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Dissipation Inequalities and Quadratic Constraints for Control, Optimization, and Learning

Lesson 3: Interconnected Systems

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Learning Objectives

In this lesson, we will:

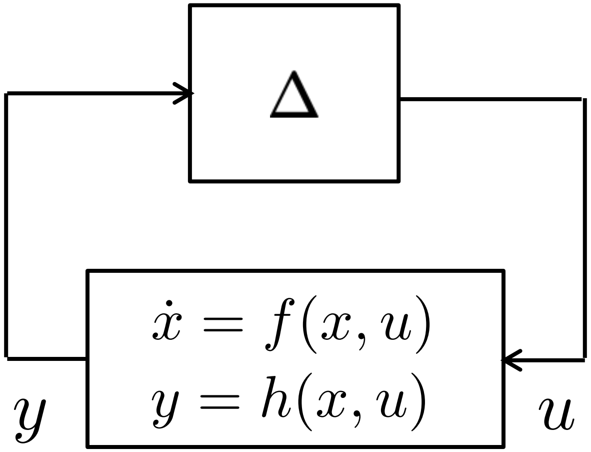
- Learn how to leverage dissipativity for modular analysis of stability and performance of interconnected systems
- Learn about computational methods to aid in the analysis
- Introduce variants of dissipativity to enable complete modularity
- Develop a deeper understanding with application examples

Outline

1. Stability of interconnections
2. Application examples
3. Performance of interconnections
4. Searching through supply rates
5. Equilibrium-independent dissipativity
6. Equilibrium-independent stability test
7. Case study: vehicle platoon
8. Delta dissipativity

Stability of Interconnections

Recall the robust stability test from Lesson 1:



Robust stability: Suppose the system $\dot{x} = f(x,u) \;\; y = h(x,u)$ is dissipative with supply rate $s(u,y)$ and pos.def. storage function V. If Δ satisfies the complementary constraint

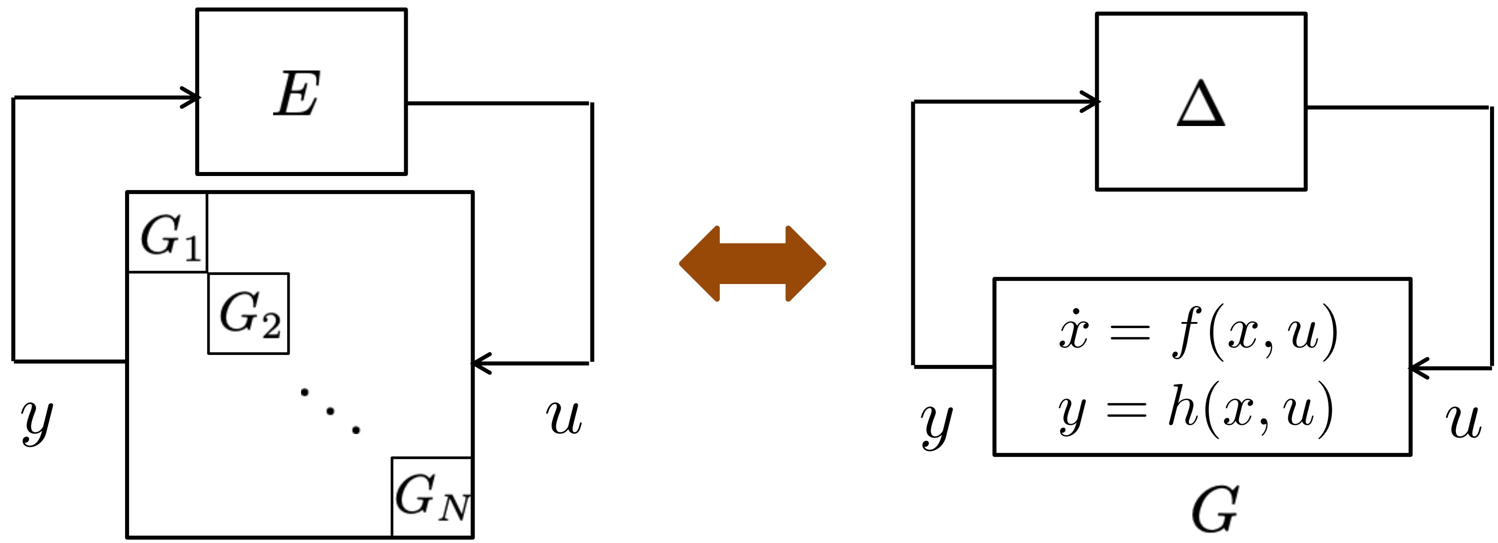
$$s(u,y) \leq 0$$

for all (u,y) such that $u = \Delta(y)$, then the origin is stable because

$$L_f V(x,u) \leq s(u,y) \leq 0.$$

Stability of Interconnections

Repurpose this criterion to study the interconnection:



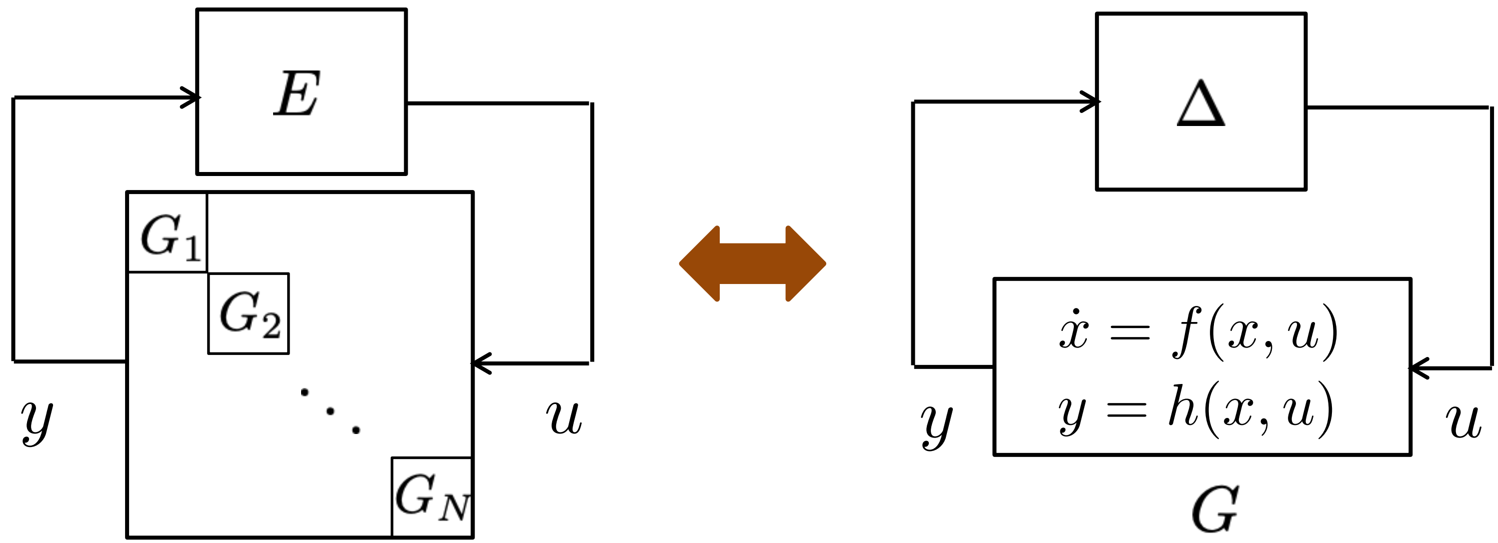
$$G_i: \; \dot{x}_i = f_i(x_i, u_i)$$
$$y_i = h_i(x_i, u_i)$$

E : interconnection matrix

View $x, u, y, f(x,u), h(x,u)$ as concatenations $i = 1, \ldots, N$ of $x_i, u_i, y_i, f_i(x_i,u_i), h_i(x_i,u_i)$

Stability of Interconnections

Repurpose this criterion to study the interconnection:



If each G_i is dissipative with supply rate $s_i(u_i, y_i)$ and pos.def. V_i then G is dissipative with

$$s(u, y) = \sum_{i=1}^{N} p_i s_i(u_i, y_i), \; p_i > 0,$$

and $V(x) = \sum_{i=1}^{N} p_i V_i(x_i)$ is a pos.def. storage function.

Stability of Interconnections

Thus, the condition $s(u, y) \leq 0$ to be satisfied by Δ becomes:

$$\exists p_i > 0 \quad \text{s.t.} \quad \underbrace{\sum_{i=1}^{N} p_i s_i(u_i, y_i)}_{s(u,y)}\Big|_{u=Ey} \leq 0 \quad \forall y$$

For quadratic supply rates $s_i(u_i, y_i) = \begin{bmatrix} u_i \\ y_i \end{bmatrix}^\top X_i \begin{bmatrix} u_i \\ y_i \end{bmatrix}$ we can turn this condition into an LMI:

$$\sum_{i=1}^{N} p_i s_i(u_i, y_i) = \begin{bmatrix} u_1 \\ y_1 \\ \vdots \\ u_N \\ y_N \end{bmatrix}^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} \underbrace{\begin{bmatrix} u_1 \\ y_1 \\ \vdots \\ u_N \\ y_N \end{bmatrix}}_{= S \begin{bmatrix} u \\ y \end{bmatrix}}$$

Define a permutation matrix S to sort inputs and outputs: $= S \begin{bmatrix} u \\ y \end{bmatrix}$

Stability of Interconnections

Then,

$$\sum_{i=1}^{N} p_i s_i(u_i, y_i) = \begin{bmatrix} u \\ y \end{bmatrix} S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \begin{bmatrix} u \\ y \end{bmatrix}$$

Substitute $u = Ey$:

$$= y^\top \begin{bmatrix} E \\ I \end{bmatrix}^\top S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \begin{bmatrix} E \\ I \end{bmatrix} y$$

Thus, if we can find $p_i > 0, i = 1, \cdots, N,$ such that

$$\begin{bmatrix} E \\ I \end{bmatrix}^\top S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \begin{bmatrix} E \\ I \end{bmatrix} \preceq 0$$

(LMI)

then the origin is stable for the interconnection and a Lyapunov function is $V(x) = \sum_{i=1}^{N} p_i V_i(x_i)$

Stability of Interconnections

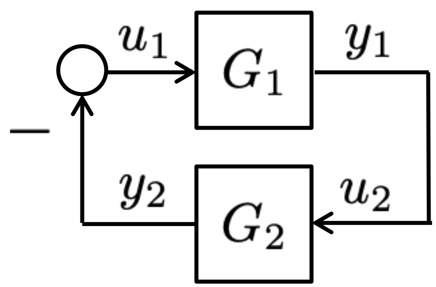
Special Case 1: Small Gain

When $X_i = \begin{bmatrix} \gamma_i^2 & 0 \\ 0 & -1 \end{bmatrix}$, (LMI) simplifies to:

$$(\Gamma E)^\top P (\Gamma E) - P \preceq 0$$

where $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_N), P = \mathrm{diag}(p_1, \cdots, p_N)$

Example:



$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{E} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Apply criterion above: we can find $p_1 > 0, p_2 > 0$ such that

$$\begin{bmatrix} p_2\gamma_2^2 & 0 \\ 0 & p_1\gamma_1^2 \end{bmatrix} - \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \preceq 0 \;\Leftrightarrow\; \begin{matrix} p_2\gamma_2^2 \le p_1 \\ p_1\gamma_1^2 \le p_2 \end{matrix} \;\Leftrightarrow\; \gamma_2^2 \le \frac{p_1}{p_2} \le \frac{1}{\gamma_1^2}$$

if and only if $\gamma_1\gamma_2 \le 1$. This is the well-known small-gain criterion.

