International Graduate School on Control, Stuttgart, May 2024

Dissipation Inequalities and Quadratic Constraints for Control, Optimization, and Learning

Lesson 2: Quadratic Constraints

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Learning Objectives

In this lesson we will

- Introduce a basic concept, the linear fractional transformation (LFT), for modeling systems with uncertainties and nonlinearities.
- Learn to bound the input/output behavior of uncertainties/nonlinearities using quadratic constraints (QCs) and integral quadratic constraints (IQCs).
- Learn to assess stability and performance using dissipation inequalities and QCs/IQCs.

Outline

1. Basic uncertainty modeling leading to LFTs
2. Static quadratic constraints (QCs)
3. Time-domain Integral Quadratic Constraints (IQCs)
4. Constructing storage functions using IQCs

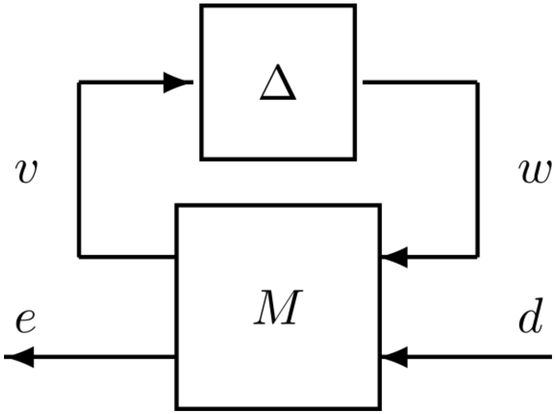
General Representation for Uncertain Systems

The basic modeling concept separates the system into 2 pieces:

1. Known "nominal" part M: This part contains dynamics that are typically easy to analyze, e.g., LTI dynamics:
2. "Uncertainty" or "perturbation" Δ: This contains components that have unknown variations and/or components that are difficult to analyze, e.g. nonlinearities.

$$\begin{bmatrix} v \\ e \end{bmatrix} = M \begin{bmatrix} w \\ d \end{bmatrix}$$

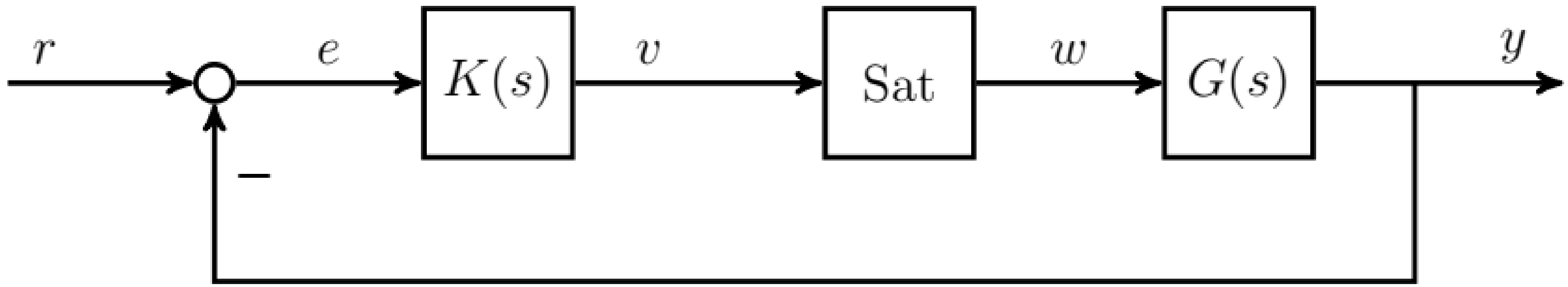
$$w = \Delta(v)$$



The feedback interconnection, denoted $F_U(M, \Delta)$, is called a Linear Fractional Transformation (LFT). This separation is a general object that greatly facilitates the analysis.

Example: Actuator Saturation

Consider the classical feedback diagram below with actuator saturation. Typical goals are to assess the closed-loop stability and analyze the gain from reference to error.

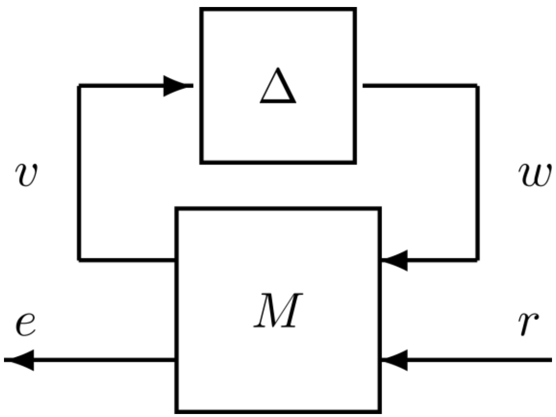


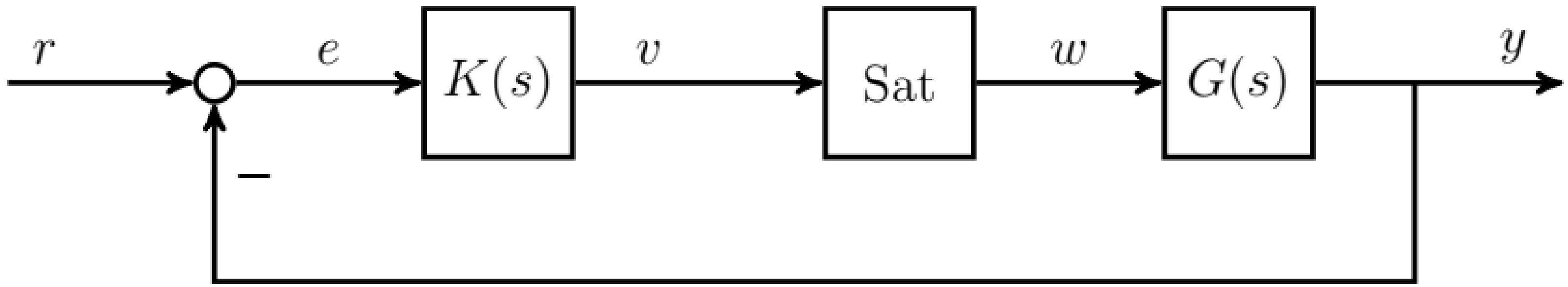
Example: Actuator Saturation

Consider the classical feedback diagram below with actuator saturation. Typical goals are to assess the closed-loop stability and analyze the gain from reference to error.

This system can be expressed as an LFT $F_U(M, \Delta)$ where:

- $w = \Delta(v)$ and Δ is the saturation.
- $\begin{bmatrix} v \\ e \end{bmatrix} = M \begin{bmatrix} w \\ r \end{bmatrix}$ and $M = \begin{bmatrix} -KG & K \\ -G & 1 \end{bmatrix}$

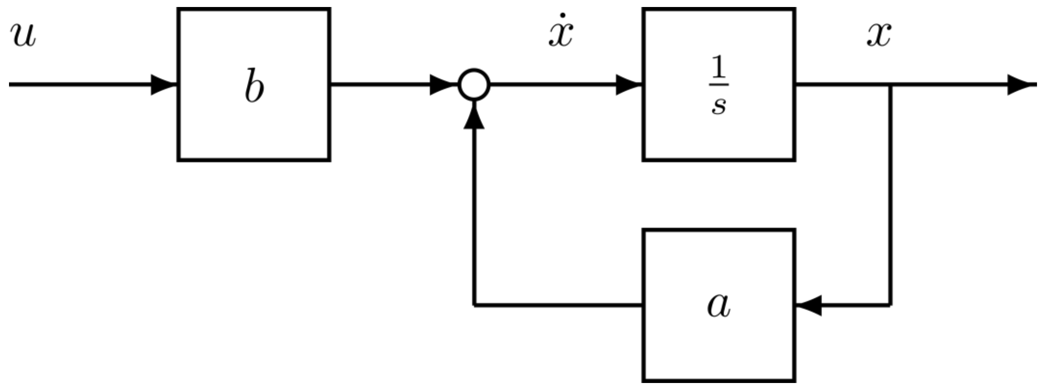




Example: Uncertain Parameters

It is common to have uncertainty in various model parameters, e.g. masses, spring constants, etc. As a simple example:

$$\dot{x}(t) = a\,x(t) + b\,u(t) \text{ where } a \in [-3,-1] \text{ and } b \in [4,6].$$



