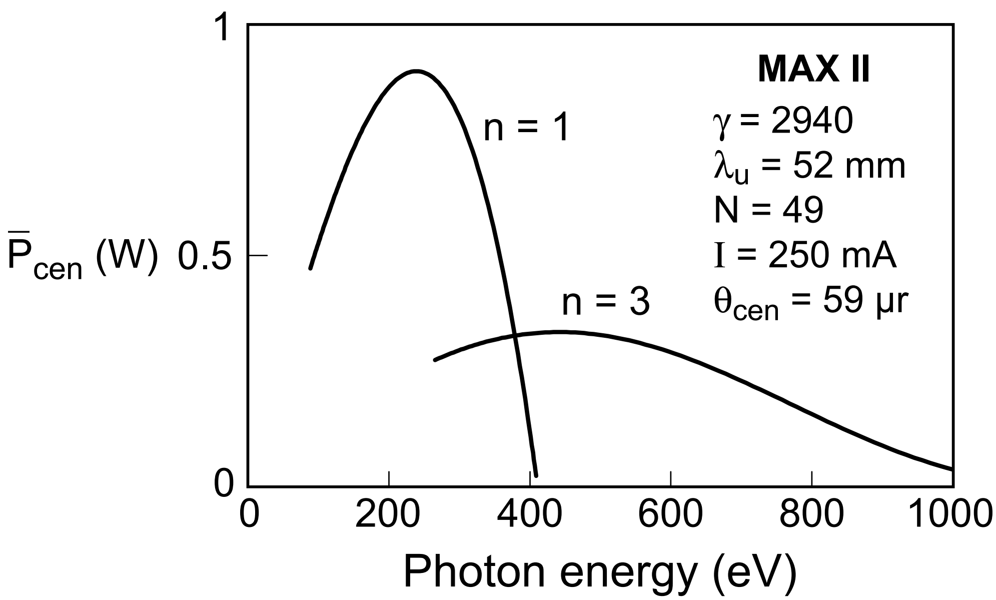


$$\theta_{cen} = \frac{1}{\gamma^* \sqrt{N}}$$

$$\left[\frac{\Delta\lambda}{\lambda}\right]_1 = \frac{1}{N}$$

$$\left[\frac{\Delta\lambda}{\lambda}\right]_3 = \frac{1}{3N}$$





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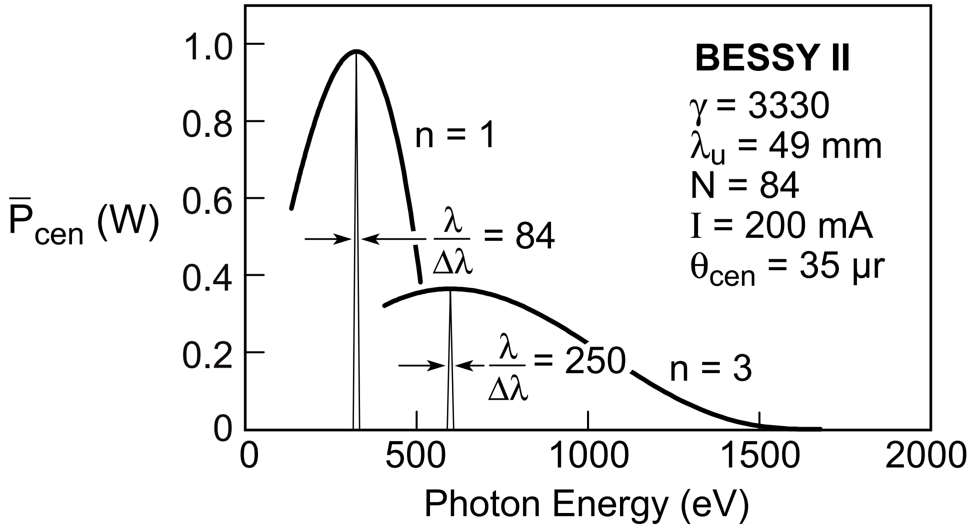