Stability of Interconnections

Special Case 2: Passivity

When $X_i = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\eta_i \end{bmatrix}$, (LMI) simplifies to:

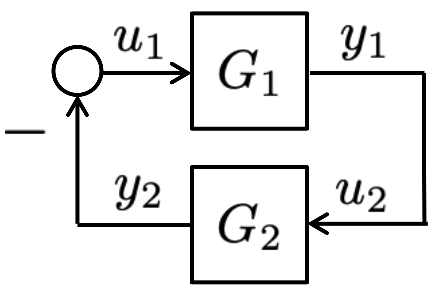
$$P(E-H) + (E-H)^\top P \preceq 0$$

where $H = \mathrm{diag}(\eta_1, \cdots, \eta_N)$, $P = \mathrm{diag}(p_1, \cdots, p_N)$

This holds with $P = I$ when $\eta_i \geq 0 \ \forall i$ and E is skew-symmetric:

$$E + E^\top = 0$$

Example:



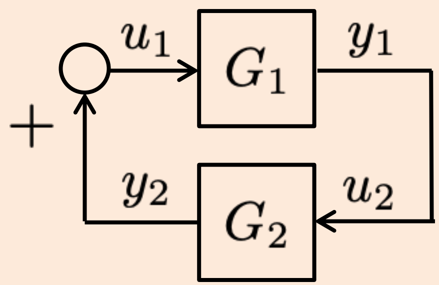
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{E} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

E is skew-symmetric; thus, negative feedback interconnection of two passive systems is stable (known as "Passivity Theorem").

Stability of Interconnections

Wake-up Problems

Consider now a *positive* feedback interconnection:



$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{E} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

1) Let each system be passive with $\eta_i = 0, \mathrm{i.e.}, H = 0$. Can you find

$$P = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \succ 0 \ \text{ s.t. } \ P(E - H) + (E - H)^\top P \preceq 0 \ \ ?$$

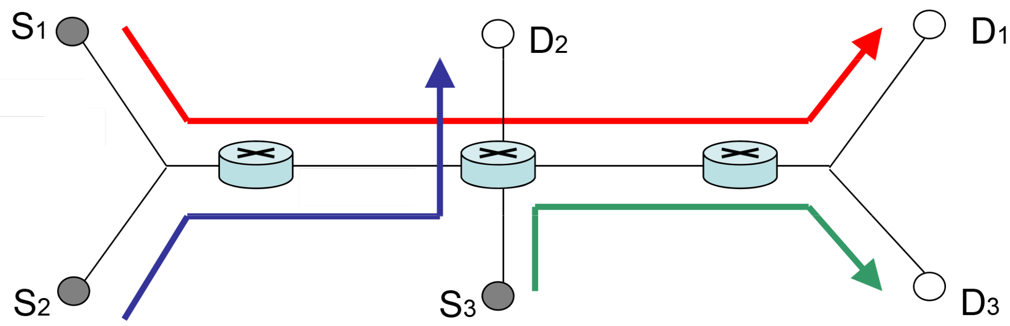
2) Now suppose G_1 and G_2 have L_2 gains γ_1, γ_2. How does the condition $(\Gamma E)^\top P (\Gamma E) - P \preceq 0$, where

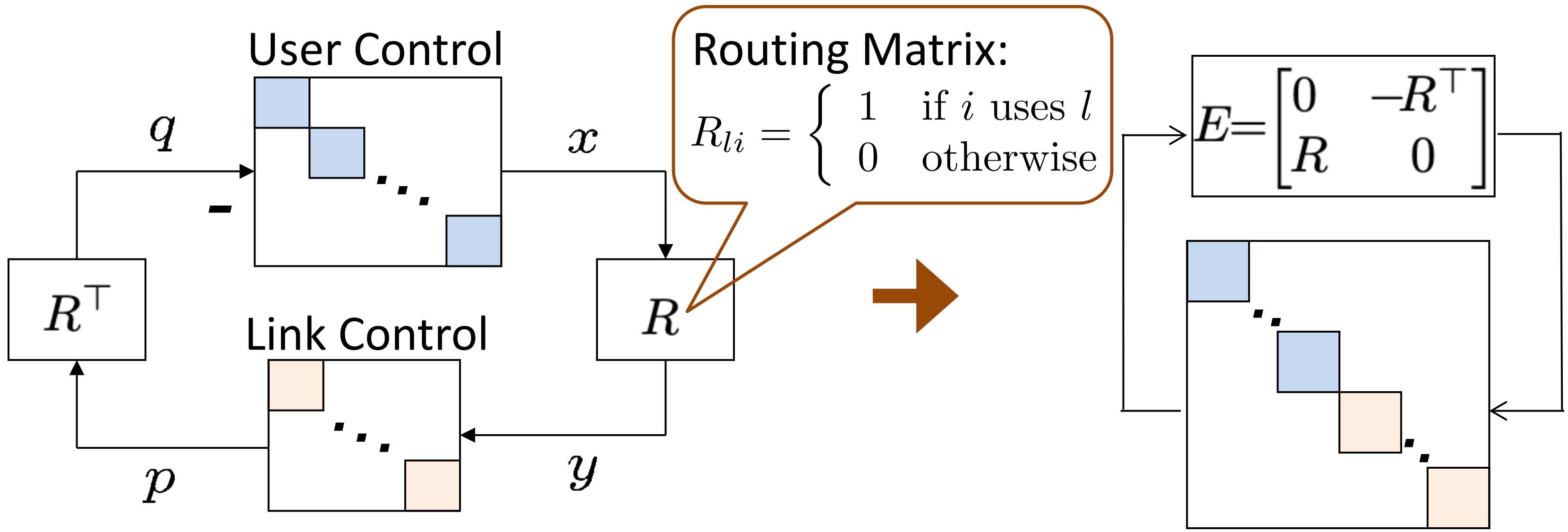
$$\Gamma = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$$

restrict the gains?

Application Examples

Internet Congestion Control



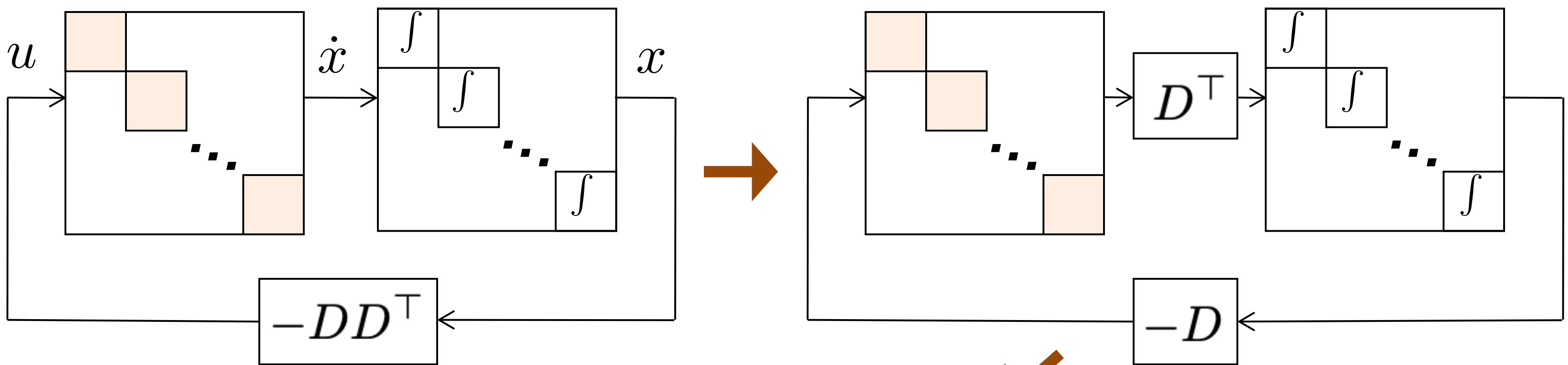


Skew symmetry of E key to stability with broad classes of user and link control protocols with passivity properties (Wen, Arcak, 2004)

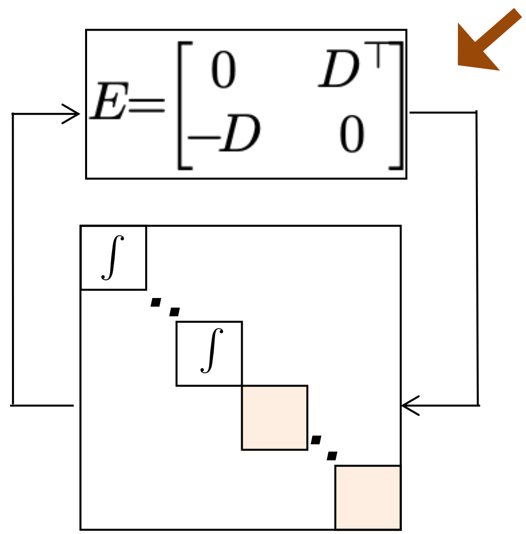
Application Examples

Multiagent Robotic Systems

Passivity intrinsic to Euler-Lagrange models of mobile robots, ships, satellites, etc. We exploit this property for motion coordination.