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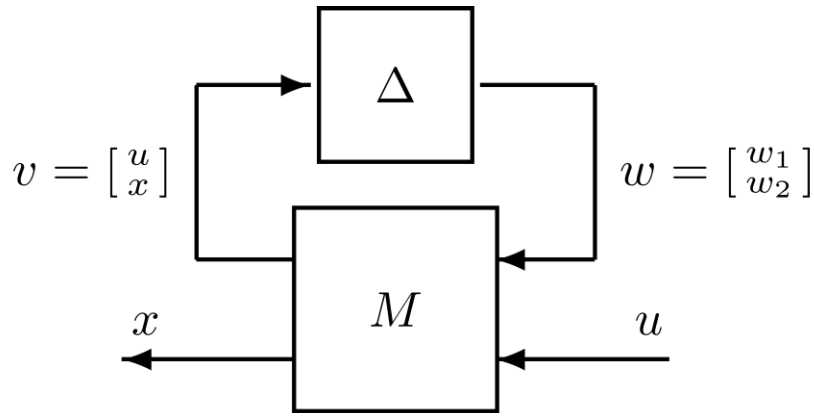
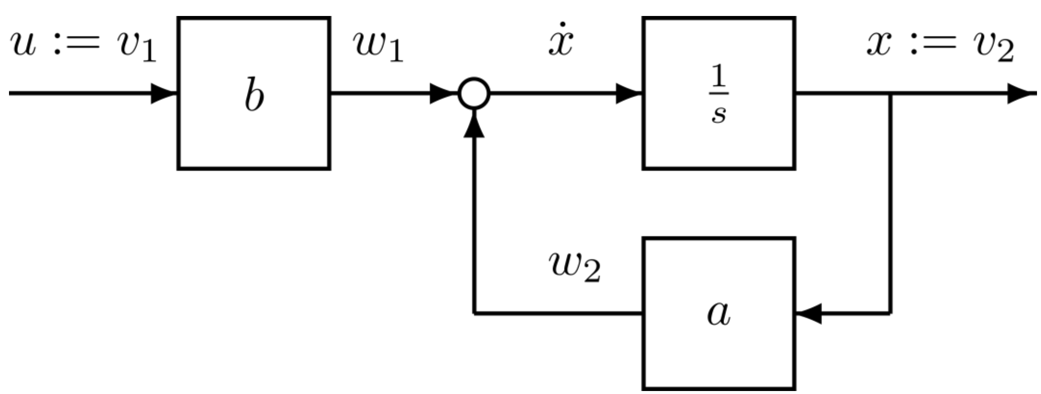
$$\dot{x}(t) = a\,x(t) + b\,u(t) \text{ where } a \in [-3,-1] \text{ and } b \in [4,6].$$

This can be expressed as $F_U(M,\Delta)$ where:

- $w = \Delta(v)$ and $\Delta = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$
- $\begin{bmatrix} v \\ x \end{bmatrix} = M \begin{bmatrix} w \\ u \end{bmatrix}$ and $M = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{s} & \frac{1}{s} & 0 \\ \frac{1}{s} & \frac{1}{s} & 0 \end{bmatrix}$



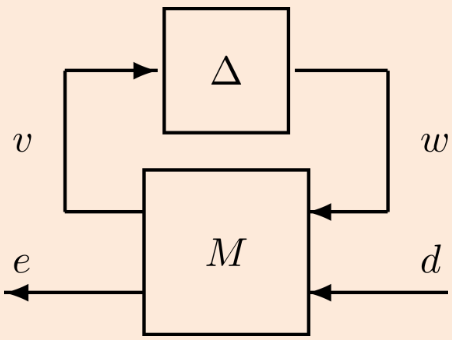


Linear Fractional Transformations (LFTs)

Wake-up Problems

True of false?

1) Consider LFTs $F_U(M_1, \Delta_1)$ and $F_U(M_2, \Delta_2)$ where



$$\Delta_1 \in \mathbb{R},\ |\Delta_1| \le 1 \text{ and } M_1 = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\Delta_2 \in \mathbb{R},\ |\Delta_2| \le 1 \text{ and } M_2 = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$$

The two LFTs represent the same uncertain system.

2) Consider LFTs $F_U(M_1, \Delta_1)$ and $F_U(M_2, \Delta_2)$ where

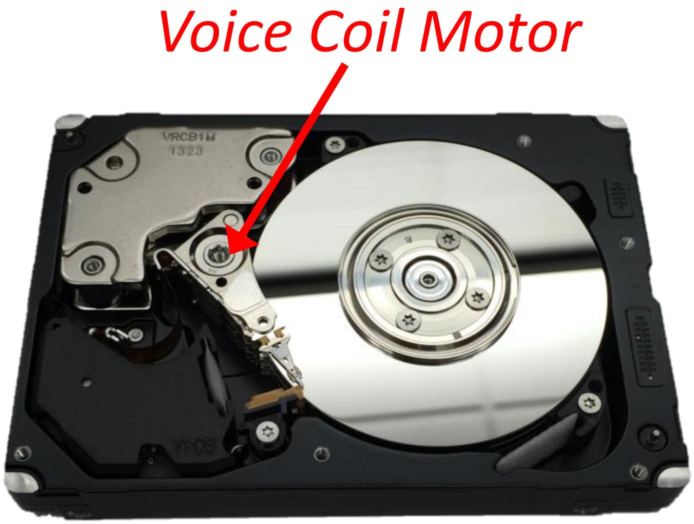
$$\Delta_1 \in \mathbb{R},\ |\Delta_1| \le 1 \text{ and } M_1 = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\Delta_2 \in \mathbb{R},\ \Delta_2 \in [1, 5] \text{ and } M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

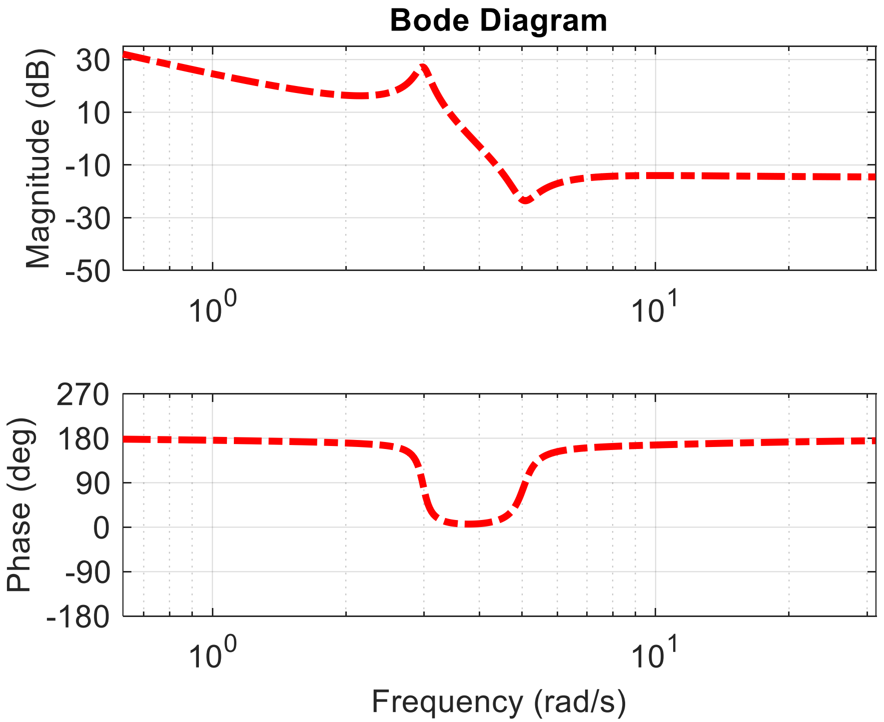
The two LFTs represent the same uncertain system.

Example: Nonparametric Uncertainty

A simplified model G_0 is often used for control design.



Nominal design model (red dashed)

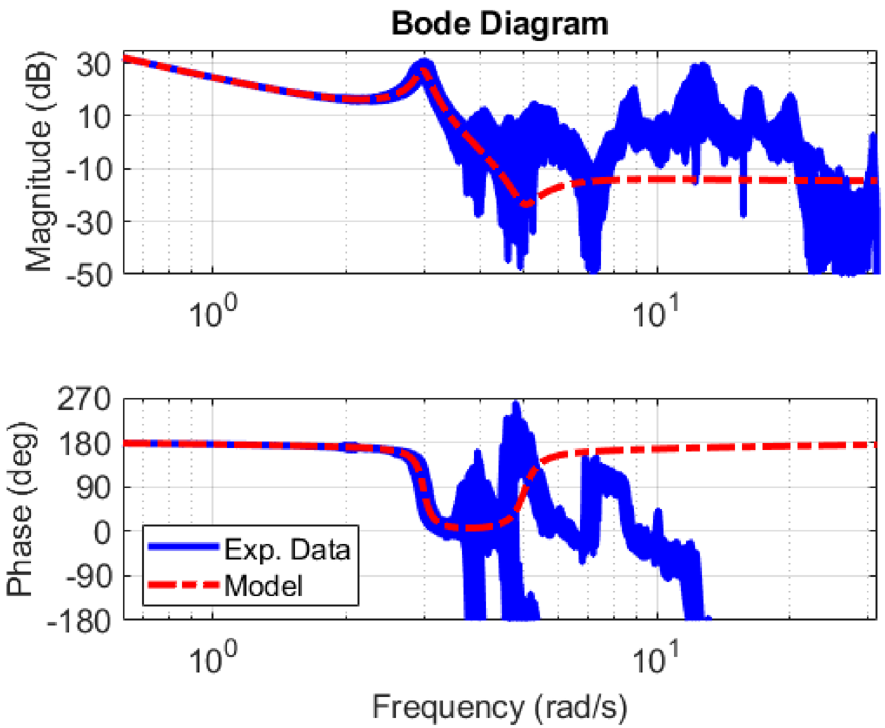


Example: Nonparametric Uncertainty

A simplified model G_0 is often used for control design.

- Actual dynamics are complex and have part-to-part variation.
- We lose model fidelity as we go to higher frequencies.

Voice Coil Motor



Nominal design model (red dashed)
Experimental responses (blue):
Courtesy of Seagate and
normalized for proprietary reasons.