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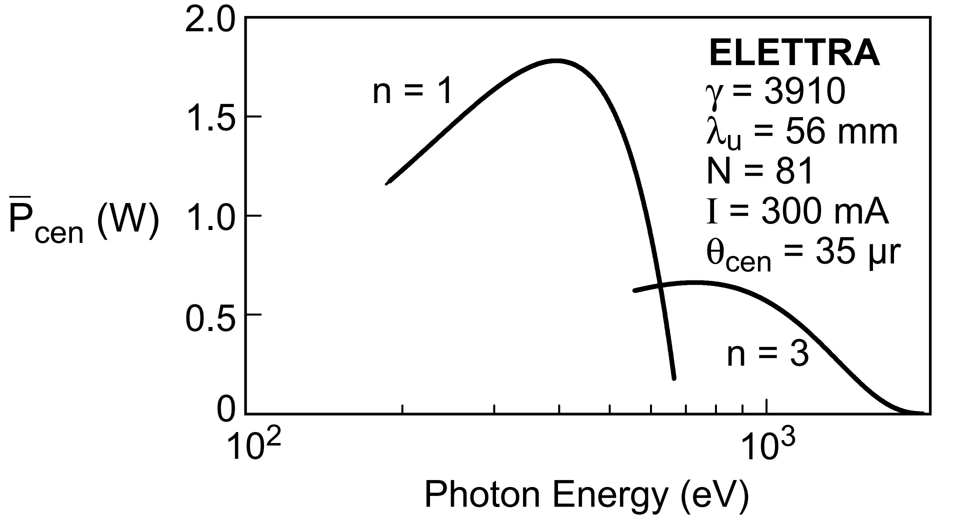
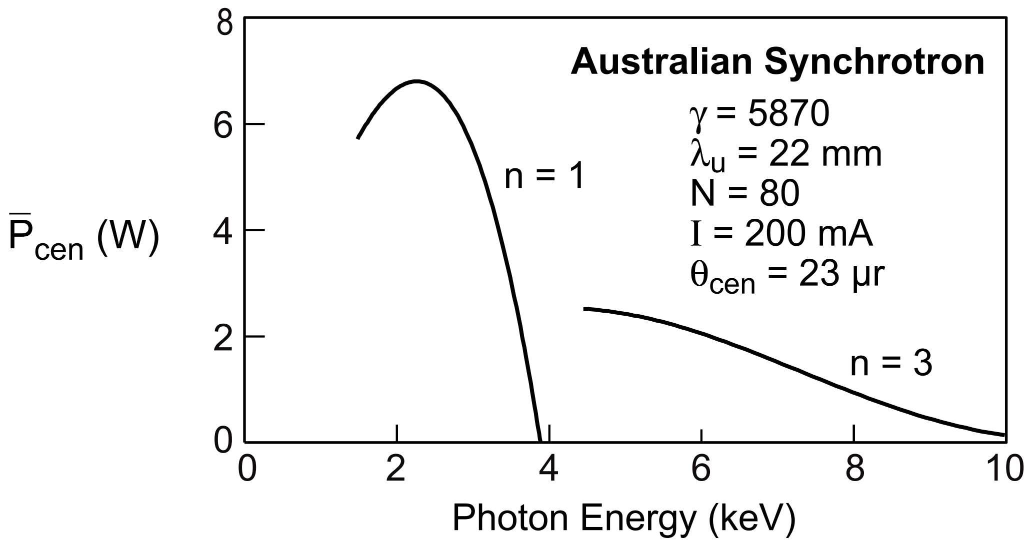
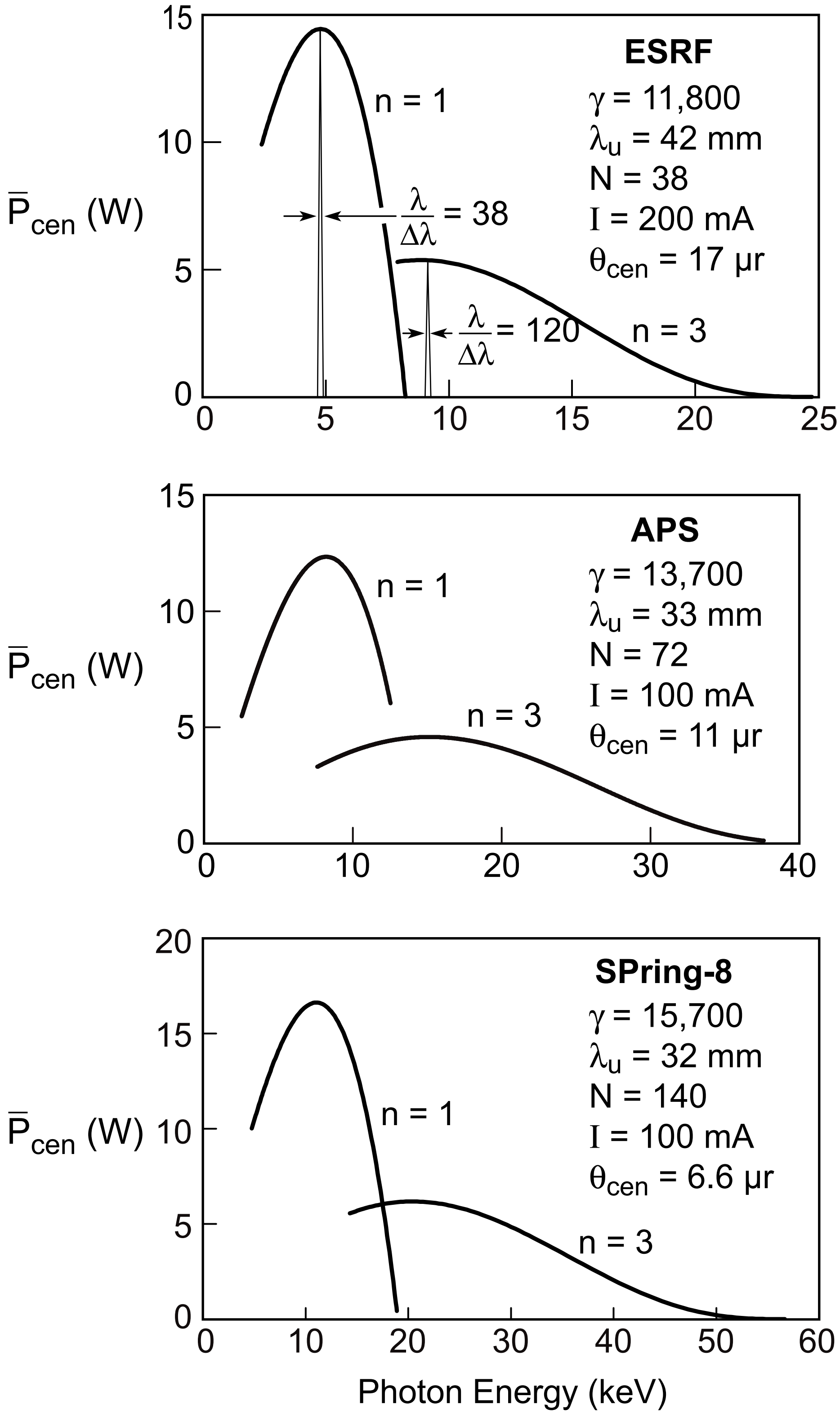