D: incidence matrix of communication graph

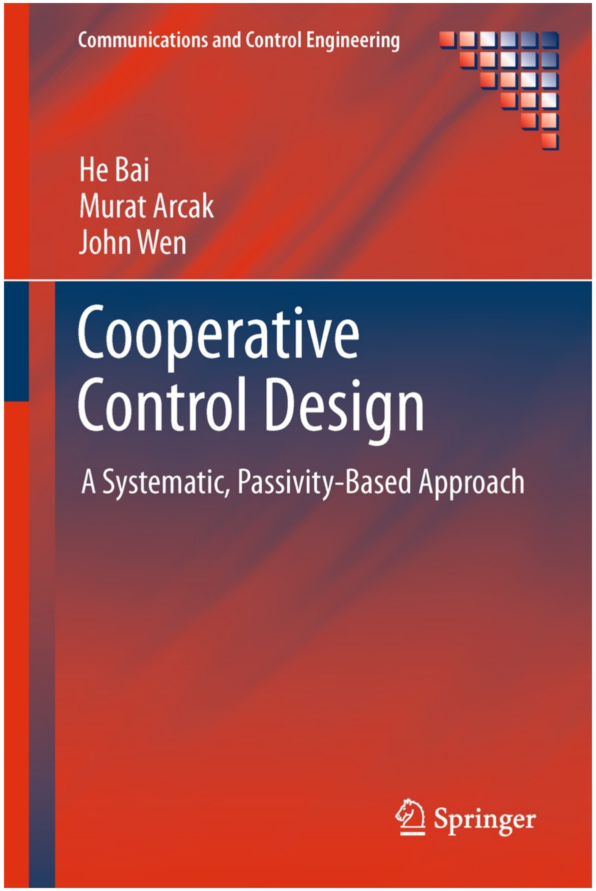


E is again skew symmetric! Stability descends from this structure and passivity of blocks.

Application Examples

Multiagent Robotic Systems

Structural property in previous slide and extensions leveraged in (Bai, Arcak,Wen, 2011) for systematic cooperative control design.



UAVs cooperatively carrying a suspended load – experiments at the Norwegian University of Science and Technology (Klausen, Meissen, Fossen, Arcak, Johansen, 2020)

Application Examples

Cyclic Interconnections

$$E = \begin{bmatrix} 0 & \cdots & 0 & \delta_1 \\ \delta_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \delta_N & 0 \end{bmatrix} \qquad \prod_{i=1}^{N} \delta_i = -1$$

(Negative feedback)

Canonical examples: ring oscillator circuits and biological analogues

Secant Criterion (Arcak, Sontag, 2006): Given output strictly passive systems with $\eta_i > 0, i = 1, \ldots, N$ and interconnection E above

$$\exists p_i > 0, i = 1, \ldots, N \quad \text{s.t.} \;\; P(E - H) + (E - H)^\top P \preceq 0$$

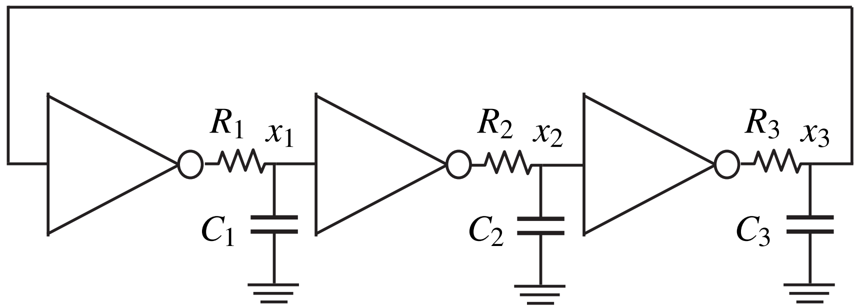
where $H = \text{diag}(\eta_1, \cdots, \eta_N)$, $P = \text{diag}(p_1, \cdots, p_N)$, if and only if

$$(\eta_1 \cdots \eta_N)^{-1} \leq \sec(\pi/N)^N$$

The bound is ∞ for $N = 2$ (recovers Passivity Thm), 8 for $N = 3$, and decreases to 1 as $N \to \infty$ (always less restrictive than small gain).

Application Examples

Example: three-stage ring oscillator circuit



$$\tau_1 \dot{x}_1 = -x_1 - h_3(x_3)$$
$$\tau_2 \dot{x}_2 = -x_2 - h_1(x_1)$$
$$\tau_3 \dot{x}_3 = -x_3 - h_2(x_2)$$

$$\tau_i = R_i C_i, \; i = 1, 2, 3$$

$$0 < x h_i(x) \le \beta_i x^2, \; x \neq 0$$

i.e., sector $[0, \beta_i]$

Decompose into subsystems:

$$\tau_i \dot{x}_i = -x_i + u_i$$
$$y_i = h_i(x_i)$$

$i = 1, 2, 3,$ w/ interconnection:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Application Examples

From Lesson 1, the subsystems are passive with storage function:

$$V_i(x_i) = \tau_i \int_0^{x_i} h_i(s)ds$$

$$V_i'(x_i)(-x_i + u_i)/\tau_i = h_i(x_i)(-x_i + u_i) = -x_i h_i(x_i) + u_i y_i$$

Recall from Lesson 2: if nonlinearity $h(\cdot)$ belongs to sector $[\alpha, \beta]$

$$\begin{bmatrix} x \\ h(x) \end{bmatrix}^\top \begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix} \begin{bmatrix} x \\ h(x) \end{bmatrix} \geq 0$$

With $\alpha = 0$ we get: $-x_i h_i(x_i) \leq -\beta_i^{-1} h_i(x_i)^2$. Thus,

$$V_i'(x_i)(-x_i + u_i)/\tau_i \leq -\eta_i y_i^2 + u_i y_i, \;\; \eta_i = \beta_i^{-1}$$

Secant criterion guarantees stability if $(\eta_1\eta_2\eta_3)^{-1} = \beta_1\beta_2\beta_3 \leq 8$

Application Examples

Wake-up Problems

1) Consider the ring oscillator example and suppose

$$h_i(x_i) = \beta \tanh(x_i), \ i = 1, 2, 3, \quad \beta \leq 2$$

What is a Lyapunov function resulting from the method discussed? (You can look up integrals on line.)

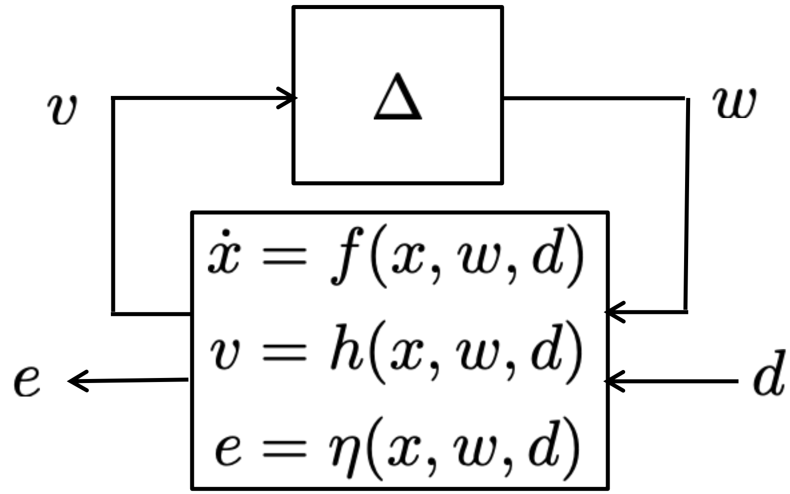
2) Now take $h_i(x_i) = \beta x_i$, which is a linear approximation around the origin, and let $\tau_i = 1, i = 1, 2, 3$. Write the model

$$\tau_1 \dot{x}_1 = -x_1 - h_3(x_3)$$
$$\tau_2 \dot{x}_2 = -x_2 - h_1(x_1)$$
$$\tau_3 \dot{x}_3 = -x_3 - h_2(x_2)$$

as $\dot{x} = Ax$. How does the eigenvalue test restrict β for stability?

Performance of Interconnections

Recall the robust performance test from Lesson 1, where the performance criterion is dissipativity with a supply rate $\sigma(d, e)$



Robust performance: If there exists storage function $x \mapsto V(x)$ s.t.

$$L_f V(x, w, d) \leq s(w, d; v, e)$$

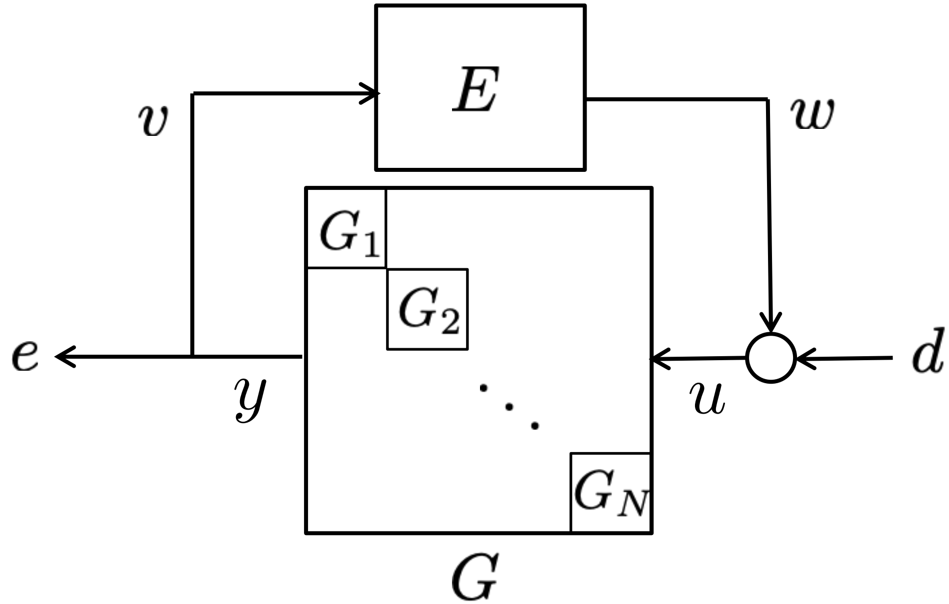
$\forall x, w, d$ and Δ restricts (v, w) such that

$$s(w, d; v, e) \leq \sigma(d, e)$$

then the interconnection is dissipative with supply rate $\sigma(d, e)$.

Performance of Interconnections

Now adapt to interconnection:



As before, each G_i is dissipative with supply rate $s_i(u_i, y_i)$ and storage function V_i. Thus, G is dissipative with supply rate

$$s(u, y) = \sum_{i=1}^{N} p_i s_i(u_i, y_i), \;\; p_i \geq 0$$

and $V(x) = \sum_{i=1}^{N} p_i V_i(x_i)$ is a storage function.