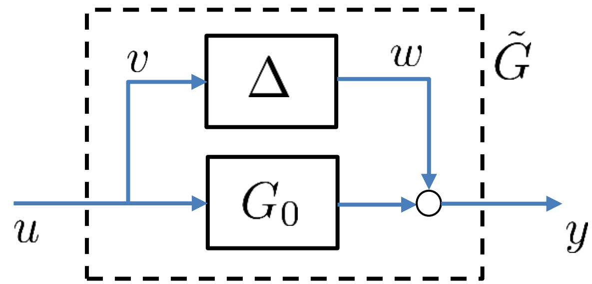
Example: Nonparametric Uncertainty

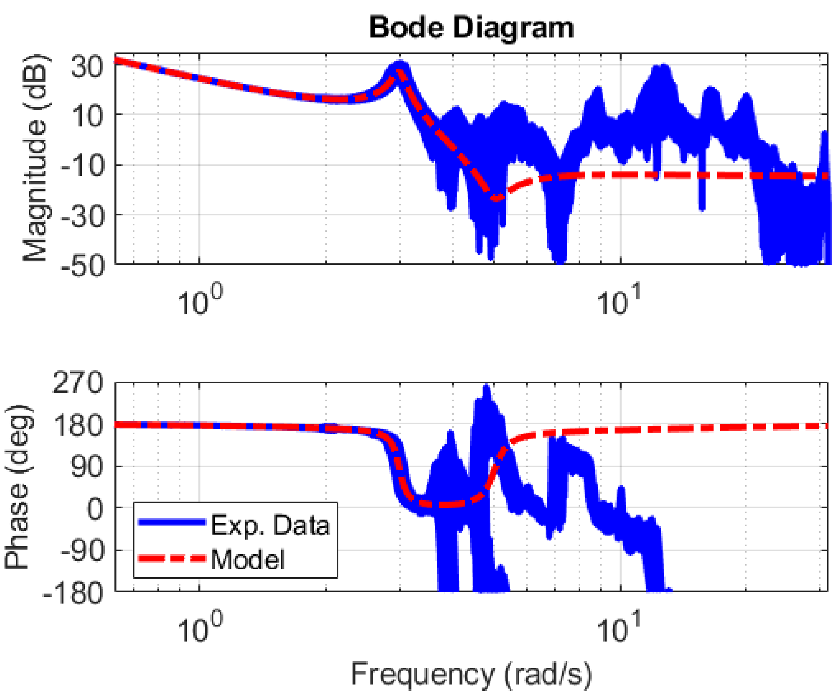
The unmodeled dynamics are captured by bounding the error in the frequency domain. This is an “additive” uncertainty.

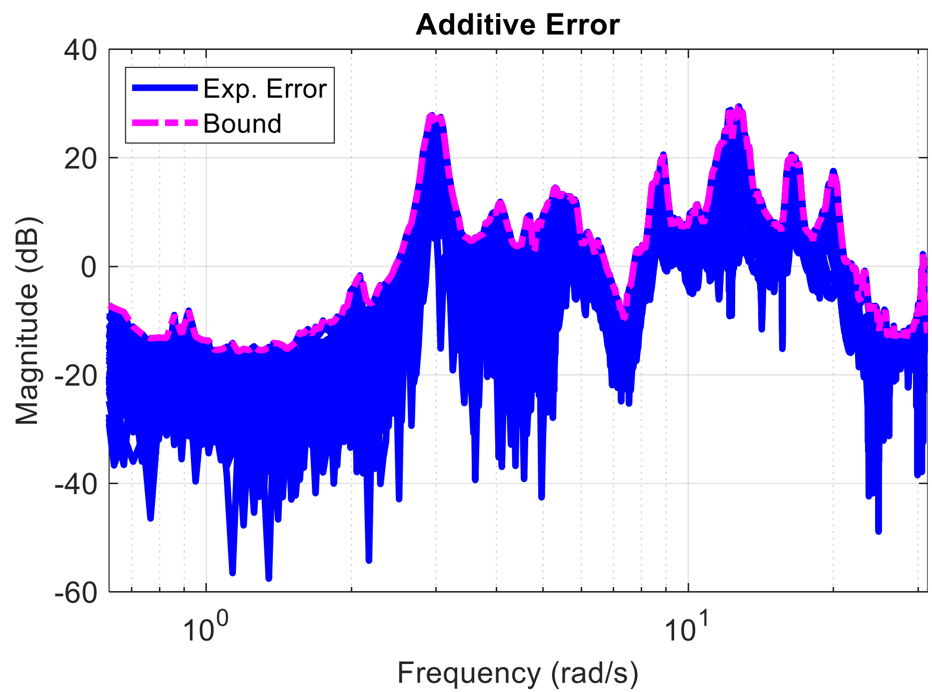
$$\underbrace{\left|\tilde{G}(\omega) - G_0(\omega)\right|}_{=:\,\Delta} \leq \alpha(\omega)$$

Absolute Error Bound

$$|\Delta(\omega)| \leq \alpha(\omega)$$







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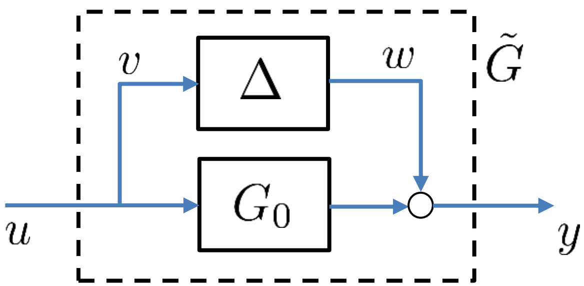
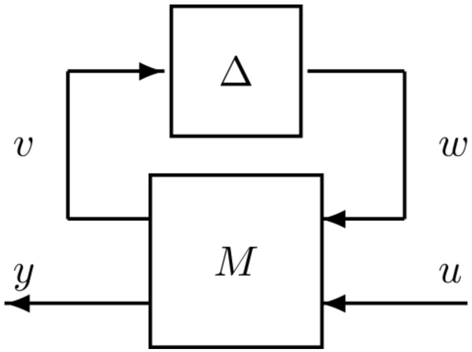


This can be expressed as an LFT $F_U(M,\Delta)$:

- $w = \Delta(v)$
- $\begin{bmatrix} v \\ y \end{bmatrix} = M \begin{bmatrix} w \\ u \end{bmatrix}$ and $M = \begin{bmatrix} 0 & 1 \\ 1 & G_0 \end{bmatrix}$



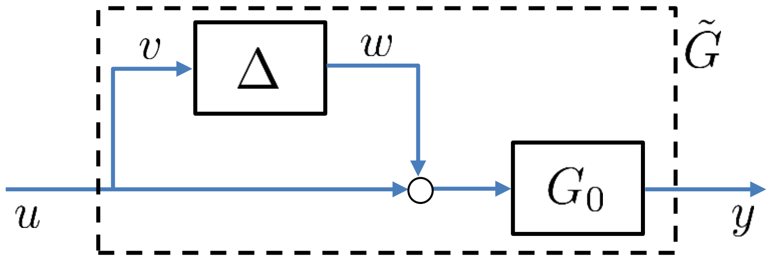
Example: Nonparametric Uncertainty

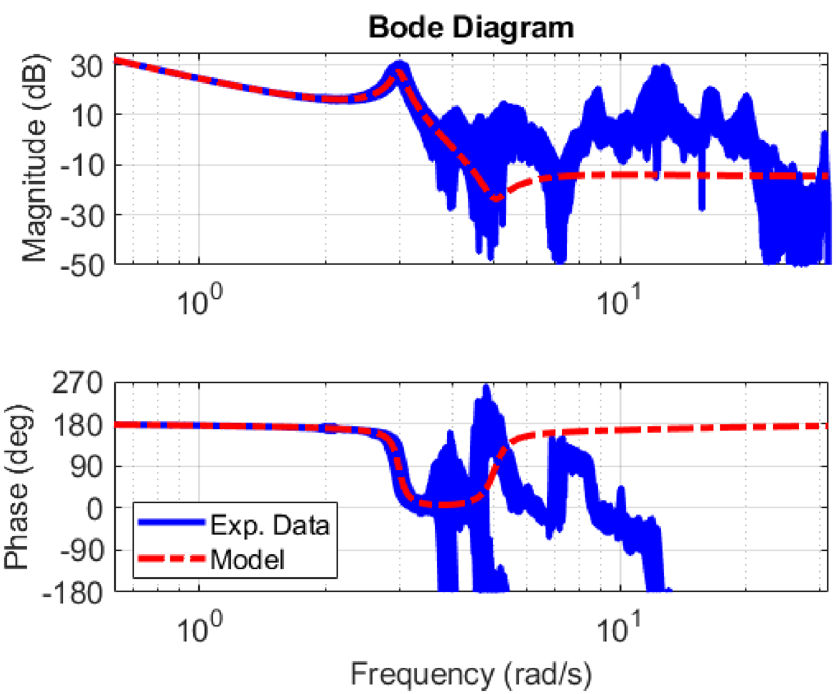
Alternatively, we can bound the relative (percent) error. This leads to a "multiplicative" uncertainty.