Power in the Central Radiation Cone For Three X-Ray Undulators



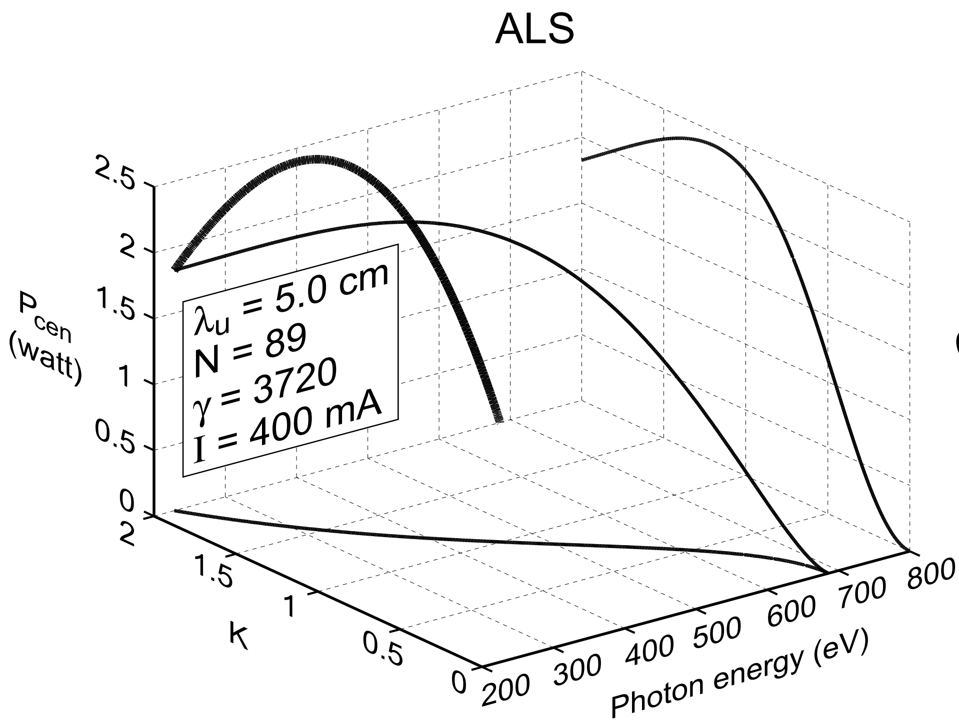
$$\theta_{cen} = \frac{1}{\gamma^* \sqrt{N}}$$