Performance of Interconnections

Moreover, the interconnection restricts u, y to:

$$u = w + d = Ee + d, \quad y = e$$

Thus, the performance condition becomes:

$$\exists p_i \geq 0 \quad \text{s.t.} \quad \textstyle\sum_{i=1}^{N} p_i s_i(u_i, y_i)\big|_{u=Ee+d, y=e} \leq \sigma(d, e)$$

For quadratic supply rates $s_i(u_i, y_i) = \begin{bmatrix} u_i \\ y_i \end{bmatrix}^\top X_i \begin{bmatrix} u_i \\ y_i \end{bmatrix}$

$$\textstyle\sum_{i=1}^{N} p_i s_i(u_i, y_i) = \begin{bmatrix} u \\ y \end{bmatrix} S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} \underbrace{S \begin{bmatrix} u \\ y \end{bmatrix}}_{= \begin{bmatrix} u_1 \\ y_1 \\ \vdots \end{bmatrix}}$$

where S is the permutation matrix defined before:

Performance of Interconnections

Since $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \begin{bmatrix} d \\ e \end{bmatrix}$ the performance condition becomes:

$$\begin{bmatrix} d \\ e \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ E^\top & I \end{bmatrix} S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \begin{bmatrix} d \\ e \end{bmatrix} \leq \sigma(d,e)$$

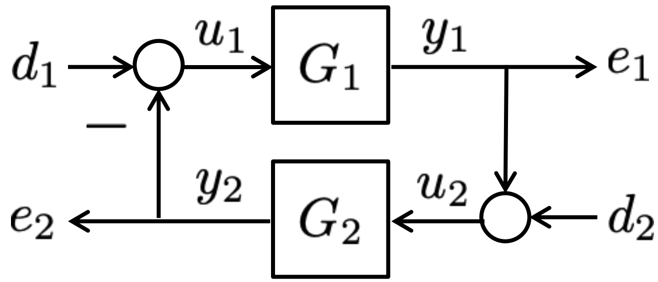
If $\sigma(d,e)$ is also quadratic: $\sigma(d,e) = \begin{bmatrix} d \\ e \end{bmatrix}^\top \Sigma \begin{bmatrix} d \\ e \end{bmatrix}$ we get the LMI:

$$\begin{bmatrix} I & 0 \\ E^\top & I \end{bmatrix} S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} - \Sigma \preceq 0$$

If $\exists p_i \geq 0$ such that this inequality holds, then the interconnection satisfies the performance criterion defined by supply rate $\sigma(d,e)$

Performance of Interconnections

Example:



$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}}^{E} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Suppose each subsystem is passive: $X_i = \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad i = 1, 2$

Then the LMI on previous slide with $P = I$ becomes:

$$\frac{1}{2}\begin{bmatrix} 0 & I \\ I & E^\top + E \end{bmatrix} - \Sigma \preceq 0$$

Since $E^\top + E = 0$ the inequality holds with $\Sigma = \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$

Thus, the negative feedback interconnection of two passive systems is itself passive – a variant of Passivity Thm with exogeneous inputs

Performance of Interconnections

Wake-up Problem

Consider the interconnection of N single-input, single-output systems, each dissipative with supply rate

$$X_i = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\eta_i \end{bmatrix}$$

Then the performance criterion simplifies to

$$\frac{1}{2}\begin{bmatrix} 0 & P \\ P & P(E-H)+(E-H)^\top P \end{bmatrix} - \Sigma \preceq 0$$

where $H = \operatorname{diag}(\eta_1, \cdots, \eta_N)$, $P = \operatorname{diag}(p_1, \cdots, p_N)$

Suppose we know $E + E^\top \preceq 0$. What is a Σ that satisfies the inequality above? What performance property does this Σ describe for the interconnection?

Searching through Supply Rates

So far we used a fixed supply rate $\{X_i\}_{i=1}^N$ for each subsystem and looked for weights $\{p_i\}_{i=1}^N$ satisfying a matrix inequality:

$$\mathcal{G}(E; p_1 X_1, \cdots, p_N X_N) \preceq 0$$

Limited flexibility. Can we search for supply rates, not just weights?

Find $\{V_i, X_i\}_{i=1}^N$ such that $\mathcal{G}(E; X_1, \cdots, X_N) \preceq 0$ **(1)**

$$\mathcal{D}_i(V_i, X_i; \xi, u) \leq 0 \quad \forall (\xi, u) \qquad \textbf{(2)}$$

where $\mathcal{D}_i(V_i, X_i; \xi, u) := \nabla V_i(\xi)^\top f_i(\xi, u) - \begin{bmatrix} u_i \\ h_i(\xi, u) \end{bmatrix}^\top X_i \begin{bmatrix} u \\ h_i(\xi, u) \end{bmatrix}$

The search for $\{V_i, X_i\}_{i=1}^N$ can be formulated as a LMI for linear systems (Lesson 1) and for polynomial systems (Lesson 5), but the combined LMI becomes intractable for large N.

Note: (2) consists of N independent constraints, coupled only by (1)

Searching through Supply Rates

Distributed Optimization Formulation:

$$\min_{x,z} \; d(x) + g(z)$$
$$s.t. \; Ax + Bz = c$$

$$z = (Z_1, \cdots, Z_N)$$
$$g(z) = \begin{cases} 0 & \text{if } \mathcal{G}(E; Z_1, \cdots, Z_N) \preceq 0 \\ \infty & \text{otherwise} \end{cases}$$

$$x = (V_1, X_1; \cdots; V_N, X_N)$$
$$d(x) = d_1(x_1) + \cdots + d_N(x_N)$$
$$d_i(x_i) = \begin{cases} 0 & \text{if } \mathcal{D}_i(V_i, X_i; \xi, u) \leq 0 \; \forall (\xi, u) \\ \infty & \text{otherwise} \end{cases}$$

$$Z_i = X_i$$

ADMM algorithm:

$$x^{k+1} = \arg\min_x \; d(x) + \|Ax + Bz^k - c + s^k\|^2$$
$$z^{k+1} = \arg\min_z \; g(z) + \|Ax^{k+1} + Bz - c + s^k\|^2$$
$$s^{k+1} = s^k + Ax^{k+1} + Bz^{k+1} - c$$

Searching through Supply Rates

Adapting to our problem (Meissen et. al, 2015):

X_i updates $i = 1, \cdots, N$:

$$X_i^{k+1} = \arg\min_{X \text{ s.t. } d_i(X)=0} \left\| X - Z_i^k + S_i^k \right\|_F^2$$

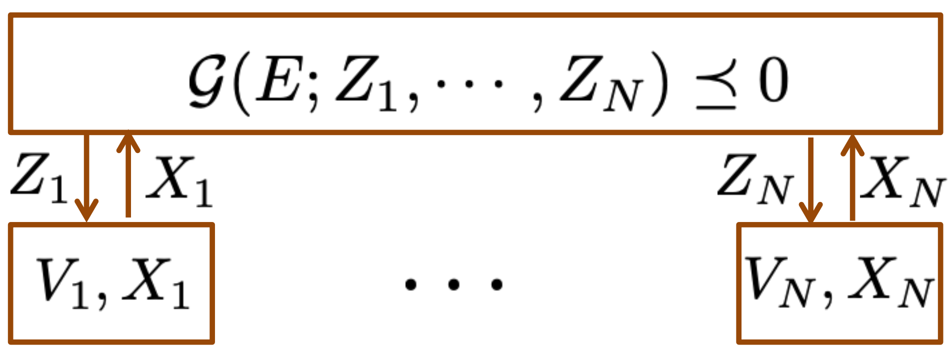
$Z_1, \cdots, Z_N$ updates:

$$Z_{1:N}^{k+1} = \arg\min_{(Z_1,\cdots,Z_N) \text{ s.t. } g(z)=0} \sum_{i=1}^{N} \left\| X_i^{k+1} - Z_i + S_i^k \right\|_F^2$$

S updates:

$$S_i^{k+1} = X_i^{k+1} - Z_i^{k+1} + S_i^k$$

enforces $Z_i = X_i$



Searching through Supply Rates

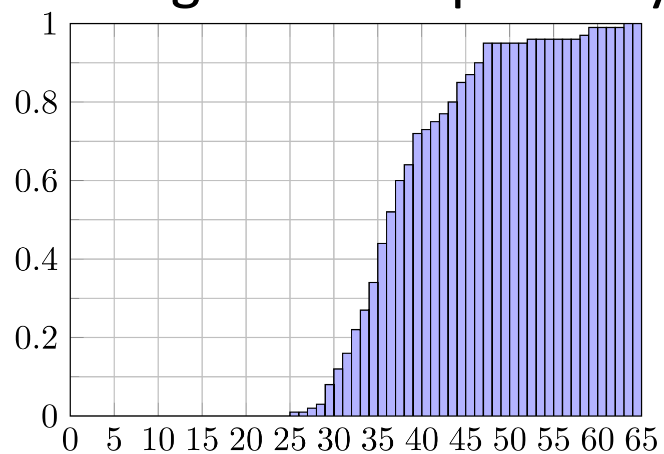
Example: We randomly generated 100 interconnection matrices $E \in \mathbb{R}^{50\times 50}$ satisfying $E + E^\top = 0$ and applied ADMM for

$$\begin{aligned} \dot{x}_i &= \begin{bmatrix} -\epsilon_i & 1 \\ -1 & -\epsilon_i \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i \qquad\qquad i = 1, \ldots, 50 \\ y_i &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_i \end{aligned}$$

The systems are passive and skew symmetry guarantees stability. We chose $\varepsilon_i > 0$ small for large L_2 gains, so not many other supply rates can satisfy the stability test.

In each trial, ADMM converged to the passivity supply rate:

Fraction of trials



Iterations to convergence

Iterations to convergence

Searching through Supply Rates

Wake-up Problem

True or False? Given linear systems

$$f_i(\xi, u) = A\xi + Bu, \;\; h_i(\xi, u) = C\xi + Du$$

with $V_i(\xi) = \xi^\top P_i \xi$, the condition

$$\mathcal{D}_i(V_i, X_i; \xi, u) \le 0 \quad \forall (\xi, u)$$

where $\mathcal{D}_i(V_i, X_i; \xi, u) := \nabla V_i(\xi)^\top f_i(\xi, u) - \begin{bmatrix} u_i \\ h_i(\xi, u) \end{bmatrix}^\top X_i \begin{bmatrix} u \\ h_i(\xi, u) \end{bmatrix}$

is a LMI with decision variables P_i, X_i.

Equilibrium-Independent Dissipativity

Stability and performance tests discussed appear to be *modular*: we can add/remove new components without having to analyze the interconnection from scratch. Instead, we use:

1) dissipativiy of blocks as abstractions of detailed dynamics;
2) LMI based on interconnection matrix for stability/performance.

A hidden obstacle to modularity:

Dissipativity of components depends on equilibrium, which itself depends on the interconnection. Do we have to analyze dissipativity all over after a change in interconnection, therefore equilibrium?