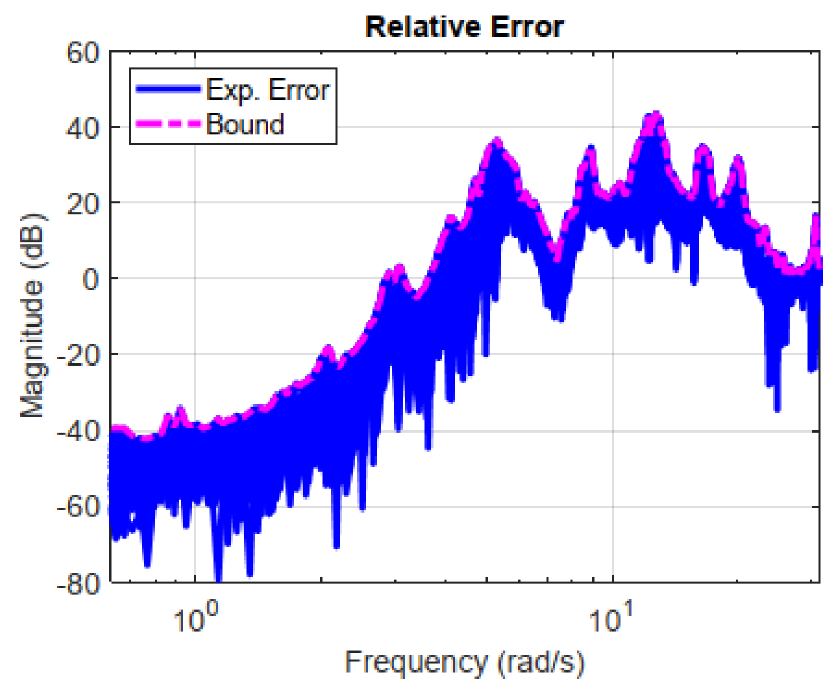
$$\left|\underbrace{\frac{\tilde{G}(\omega) - G_0(\omega)}{G_0(j\omega)}}_{=:\,\Delta}\right| \le \alpha(\omega)$$

Relative Error Bound

$|\Delta(\omega)| \le \alpha(\omega)$





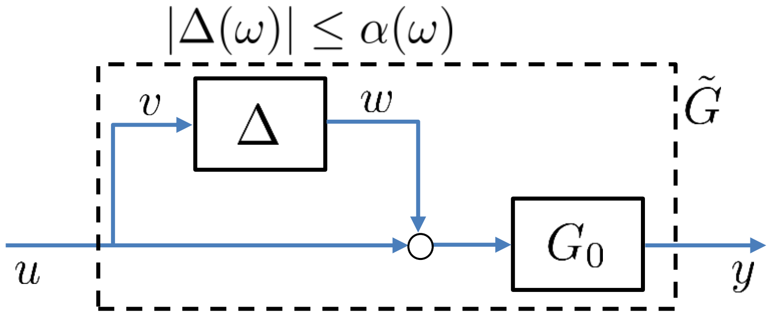


Example: Nonparametric Uncertainty

Alternatively, we can bound the relative (percent) error. This leads to a "multiplicative" uncertainty.

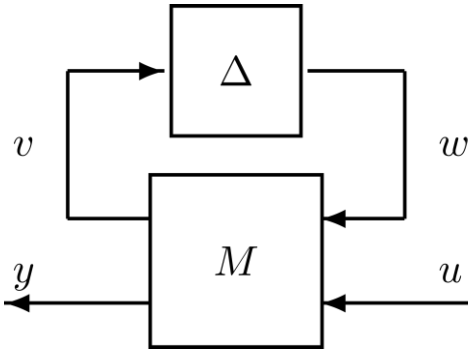
$$\underbrace{\left| \frac{\tilde{G}(\omega) - G_0(\omega)}{G_0(j\omega)} \right|}_{=:\,\Delta} \leq \alpha(\omega)$$

Relative Error Bound



This can also be expressed as an LFT $F_U(M, \Delta)$:

- $w = \Delta(v)$
- $\begin{bmatrix} v \\ y \end{bmatrix} = M \begin{bmatrix} w \\ u \end{bmatrix}$ and $M = \begin{bmatrix} 0 & 1 \\ G_0 & G_0 \end{bmatrix}$



Multiplicative uncertainty uses a non-dimensional error bound, e.g. a bound of 0.1 corresponds to 10% uncertainty.

Generic LFT Formula

We can express the LFT $F_U(M, \Delta)$ based on the uncertainty Δ and the partitioned nominal system M:

$$\begin{bmatrix} v \\ e \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w \\ d \end{bmatrix} \text{ and } w = \Delta v$$

Combine the first plant equation with the uncertainty equation:

$$v = M_{11}\Delta v + M_{12}d \Longrightarrow v = (I - M_{11}\Delta)^{-1}M_{12}d$$

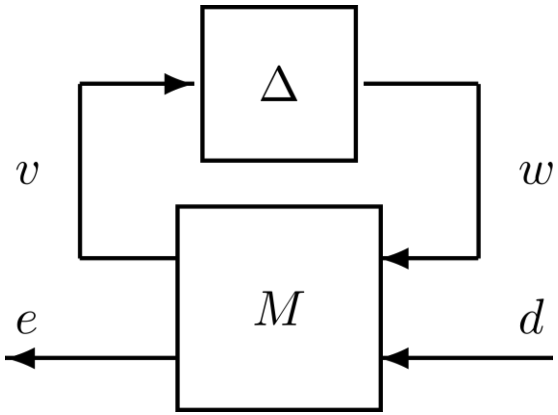
Substitute this into the second plant equation:

$$e = M_{21}\Delta v + M_{22}d \Longrightarrow e = \left[M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}\right] d$$

This gives the following general expression for the uncertain system:

$$F_U(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$$

This expression appears in various derivations.



Linear Fractional Transformation

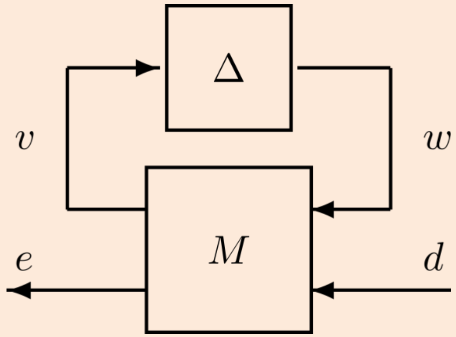
A linear fractional transformation in complex analysis refers to an invertible function $f: \mathbb{C} \to \mathbb{C}$ of the form:

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ so that f is invertible.

This function is a *__transformation__* from z to $f(z)$ defined by a *__fraction__* (ratio) where the numerator and denominator are *__linear__* in z.

Our LFT is a generalization:



$$F_U(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$$

It simplifies to the form above when $\Delta \in \mathbb{C}$ and $M \in \mathbb{C}^{2\times 2}$.

Interconnections of LFTs

An important property of LFTs is that typical algebraic operations preserve LFT structure, e.g.

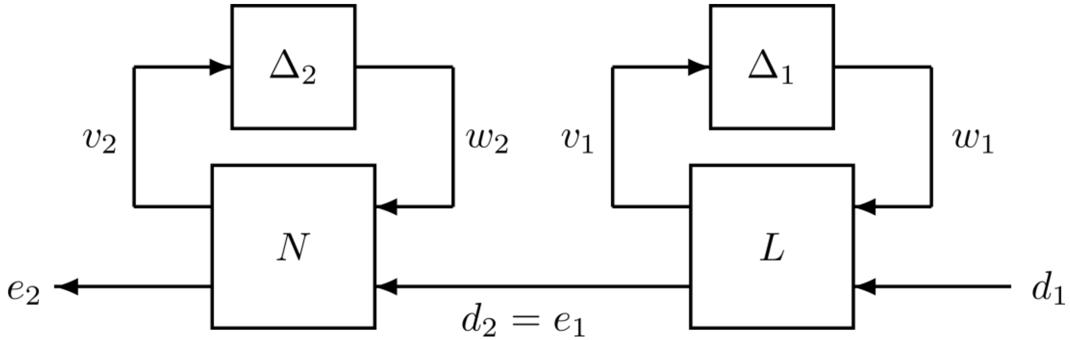
- frequency response,
- inverses,
- cascade (serial) connections,
- parallel connections, and
- feedback connections

A few examples are given on the following slides.

Hence, typical interconnections of LFTs are still in the form of an LFT. For this reason, the LFT is an excellent choice for a general hierarchical representation of uncertainty.

Cascade (Serial) Connections of LFTs

Consider the serial connection of LFTs $F_U(L, \Delta_1)$ and $F_U(N, \Delta_2)$.



The output of the first LFT is the input to the second, i.e. $d_2 = e_1$.