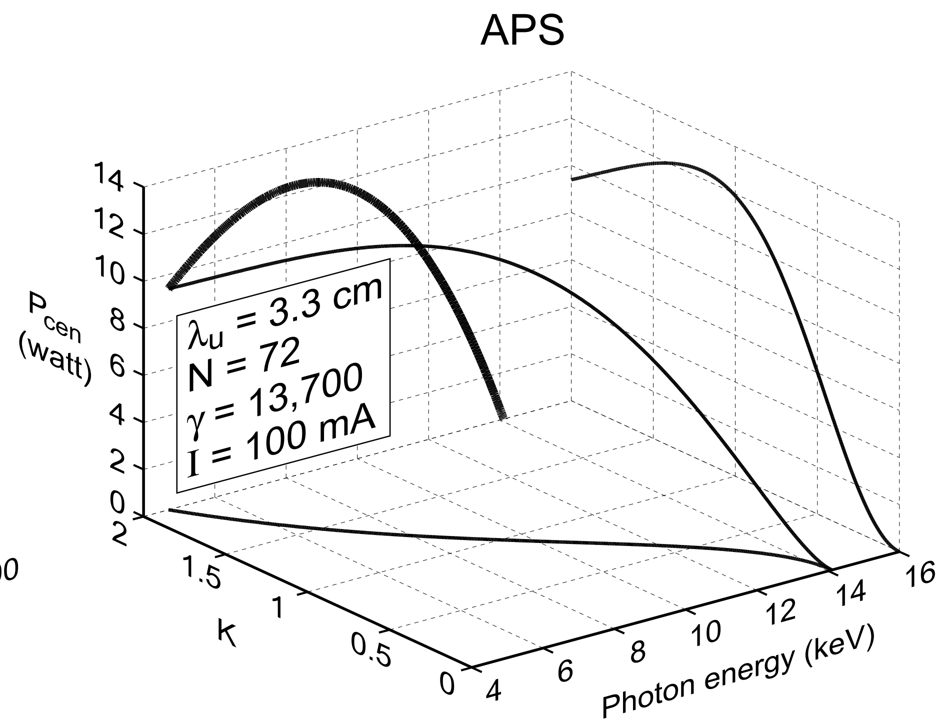
$$\left[\frac{\Delta\lambda}{\lambda}\right]_1 = \frac{1}{N}$$

$$\left[\frac{\Delta\lambda}{\lambda}\right]_3 = \frac{1}{3N}$$

Power Radiated into the Central Radiation Cone

($\theta_{cen} = 1/\gamma^* \sqrt{N}$, $\lambda/\Delta\lambda = N$, $n = 1$)

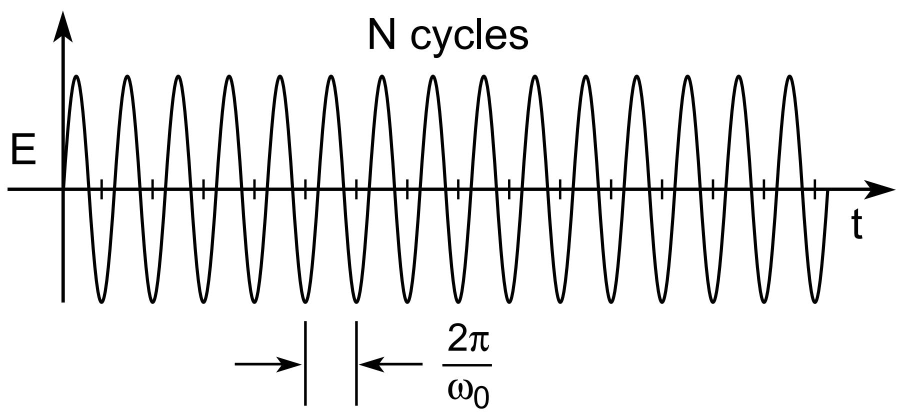




Spectral Bandwidth of Undulator Radiation from a Single Electron

(On-axis radiation, $\theta = 0$)

Radiated Wavetrain



Spectral Distribution