Example: Lotka-Volterra population model for interacting species

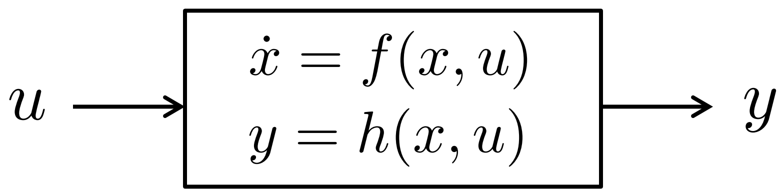
$$\dot{x}_i = \left(\lambda_i - \gamma_i x_i + \sum_{j \neq i} e_{ij} x_j \right) x_i, \quad i = 1, \ldots, N$$

Equilibrium depends on the interconnection coefficients e_{ij}.

Equilibrium-Independent Dissipativity

A stronger property that eliminates this problem (Hines et. al, 2011):

Equilibrium-Independent Dissipativity (EID): Dissipativity relative to any point that may become an equilibrium under an input bias.



Suppose, for all $\bar{x} \in \bar{\mathcal{X}} \subset \mathcal{X}$ there exists unique $\bar{u}$ s.t. $f(\bar{x}, \bar{u}) = 0$.

We call the system EID if ∃ storage function $V : \mathcal{X} \times \bar{\mathcal{X}} \to \mathbb{R}_{\geq 0}$ s.t. $\forall x \in \mathcal{X}, \bar{x} \in \bar{\mathcal{X}}, u \in \mathcal{U}$,

$$V(\bar{x}, \bar{x}) = 0, \quad \nabla_x V(x, \bar{x})^\top f(x, u) \leq s(u - \bar{u}, y - \bar{y})$$

where $\bar{u}, \bar{y}$ are functions of $\bar{x}$ through $f(\bar{x}, \bar{u}) = 0$, $\bar{y} = h(\bar{x}, \bar{u})$.

Equilibrium-Independent Dissipativity

Example: $\dot{x} = u,\ y = x,\ x \in \mathcal{X} = \mathbb{R}^n$

For every $\bar{x} \in \bar{\mathcal{X}} = \mathbb{R}^n$, $\bar{u} = 0$ is unique sol'n to $f(x, u) = u = 0$

Let $V(x, \bar{x}) = \frac{1}{2}\|x - \bar{x}\|^2$. Then, $\nabla_x V(x, \bar{x}) = x - \bar{x} = y - \bar{y}$ and

$$\nabla_x V(x, \bar{x})^\top u = (y - \bar{y})^\top u = (y - \bar{y})^\top (u - \bar{u})$$

Example: Linear system

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$
$$y = Cx + Du$$

Take $\bar{\mathcal{X}}$ to be projection of the null space of $[A,\ B]$ onto the span of first n unit vectors in $\mathbb{R}^{n+m}$. If B has full column rank, then for each $\bar{x} \in \bar{\mathcal{X}}$ there exists unique $\bar{u}$ s.t. $A\bar{x} + B\bar{u} = 0$.

Equilibrium-Independent Dissipativity

Note from $A\bar{x} + B\bar{u} = 0,\ \bar{y} = C\bar{x} + D\bar{u}$:

$$\begin{aligned} Ax + Bu &= A(x - \bar{x}) + B(u - \bar{u}) \\ y - \bar{y} &= C(x - \bar{x}) + D(u - \bar{u}) \end{aligned} \qquad \textbf{(1)}$$

Suppose the LMI for standard dissipativity from Lesson 1 holds:

$$\begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix}^\top - \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top X \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \preceq 0 \qquad \textbf{(2)}$$

$V(x, \bar{x}) = (x - \bar{x})^\top P(x - \bar{x})$ gives $\nabla_x V(x, \bar{x}) = 2P(x - \bar{x})$ and, from (1)-(2):

$$\begin{aligned} \nabla_x V(x, \bar{x})^\top (Ax + Bu) &= \begin{bmatrix} x - \bar{x} \\ u - \bar{u} \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} \begin{bmatrix} x - \bar{x} \\ u - \bar{u} \end{bmatrix} \\ &\leq \begin{bmatrix} u - \bar{u} \\ y - \bar{y} \end{bmatrix}^\top X \begin{bmatrix} u - \bar{u} \\ y - \bar{y} \end{bmatrix} \end{aligned}$$

Thus dissipativity equivalent to EID for linear systems.

Equilibrium-Independent Dissipativity

Example: $\dot{x} = f(x) + g(x)u,\ y = h(x),\ x \in \mathcal{X} = \mathbb{R}$

equilibrium-independent dissipative with supply rate $X = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\varepsilon \end{bmatrix}$

if $g(x) > 0\, \forall x$, h increasing, $\phi = \frac{f}{g} + \varepsilon h$ nonincreasing function:

$$V(x,\bar{x}) = \int_{\bar{x}}^{x} \frac{h(s) - h(\bar{x})}{g(s)} ds$$

$$\nabla_x V(x,\bar{x}) f(x,u) = \frac{h(x) - h(\bar{x})}{g(x)} (f(x) + g(x)u)$$

$$= (y - \bar{y}) \left(\frac{f(x)}{g(x)} + u - \frac{f(\bar{x})}{g(\bar{x})} - \bar{u} \right)$$

$$= (y - \bar{y})(\phi(x) - \phi(\bar{x}) + u - \bar{u} - \varepsilon(y - \bar{y}))$$

$$\leq (y - \bar{y})(u - \bar{u}) - \varepsilon(y - \bar{y})^2$$

$y - \bar{y}$ has same sign as $x - \bar{x}$, opposite to that of $\phi(x) - \phi(\bar{x})$

Equilibrium-Independent Dissipativity

Wake-up Problem

Consider the following system, defined on $\mathcal{X} = (0, \infty)$:

$$\dot{x} = (\lambda - \gamma x + u)x$$
$$y = x$$

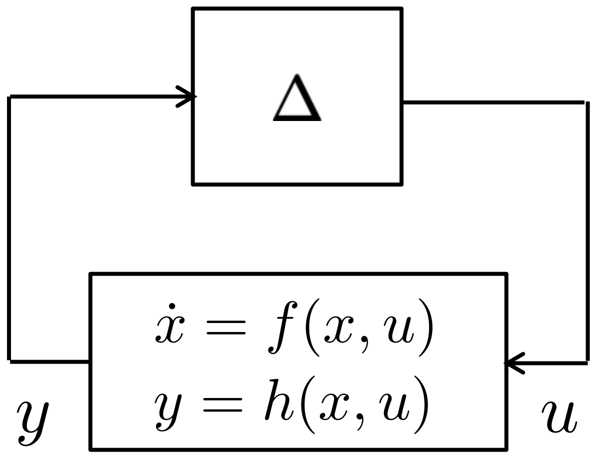
Find f, g, h such that this system is in the form:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

What is the largest ε such that $\phi = \frac{f}{g} + \varepsilon h$ is nonincreasing, so that the system is EID with supply rate defined by $X = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\varepsilon \end{bmatrix}$?

Can you show that the dissipation inequality holds with equality?

Equilibrium-Independent Stability Test



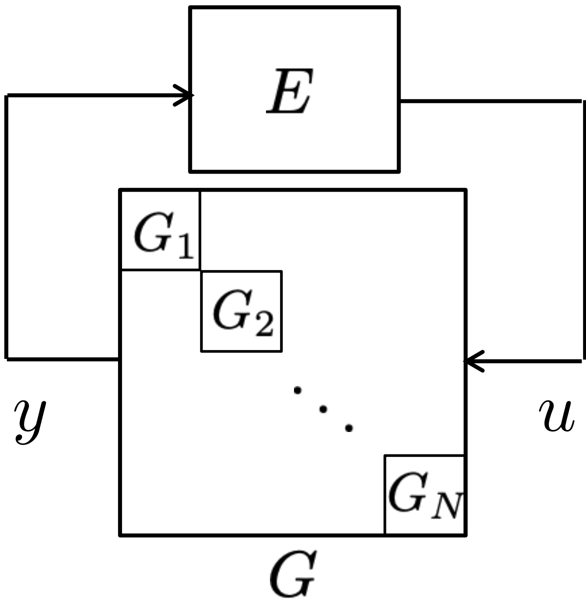
Suppose the system $\dot{x} = f(x,u)$, $y = h(x,u)$ is EID with supply rate s and storage function V such that $V(x,\bar{x}) > 0 \; x \neq \bar{x}$, and Δ satisfies

$$s(u-\bar{u}, y-\bar{y}) \leq 0$$

for all (u,y) such that $u = \Delta(y)$ and for all $(\bar{u},\bar{y})$ corresponding to a $\bar{x} \in \bar{\mathcal{X}}$. If x^* is an equilibrium for the interconnection then it is stable and $V(\cdot, x^*)$ is a Lyapunov function.

Equilibrium-Independent Stability Test

Adapt to:



E : interconnection matrix

$$G_i: \; \dot{x}_i = f_i(x_i, u_i)$$
$$y_i = h_i(x_i, u_i)$$

each one EID with supply rate

$$\begin{bmatrix} u_i - \bar{u}_i \\ y_i - \bar{y}_i \end{bmatrix}^\top X_i \begin{bmatrix} u_i - \bar{u}_i \\ y_i - \bar{y}_i \end{bmatrix}$$

Then, G is EID with following supply rate for $p_i > 0, i = 1, \ldots, N$:

$$\begin{bmatrix} u - \bar{u} \\ y - \bar{y} \end{bmatrix}^\top S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \underbrace{\begin{bmatrix} u - \bar{u} \\ y - \bar{y} \end{bmatrix}}_{= \begin{bmatrix} E \\ I \end{bmatrix} (y - \bar{y})}$$

Substitute $u = Ey, \; \bar{u} = E\bar{y}$:

Equilibrium-Independent Stability Test

$$= (y-\bar{y})^\top \begin{bmatrix} E \\ I \end{bmatrix}^\top S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \begin{bmatrix} E \\ I \end{bmatrix} (y-\bar{y})$$

Theorem: Suppose each subsystem is EID with quadratic supply rate defined by X_i and storage function V_i s.t. $V_i(x_i, \bar{x}_i) > 0, x_i \neq \bar{x}_i$. Suppose, further, there exist $p_i > 0, i = 1, \cdots, N$, such that

$$\begin{bmatrix} E \\ I \end{bmatrix}^\top S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \begin{bmatrix} E \\ I \end{bmatrix} \preceq 0 \qquad \textbf{(LMI)}$$