Cascade (Serial) Connections of LFTs

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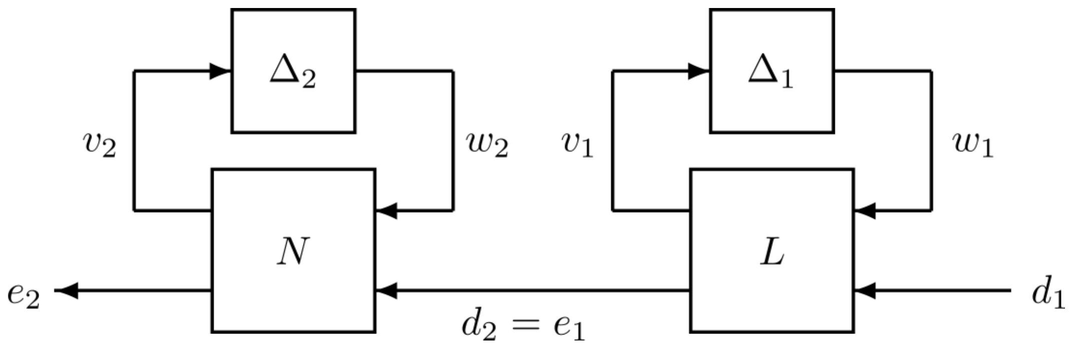
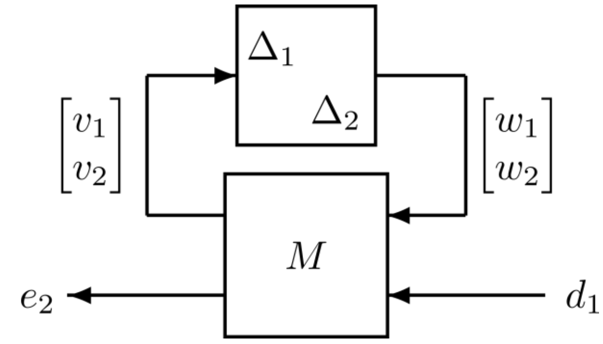




The individual LFTs are:

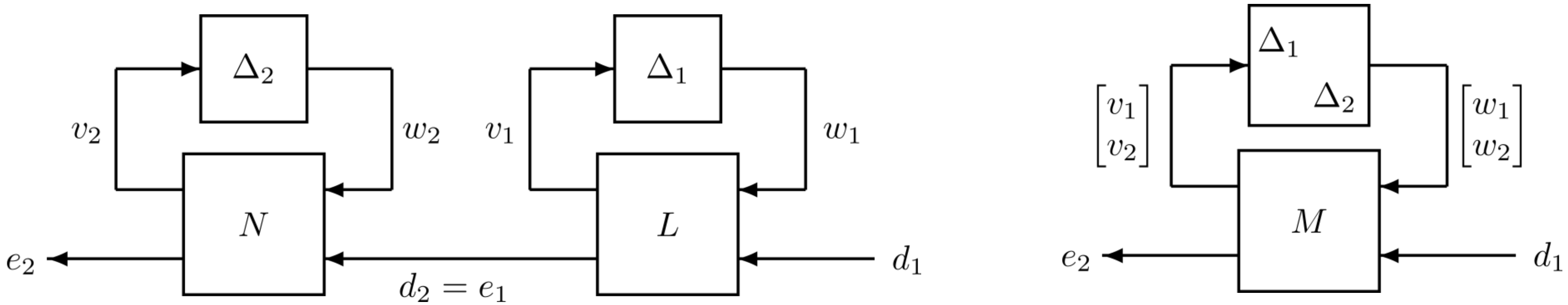
$$\begin{bmatrix} v_2 \\ e_2 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} w_2 \\ d_2 \end{bmatrix}, \quad w_2 = \Delta_2 v_2 \qquad \begin{bmatrix} v_1 \\ e_1 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ d_1 \end{bmatrix}, \quad w_1 = \Delta_1 v_1$$

The combined LFT $F_U(M, \Delta)$ is:

$$\begin{bmatrix} v_1 \\ v_2 \\ e_2 \end{bmatrix} := \underbrace{\begin{bmatrix} L_{11} & 0 & L_{12} \\ N_{12}L_{21} & N_{11} & N_{12}L_{22} \\ N_{22}L_{21} & N_{21} & N_{22}L_{22} \end{bmatrix}}_{:=M} \begin{bmatrix} w_1 \\ w_2 \\ d_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \underbrace{\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}}_{:=\Delta} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Cascade (Serial) Connections of LFTs

Consider the serial connection of LFTs $F_U(L, \Delta_1)$ and $F_U(N, \Delta_2)$.

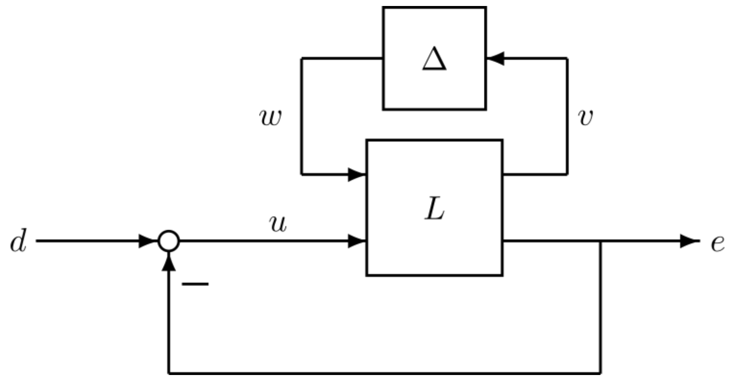


Uncertainties Δ_1 and Δ_2 in the individual LFTs combine into a single uncertainty Δ with block diagonal structure.

In other words, component level uncertainty leads to "structured" uncertainty at the system level.

Feedback Connections of LFTs

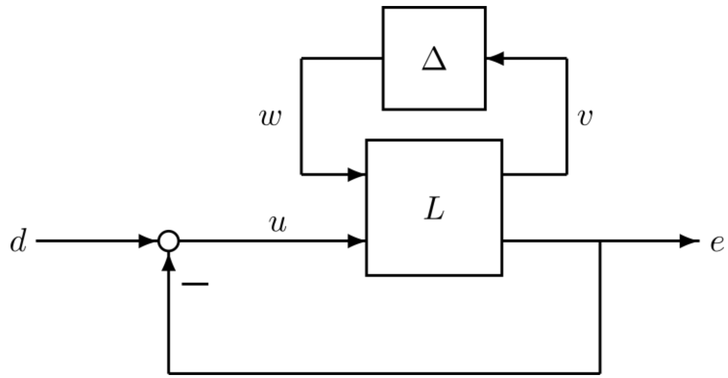
Consider the feedback connection of an LFT $F_U(L, \Delta)$.

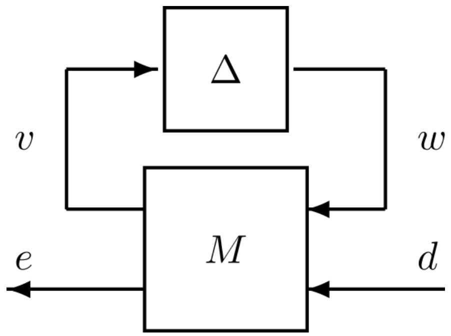


The LFT output is connected to the input via negative feedback.

Feedback Connections of LFTs

Consider the feedback connection of an LFT $F_U(L, \Delta)$.





The individual LFT is:

$$\begin{bmatrix} v \\ e \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$w = \Delta v$$

The combined LFT is:

$$\begin{bmatrix} v \\ e \end{bmatrix} = \underbrace{\begin{bmatrix} L_{11} - L_{12}(I + L_{22})^{-1} L_{21} & L_{12}(I + L_{22})^{-1} \\ (I + L_{22})^{-1} L_{21} & L_{22}(I + L_{22})^{-1} \end{bmatrix}}_{:=M} \begin{bmatrix} w \\ d \end{bmatrix} \quad \text{and} \quad w = \Delta v$$

There are some additional technical (well-posedness) conditions required for the inverses to exist.

Linear Fractional Transformations (LFTs)

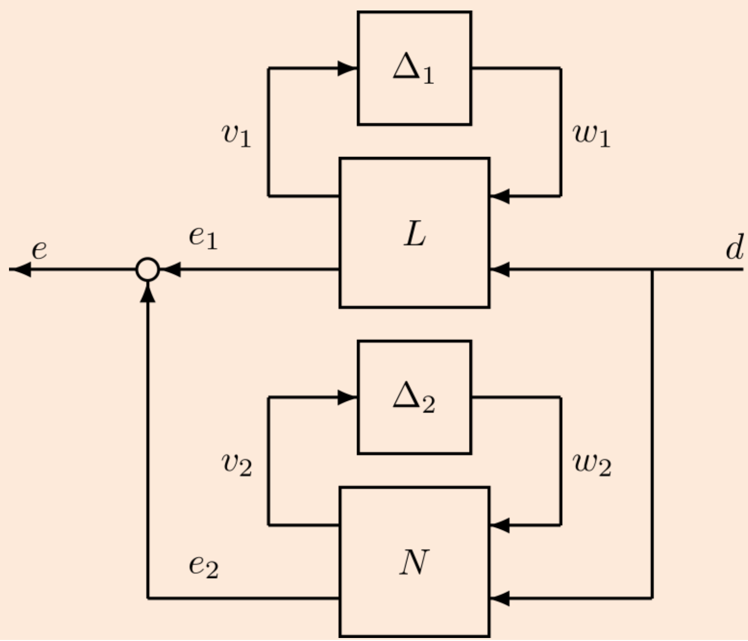
Wake-up Problem

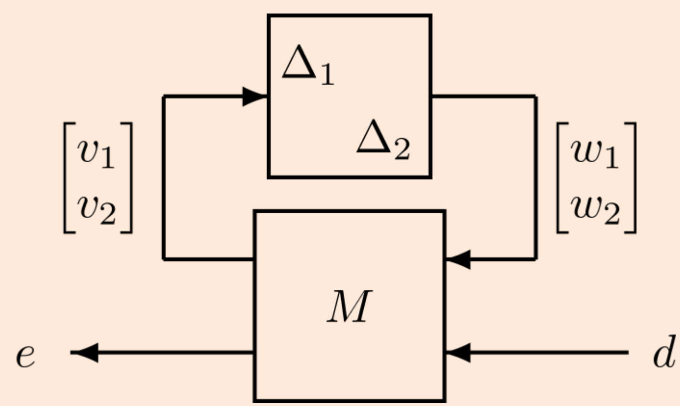
Consider the parallel connection of LFTs $\mathrm{F_U}(\mathrm{L}, \Delta_1)$ and $\mathrm{F_U}(\mathrm{N}, \Delta_2)$.

Express the combined LFT in terms of the individual LFTs:

$$\begin{bmatrix} v_2 \\ e_2 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} w_2 \\ d \end{bmatrix}, \quad w_2 = \Delta_2 v_2$$

$$\begin{bmatrix} v_1 \\ e_1 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ d \end{bmatrix}, \quad w_1 = \Delta_1 v_1$$



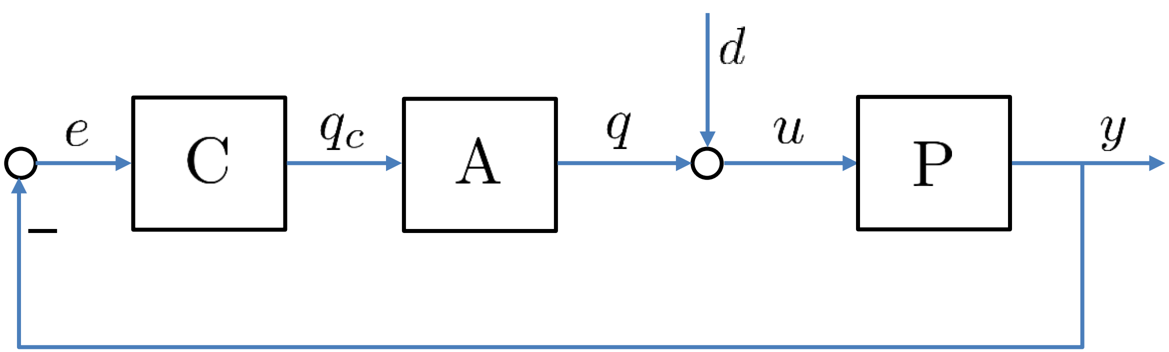


Example: SISO Feedback System

- Unstable plant with uncertain pole and input gain: $P(s) = \frac{b}{s-a}$ where $a \in [0.8, 1.1]$ and $b \in [1.7, 2.6]$

- First-order actuator with additive dynamic uncertainty $A(s) = A_0(s) + E(s)$ where $A_0(s) = \frac{10}{s+10}$ & $|E(j\omega)| \leq 0.1,\ E$ stable

- Proportional-Integral control $C(s) = \frac{3s+4.5}{s}$



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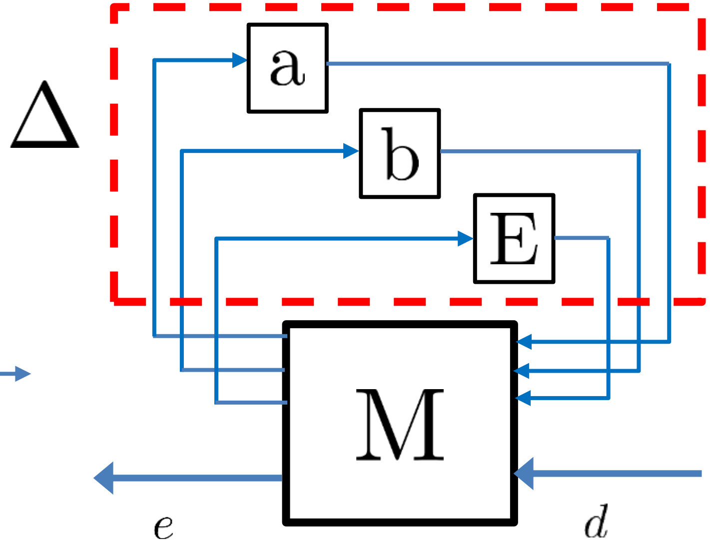
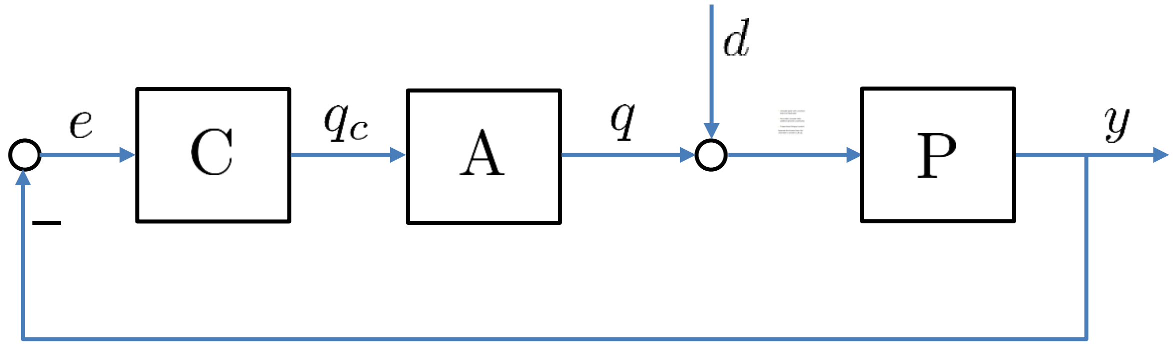


Separate the known from the uncertain to create $F_U(M, \Delta)$.



Uncertainty Normalization

The uncertainty can be normalized as follows:

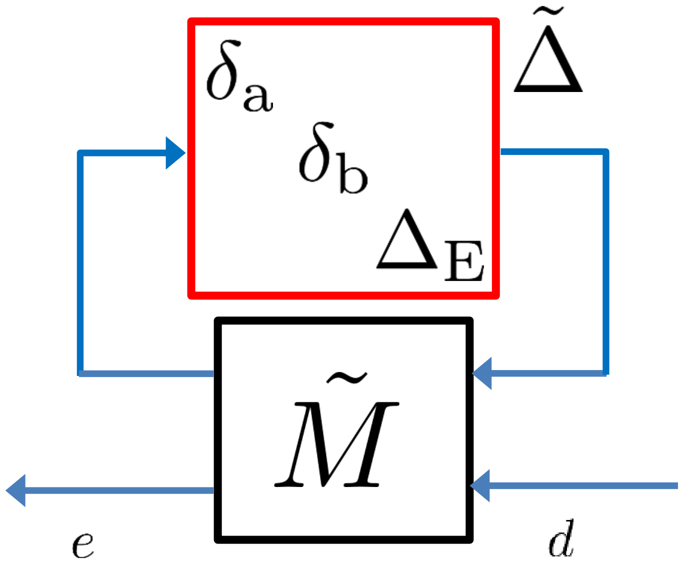
$$a \in [0.8, 1.1] \Rightarrow a = 0.95 + 0.15\delta_a \text{ with } |\delta_a| \leq 1.$$

$$b \in [1.7, 2.6] \Rightarrow b = 2.15 + 0.45\delta_b \text{ with } |\delta_b| \leq 1.$$

$$|E(j\omega)| \leq 0.1 \Rightarrow E = 0.1\Delta_E \text{ with } |\Delta_E(j\omega)| \leq 1.$$

This normalization often performed but is not necessarily required for our analyses later.

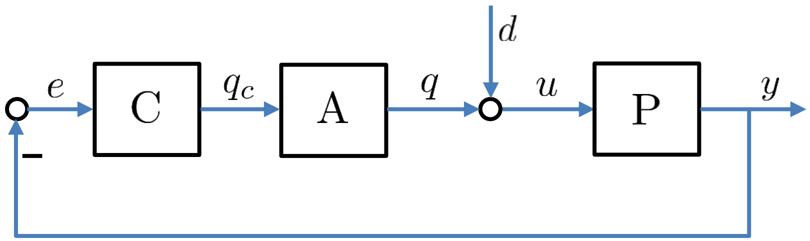
This yields an LFT in normalized form, $F_U(\widetilde{M}, \widetilde{\Delta})$.



Numerical Algorithms and Software

Reliable software to create uncertainty models & perform analyses.