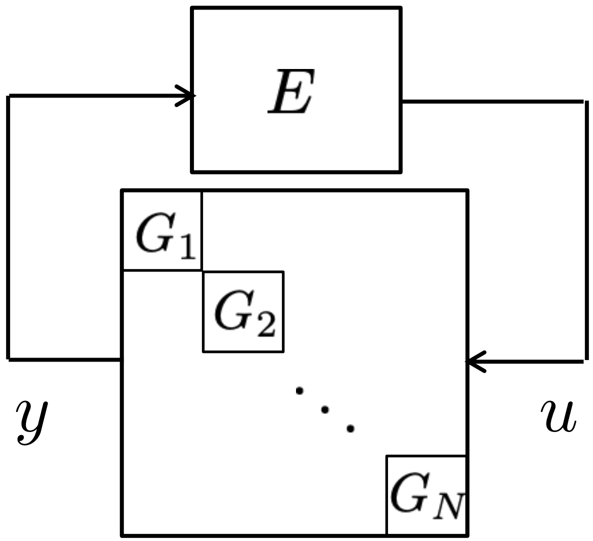
Under these conditions, if the interconnection admits an equilibrium x^*, then it is stable with Lyapunov function

$$V(x) = \sum_{i=1}^{N} p_i V_i(x_i, x_i^*)$$

Equilibrium-Independent Stability Test

Example: Lotka-Volterra population model for interacting species

$$\dot{x}_i = \left(\lambda_i - \gamma_i x_i + \sum_{j \neq i} e_{ij} x_j\right) x_i, \quad i = 1, \ldots, N$$



$$E = (e_{ij})$$

$$G_i: \quad \dot{x}_i = (\lambda_i - \gamma_i x_i + u_i) x_i$$

$$y_i = x_i$$

From "wake-up problem," EID with

$$X_i = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\gamma_i \end{bmatrix}$$

$$V_i(x_i, \bar{x}_i) = \int_{\bar{x}_i}^{x_i} \frac{h_i(s) - h_i(\bar{x}_i)}{g_i(s)} ds = \int_{\bar{x}_i}^{x_i} \frac{s - \bar{x}_i}{s} ds$$

$$= x_i - \bar{x}_i - \bar{x}_i \ln\left(\frac{x_i}{\bar{x}_i}\right)$$

Equilibrium-Independent Stability Test

LMI with this supply rate is: $P(E-\Gamma)+(E-\Gamma)^\top P \preceq 0$

where $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_N), P = \mathrm{diag}(p_1, \cdots, p_N)$

If $p_i > 0,\ i = 1, \ldots, N$ exist solving this LMI, and if the model

$$\dot{x}_i = \left(\lambda_i - \gamma_i x_i + \sum_{j \neq i} e_{ij} x_j \right) x_i, \quad i = 1, \ldots, N$$

admits an equilibrium x^*, then it is stable with Lyapunov function

$$V(x) = \sum_{i=1}^{N} p_i \left\{ x_i - x_i^* - x_i^* \ln\left(\frac{x_i}{x_i^*} \right) \right\}.$$

If LMI holds with strict inequality, then x^* is asymptotically and the region of attraction is the positive orthant $(0, \infty)^N$.

Equilibrium-Independent Stability Test

Special case: Predator-prey model $e_{12}e_{21} < 0$

Note $E - \Gamma = \begin{bmatrix} -\gamma_1 & e_{12} \\ e_{21} & -\gamma_2 \end{bmatrix}$ and take $P = \begin{bmatrix} |e_{21}| & 0 \\ 0 & |e_{12}| \end{bmatrix}$. Then,

$$P(E-\Gamma) + (E-\Gamma)^\top P = \begin{bmatrix} -2\gamma_1|e_{21}| & 0 \\ 0 & -2\gamma_2|e_{12}| \end{bmatrix} \preceq 0$$

If an equilibrium x^* in positive quadrant exists, it is stable:

$$V(x) = \sum_{i=1}^{2} p_i \left\{ x_i - x_i^* - x_i^* \ln\left(\frac{x_i}{x_i^*}\right)\right\}$$

If $\gamma_1 > 0, \gamma_2 > 0$ then globally asymptotically stable with respect to the positive quadrant.

In the classical predator-prey model $\gamma_1 = \gamma_2 = 0$; thus

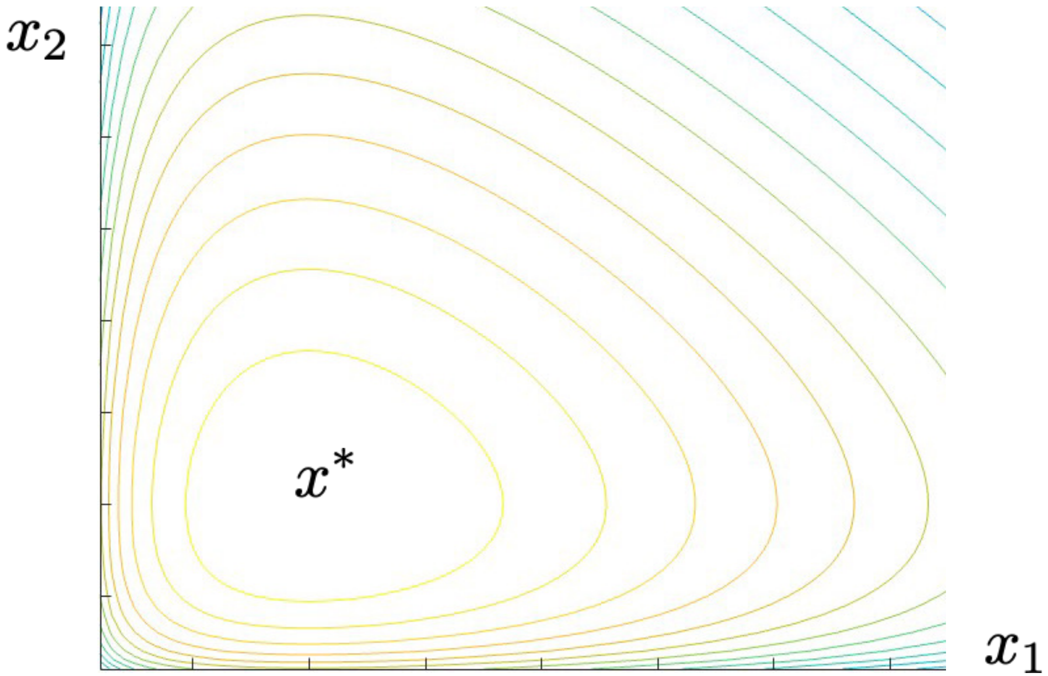
$$P(E-\Gamma) + (E-\Gamma)^\top P = 0$$

Equilibrium-Independent Stability Test

In addition, the subsystems are "lossless" (dissipation inequality holds with equality). Thus, the Lyapunov function

$$V(x) = \sum_{i=1}^{2} p_i \left\{ x_i - x_i^* - x_i^* \ln\left(\frac{x_i}{x_i^*}\right) \right\}$$

is constant along trajectories. Contours of V are periodic orbits:

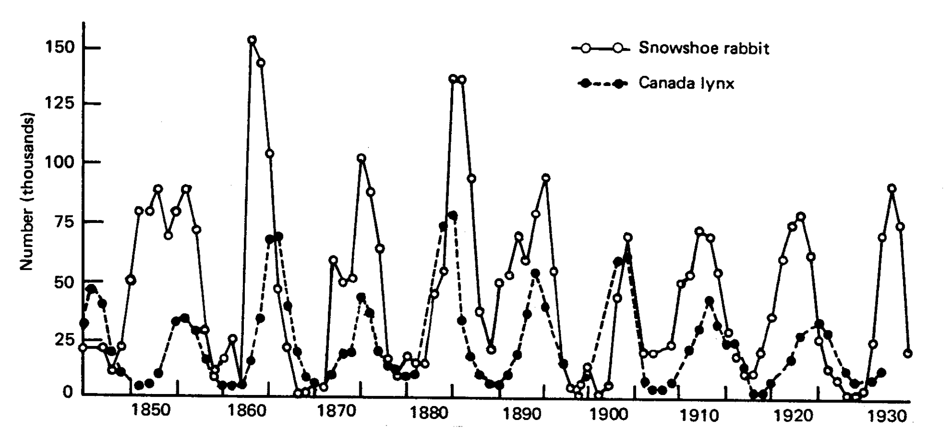


Equilibrium-Independent Stability Test

Lotka-Volterra in the Wild

The Hudson Bay Company's pelt records from 1845 to 1935 indicated oscillations in Canadian lynx and snowshoe hare populations.

Researchers have used this data to justify the Lotka-Volterra model and to fit parameters.



Case Study: Vehicle Platoon

Velocities and positions of vehicles $i = 1, \cdots, N$ governed by:

$$\dot{v}_i(t) = -v_i(t) + v_i^0 + u_i(t)$$
$$\dot{x}_i(t) = v_i(t)$$

Introduce undirected graph s.t. vertices i and j are connected with an edge if i and j have access to relative position $x_i - x_j$. Select one end of edge to be head, the other to be the tail, and define the incidence matrix:

$$D_{il} = \begin{cases} 1 & \text{if vertex } i \text{ is the head of edge } l \\ -1 & \text{if vertex } i \text{ is the tail of edge } l \\ 0 & \text{otherwise} \end{cases}$$

Then the vector of relative positions is given by $z = D^\top x$.

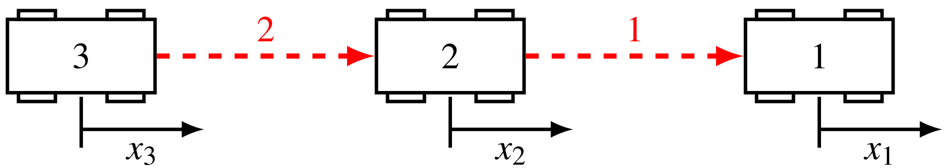
Case Study: Vehicle Platoon

Coordination feedback:

$$u = -D \begin{bmatrix} h_1(z_1) \\ \vdots \\ h_L(z_L) \end{bmatrix} \quad L : \# \text{ of edges}$$

$h_\ell : \mathbb{R} \to \mathbb{R},\ \ell = 1, \ldots, L$ onto and increasing functions that play the role of "virtual" spring forces.