- Matlab’s Robust Control Toolbox (Safonov & Chiang), (Balas, Doyle, Glover, Packard, & Smith), (Gahinet, Nemirovski, Laub, & Chilali)
- ONERA’s Systems Modeling, Analysis and Control Toolbox (Biannic, Burlion, Demourant, Ferreres, Hardier, Loquen, & Roos)



Example Matlab code to assess robustness of simple feedback loop.

```
% Unstable plant with parametric uncertainty
a = ureal('a',0.95, 'Range', [0.8  1.1]);
b = ureal('b',2.15, 'Range', [1.7  2.6]);
P = tf(b, [1 -a]);

% Actuator with non-parametric (dynamic) unc.
nomAct = tf(10, [1 10]);
DeltaA = ultidyn('DeltaA',[1 1],'Bound',0.1);
A = nomAct + DeltaA;

% Uncertain closed-loop (d->e) with PI control
C = tf([3 4.5],[1 0]);
Td2e = feedback(-P, A*C);

% Extract LFT model with normalized uncertainty
[M,DeltaNormalized] = lftdata(Td2e);
```

Static Quadratic Constraints (QCs)

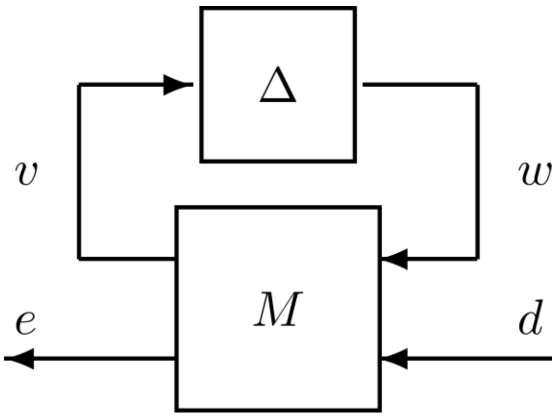
We introduced the LFT with Δ containing components that have unknown variations and/or are difficult to analyze.

The next step is to bound the input / output behavior Δ. A useful starting point is a static quadratic constraint on (v, w).

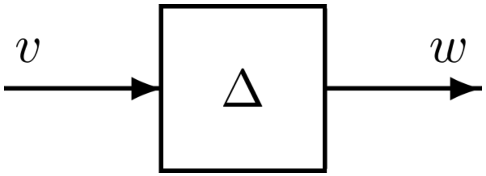
Definition: Δ satisfies the static QC defined by a matrix $J = J^\top \in \mathbb{R}^{(n_v+n_w)\times(n_v+n_w)}$ if each input/output pair $w = \Delta(v)$ satisfies:

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top J \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0 \quad \forall t \geq 0$$

Static QCs will provide useful bounds to combine with dissipation inequalities.

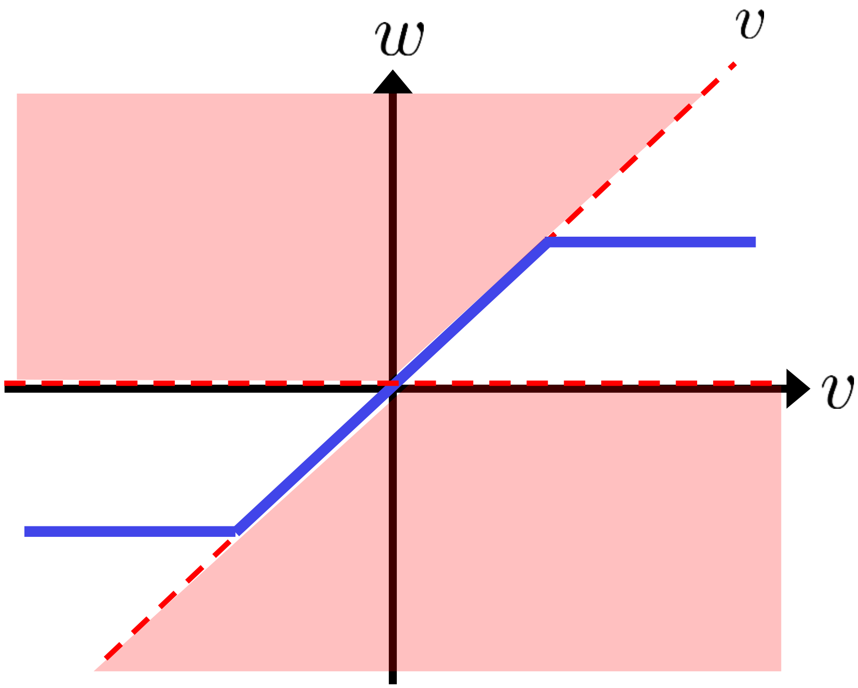


Example: Saturation



Suppose Δ is a unit saturation:

$$w(t) = \begin{cases} v(t) & \text{if } |v(t)| \leq 1 \\ \text{sign}(v(t)) & \text{otherwise} \end{cases}$$



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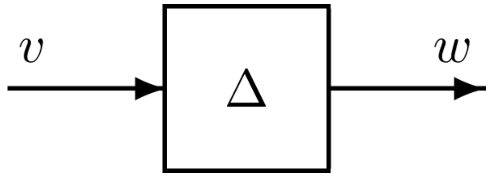

Suppose Δ is a unit saturation:

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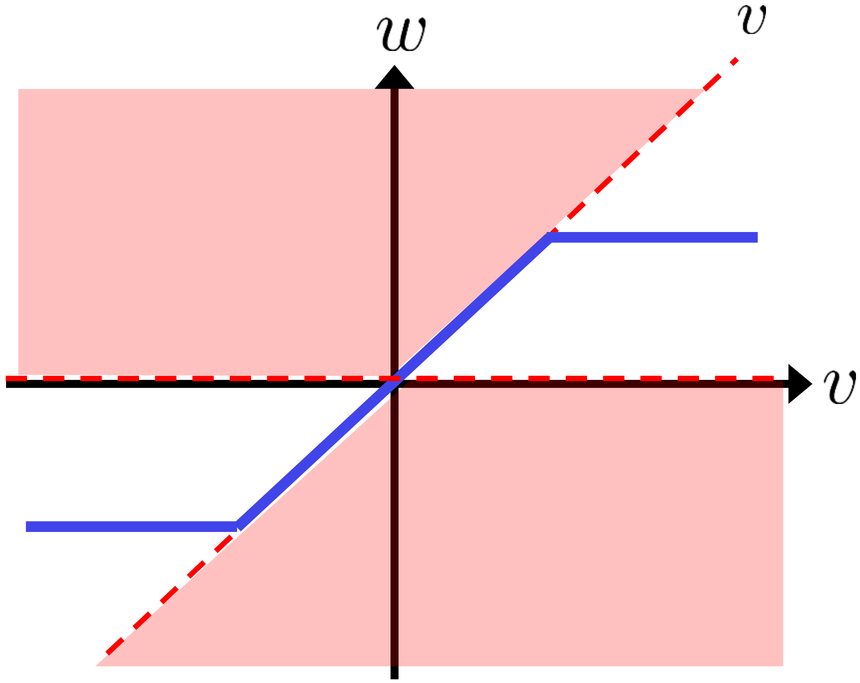


$$w(t) \cdot (v(t) - w(t)) \geq 0$$

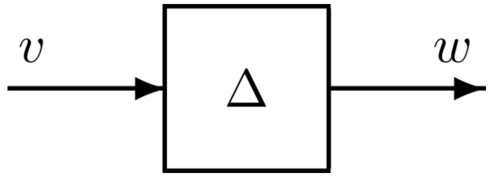
$\Updownarrow$

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}}_{:=J} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0$$

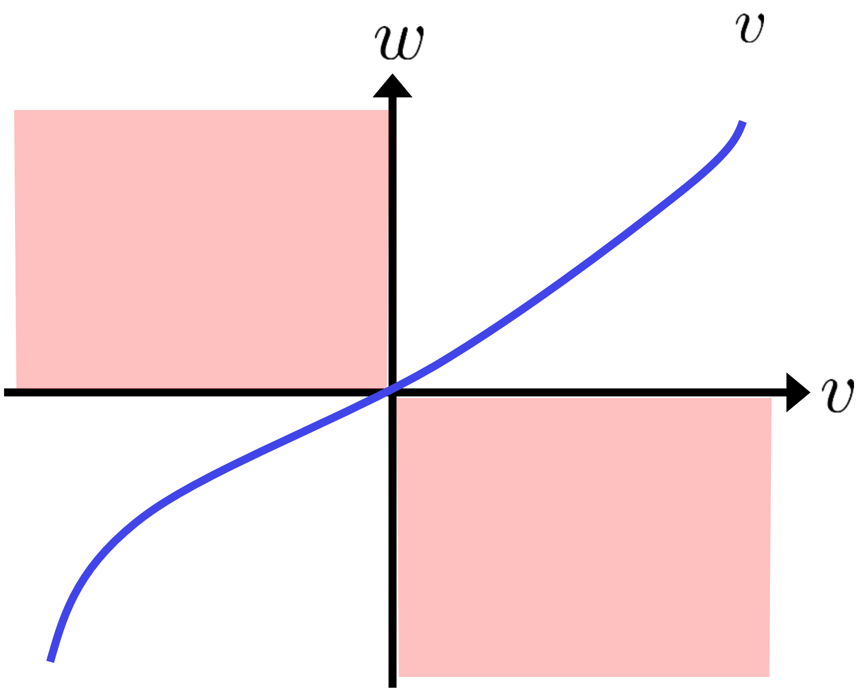
$\Updownarrow$

Δ satisfies the static QC defined by J.

Example: First/Third Quadrant Nonlinearity



Suppose Δ is a nonlinearity, $w = f(v,t)$, whose graph lies in the first/third quadrant.