Example:



$$D = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = D^\top x = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix}$$

$$u_1 = -h_1(z_1) \quad u_2 = h_1(z_1) - h_2(z_2) \quad u_3 = h_2(z_2)$$

Case Study: Vehicle Platoon

Relative position evolves according to:

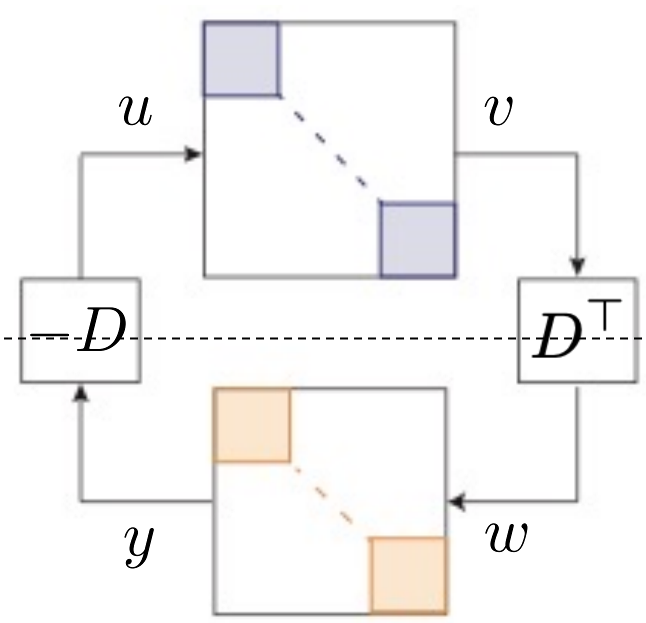
$$\dot{z} = D^\top v =: w \qquad w: \text{ vector of relative velocities}$$

Define $y := \begin{bmatrix} h_1(z_1) \\ \vdots \\ h_L(z_L) \end{bmatrix}$ and rewrite controller as $u = -Dy$. Then…

Closed-loop System:

N subsystems for vehicle velocities

L subsystems for relative positions

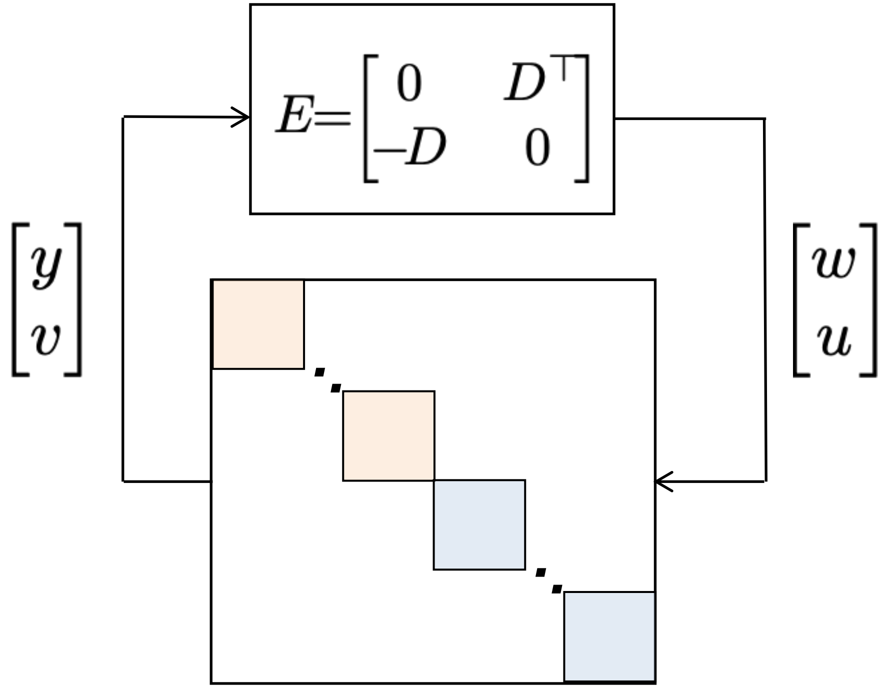


$$\dot{v}_i = -v_i + v_i^0 + u_i$$

$$\dot{z}_\ell = w_l, \;\; y_\ell = h_\ell(z_\ell)$$

Case Study: Vehicle Platoon

Equivalently:



E is skew symmetric. Thus, if an equilibrium (v^*, z^*) exists, then stability follows from equilibrium-independent passivity of the subsystems:

$$\dot{v}_i = -v_i + v_i^0 + u_i, \;\; i = 1, \cdots, N$$

$$\dot{z}_\ell = w_l, \;\; y_\ell = h_\ell(z_\ell), \;\; \ell = 1, \cdots, L$$

Case Study: Vehicle Platoon

These subsystems are indeed equilibrium-independent passive:

Recall: $\dot{x} = f(x) + g(x)u, \; y = h(x)$ is EID with $X = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\varepsilon \end{bmatrix}$

if $g(x) > 0 \, \forall x, \; h$ increasing, $\phi = \dfrac{f}{g} + \varepsilon h$ nonincreasing function

Storage function: $V(x, \bar{x}) = \displaystyle\int_{\bar{x}}^{x} \frac{h(s) - h(\bar{x})}{g(s)} ds$

$\dot{v}_i = -v_i + v_i^0 + u_i, \; i = 1, \cdots, N$ satisfies these with $\varepsilon_i = 1$

$\dot{z}_\ell = w_l, \; y_\ell = h_\ell(z_\ell), \; \ell = 1, \cdots, L$ satisfies them with $\varepsilon_\ell = 0$

(LMI) holds with $P = I$. Thus, a Lyapunov function is

$$V(v, z) = \frac{1}{2} \sum_{i=1}^{N} (v_i - v_i^*)^2 + \sum_{\ell=1}^{L} \int_{z_\ell^*}^{z_\ell} (h_\ell(s) - h_\ell(z_\ell^*)) ds$$

Case Study: Vehicle Platoon

Existence of (unique) equilibrium: if (v^*, z^*) exists, it must satisfy

$$0 = -v^* + v^0 - D \begin{bmatrix} h_1(z_1^*) \\ \vdots \\ h_L(z_L^*) \end{bmatrix} \qquad \textbf{(1)}$$

$$0 = D^\top v^* \qquad \textbf{(2)}$$

For a connected graph $D^\top v^* = 0 \;\Rightarrow\; v^* = \alpha \mathbf{1}$

Substitute $v^* = \alpha \mathbf{1}$ in (1) and multiply from the left by $\mathbf{1}^\top$:

$$0 = -\alpha \mathbf{1}^\top \mathbf{1} + \mathbf{1}^\top v^0 = -\alpha N + \sum_{i=1}^N v_i^0$$

Thus, $\alpha = \frac{1}{N} \sum_{i=1}^N v_i^0$ and (1) becomes:

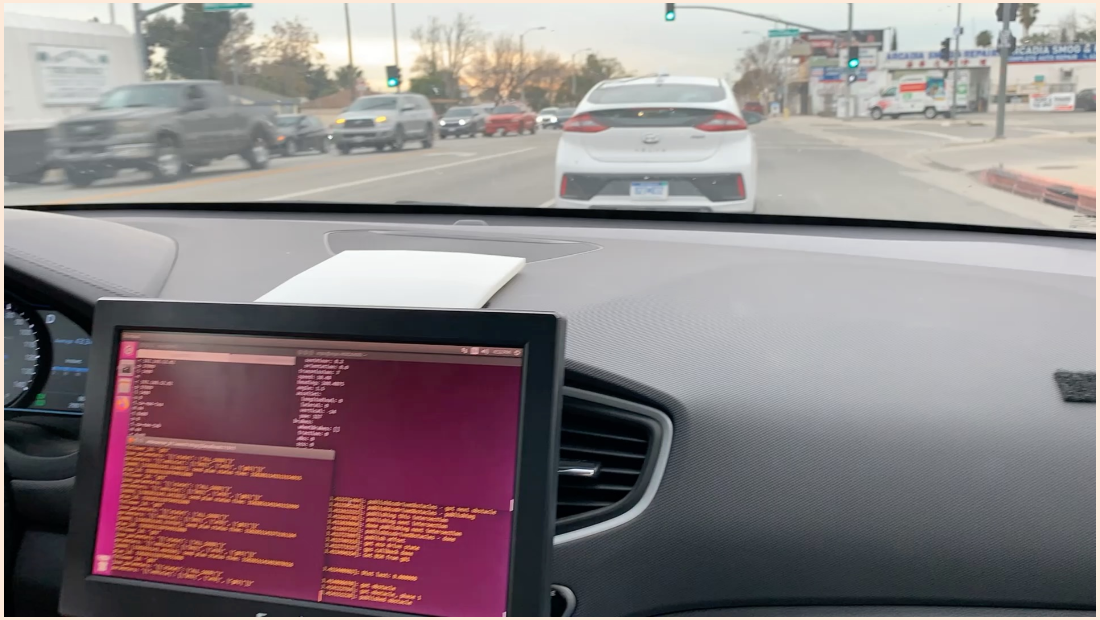
$$-\frac{1}{N} \sum_{i=1}^N v_i^0 + v_i^0 - \sum_{\ell=1}^L D_{i\ell} h_\ell(z_\ell^*) = 0 \quad i = 1, \ldots, N$$

For acyclic graphs we can solve for $h_\ell(z_\ell^*)$ from this, then find z_ℓ^*.

Case Study: Vehicle Platoon

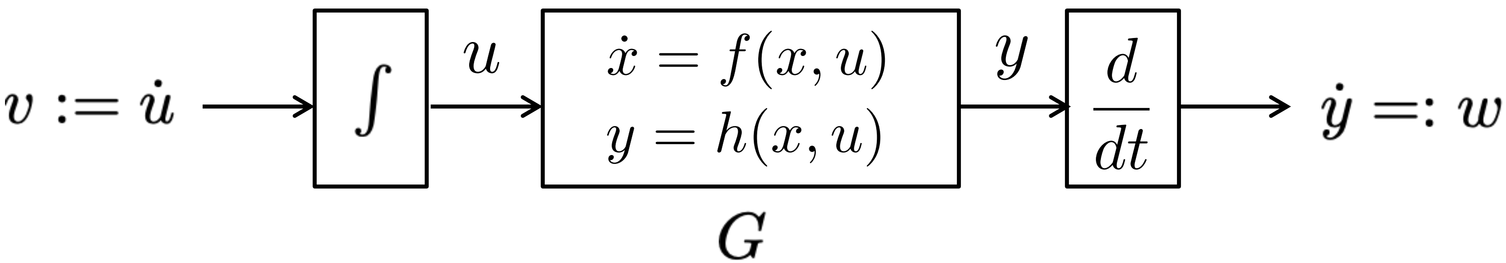
Why Platoons?

Platoons increase utilization of road and intersection capacities by enabling safe tailgating! The prevailing approach is to further safeguard a controller like this one, or an MPC control, with control barrier functions.



Delta Dissipativity

Dissipativity with respect to input/output derivatives $(\dot{u}, \dot{y})$:



The system G is **delta dissipative** if there exists storage function $S : \mathcal{X} \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that $S(x, u) = 0 \Leftrightarrow f(x, u) = 0$ and

$$\nabla_x S(x, u)^\top f(x, u) + \nabla_u S(x, u)^\top v \leq s(v, w) \quad \forall x, u, v$$

where $w := \nabla_x h(x, u)^\top f(x, u) + \nabla_u h(x, u)^\top v.$

Note: In EID, the storage function $V(\cdot, \bar{x})$ depends on the equilibrium candidate. Here, it depends on input and vanishes when $f(x, u) = 0$. Thus, equilibrium independence is implicit.

Delta Dissipativity

Example: Linear system $\dot{x} = Ax + Bu,\ y = Cx + Du$

Take $S(x,u) = (Ax+Bu)^\top P(Ax+Bu)$ and check the condition

$$\nabla_x S(x,u)^\top (Ax+Bu) + \nabla_u S(x,u)^\top v \leq \begin{bmatrix} v \\ w \end{bmatrix}^\top X \begin{bmatrix} v \\ w \end{bmatrix} \qquad \textbf{(1)}$$

where $w = C(Ax+Bu) + Dv$.

Right side of (1): $\begin{bmatrix} Ax+Bu \\ v \end{bmatrix}^\top \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top X \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} Ax+Bu \\ v \end{bmatrix}$

Left side of (1): $\begin{bmatrix} Ax+Bu \\ v \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} \begin{bmatrix} Ax+Bu \\ v \end{bmatrix}$

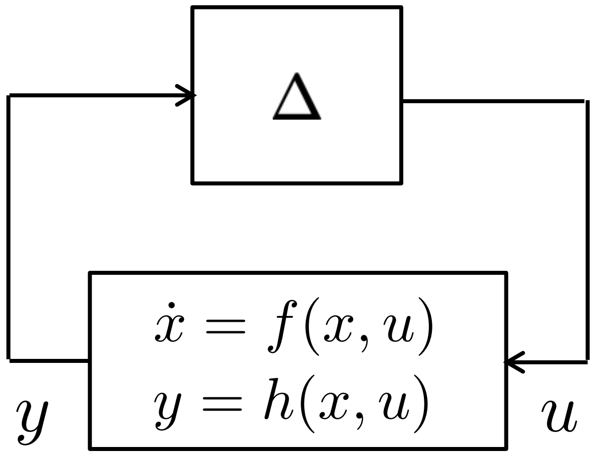
Thus, (1) boils down to the same LMI as dissipativity and EID:

$$\begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix}^\top - \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top X \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \preceq 0$$

Delta Dissipativity

Stability from Delta Dissipativity

Assumption: Δ is a static map and the interconnection is well posed; that is, $u = \Delta(h(x,u))$ has implicit solution $u = g(x)$.



Suppose the system $\dot{x} = f(x,u) \;\; y = h(x,u)$ is delta dissipative with supply rate s and storage function S, and Δ satisfies the constraint

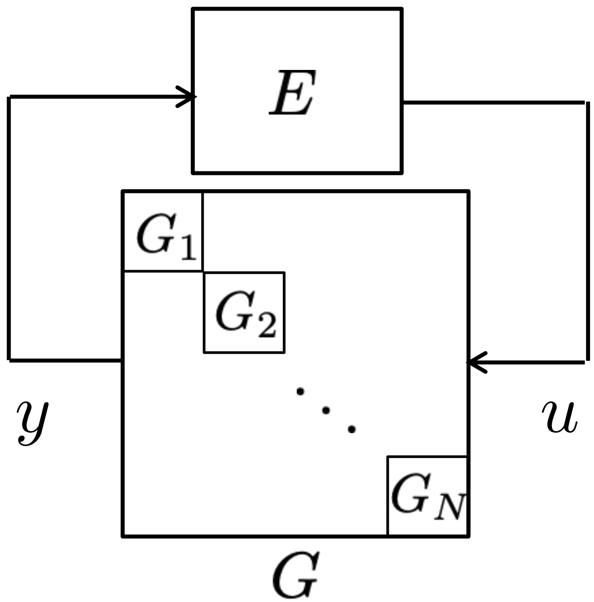
$$s(\dot{u}(t), \dot{y}(t)) \leq 0 \quad \forall t$$

for all differentiable signals $u(\cdot), v(\cdot)$ s.t. $u(t) = \Delta(v(t))$. Under these conditions, if the interconnection has an equilibrium, then it is stable and a Lyapunov function is given by $V(x) = S(x, g(x))$.

Note: since $S(x,u)$ nonnegative and vanishes only when $f(x,u) = 0$ $V(x) > 0$ except when $f(x, g(x)) = 0$, i.e., when at equilibrium.

Delta Dissipativity

Adapt to:



E : interconnection matrix

$$G_i : \begin{aligned} \dot{x}_i &= f_i(x_i, u_i) \\ y_i &= h_i(x_i, u_i) \end{aligned}$$

delta dissipative with supply rate

$$\begin{bmatrix} \dot{u}_i \\ \dot{y}_i \end{bmatrix}^\top X_i \begin{bmatrix} \dot{u}_i \\ \dot{y}_i \end{bmatrix}$$

Then, for any $p_i > 0, i = 1, \ldots, N,$ G is delta dissipative with:

$$\begin{bmatrix} \dot{u} \\ \dot{y} \end{bmatrix}^\top S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \underbrace{\begin{bmatrix} \dot{u} \\ \dot{y} \end{bmatrix}}$$

Substitute $u = Ey$, $\dot{u} = E\dot{y}$: $\quad = \begin{bmatrix} E \\ I \end{bmatrix} \dot{y}$

Delta Dissipativity

$$= \dot{y}^\top \begin{bmatrix} E \\ I \end{bmatrix}^\top S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \begin{bmatrix} E \\ I \end{bmatrix} \dot{y}$$

Thus, we arrive at the same condition for stability of the equilibrium of the interconnection: there exist $p_i > 0, i = 1, \cdots, N$, s.t.

$$\begin{bmatrix} E \\ I \end{bmatrix}^\top S^\top \begin{bmatrix} p_1 X_1 & & \\ & \ddots & \\ & & p_N X_N \end{bmatrix} S \begin{bmatrix} E \\ I \end{bmatrix} \preceq 0$$

(LMI)

Lyapunov function: $V(x) = \sum_{i=1}^{N} p_i S_i(x_i, g_i(x))$

where g is obtained from the solution $u = g(x)$ of $u = Eh(x, u)$.

Although EID and delta dissipativity may appear interchangeable, there are systems where one holds but not the other (Lesson 6).

Delta Dissipativity

Origins of Delta Dissipativity in Game Theory

Delta passivity was introduced in a study of "population games":

- Fox and Shamma, Population games, stable games, and passivity, Games, vol.4, pp. 561-583, 2013

This notion is implicit in a proof of convergence to Nash equilibria in:

- Hofbauer and Sandholm, Stable games and their dynamics, J. of Econ. Theory, vol.144, pp. 1665-1693, 2009

By making the connection to passivity, Fox and Shamma opened the door to new results for population games, discussed in Lesson 6.

Delta Dissipativity

Origins of Delta Dissipativity in Game Theory

Delta passivity was introduced in a study of "population games":

- Fox and Shamma, Population games, stable games, and passivity, Games, vol.4, pp. 561-583, 2013

This notion is implicit in a proof of convergence to Nash equilibria in:

- Hofbauer and Sandholm, Stable games and their dynamics, J. of Econ. Theory, vol.144, pp. 1665-1693, 2009

By making the connection to passivity, Fox and Shamma opened the door to new results for population games, discussed in Lesson 6.

Outside of game theory, an identical notion appeared later in:

- Kosaraju, Kawano and Scherpen, "Krasovskii's passivity", *IFAC-PapersOnLine*, vol. 52, no. 16, pp. 466-471, 2019

Stability/performance criteria for interconnections derived in:

- Schweidel and Arcak, Compositional analysis of interconnected systems using delta dissipativity, L-CSS, vol.6, pp. 662-667, 2022

Summary

In this lesson:

- We leveraged dissipativity for compositional stability/performance verification of interconnected systems
- Introduced computationally efficient methods: LMIs, ADMM, etc.
- Achieved complete modularity of the method with new notions: equilibrium independent dissipativity, delta dissipativity
- Presented examples from congestion control, multiagent systems, oscillator circuits, ecological models.

Key features of the methodology:

- Modularity (by dissipativity & variants)
- Scalability (by decomposition, ADMM)
- Substitutability: can replace subsystems without losing system-level guarantees

Further Reading

- Arcak, Meissen, Packard, Networks of Dissipative Systems, Springer 2016
- Arcak, Compositional design and verification of large scale systems using dissipativity theory, IEEE Control Systems Mag., vol.42, no.2, pp. 51-62, 2022
- Wen and Arcak, A unifying passivity framework for network flow control, IEEE Trans. Automatic Control, vol.49, no.2, pp. 162-174, 2004
- Bai, Arcak, Wen, Cooperative Control Design, Springer, 2011
- Klausen, Meissen, Fossen, Arcak, Johansen, Cooperative control for multirotors transporting an unknown suspended load under environmental disturbances, IEEE Trans. Control Systems Technology, vol.28, no.2, pp. 653-660, 2020
- Arcak, Sontag, Diagonal stability of a class of cyclic systems and its connection with the secant criterion, Automatica, vol.42, no.9, pp.1531-1537, 2006
- Hines, Arcak, Packard, Equilibrium-independent passivity: A new definition and numerical certification. Automatica, vol.47, no.9, pp. 1949-1956, 2011
- Meissen, Lessard, Arcak, Packard, Compositional performance certification of interconnected systems using ADMM, Automatica, vol.61, pp. 55-63, 2015
- Schweidel and Arcak, Compositional analysis of interconnected systems using delta dissipativity, L-CSS, vol.6, pp. 662-667, 2022

Self-Study Problems

1) Antelopes, hyenas, and lions:

Consider the Lotka-Volterra model for three species, where species 2 and 3 both prey on species 1:

$$e_{12} < 0 \quad e_{13} < 0 \quad e_{21} > 0 \quad e_{31} > 0$$

but they are neutral to each other:

$$e_{23} = e_{32} = 0$$

Recall also that the diagonal entries of E are zero. Investigate whether a diagonal $P \succ 0$ exists such that

$$P(E - \Gamma) + (E - \Gamma)^\top P \preceq 0$$

where $\Gamma \succeq 0$ is a diagonal matrix of parameters appearing in the model. Your answer should not depend on specific values of E and Γ, but only their sparsity and sign structure.

Self-Study Problems

2) A cyclic interconnection

Consider the system

$$\dot{x}_1 = f(x_1) - h(x_3)$$
$$\dot{x}_2 = f(x_2) - h(x_1)$$
$$\dot{x}_2 = f(x_2) - h(x_2)$$

where f is a strictly decreasing function and h is an increasing function. If f is onto, there exists a unique equilibrium.

a) Decompose this system into a cyclic interconnection of three first order subsystems.

b) Provide a condition on f and h such that the equilibrium is guaranteed to be stable (without knowledge of where the equilibrium is).