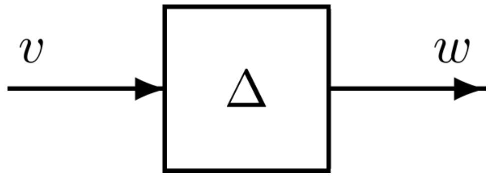
$$w(t) \cdot v(t) \geq 0$$

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{:=J} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0$$

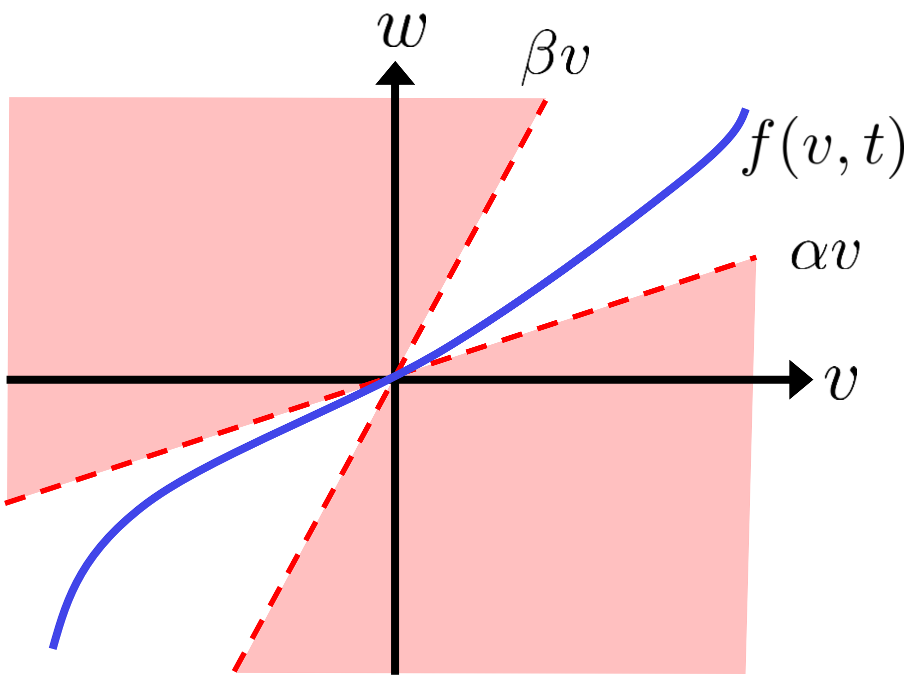
Δ satisfies the static QC defined by J.

The function is passive (pointwise in time).

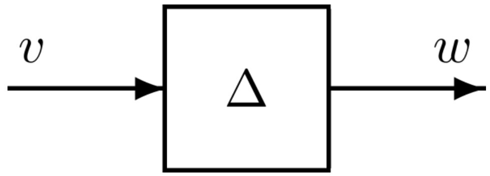
Example: Sector-bounded Nonlinearity



Suppose Δ is a nonlinearity, $w = f(v,t)$, whose graph lies in the sector $[\alpha, \beta]$.



Example: Sector-bounded Nonlinearity



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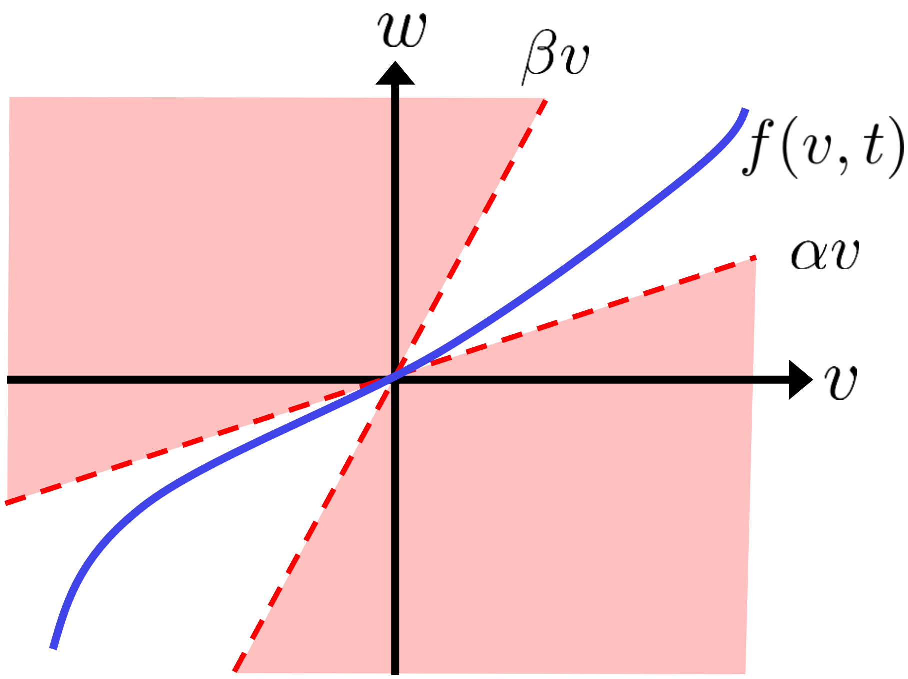


$$(w(t) - \alpha v(t)) \cdot (\beta v(t) - w(t)) \geq 0$$

$\Updownarrow$

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \underbrace{\begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix}}_{:=J} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0$$

$\Updownarrow$

Δ satisfies the static QC defined by J.

Static QCs

Wake-up Problems

Specify a sector that contains each function $w = f(v)$ below.

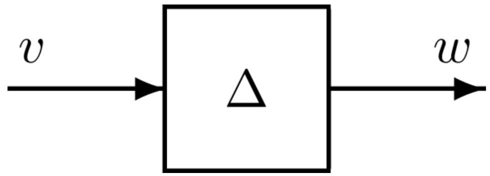
a) Unit Deadzone: $w = \begin{cases} 0 & \text{if } |v| \leq 1 \\ v - 1 & v \geq 1 \\ v + 1 & v \leq -1 \end{cases}$

b) ReLU: $w = \max(v, 0)$

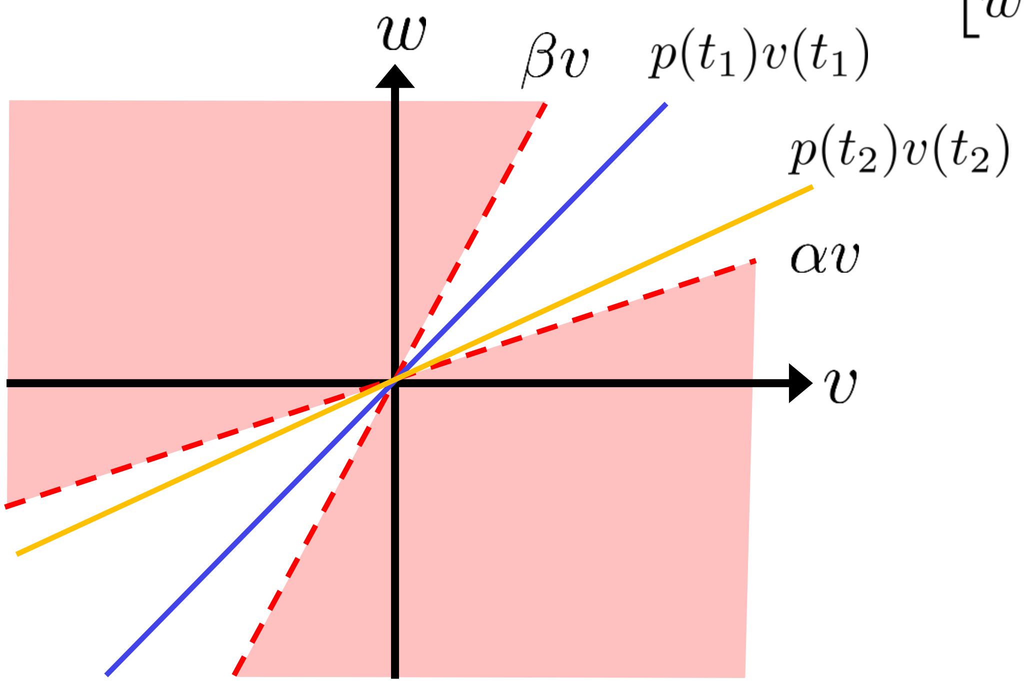
c) Leaky ReLU: $w = \begin{cases} v & \text{if } v \geq 0 \\ \epsilon v & v < 0 \end{cases}$ where $0 < \epsilon < 1$.

d) Cubic nonlinearity: $w = v^3$

Example: Time-Varying Real Parameter



Suppose $w = p(t)v$ where $p(t) \in [\alpha, \beta]$ for all $t \geq 0$.



$$(w(t) - \alpha v(t)) \cdot (\beta v(t) - w(t)) \geq 0$$

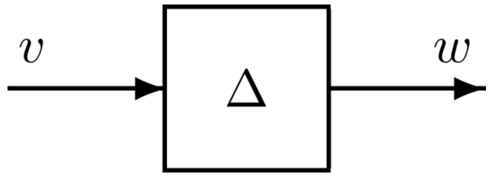
$\Updownarrow$

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \underbrace{\begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix}}_{:=J} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0$$

$\Updownarrow$

Δ satisfies the static QC defined by J.

Example: Time-Varying Real Parameter



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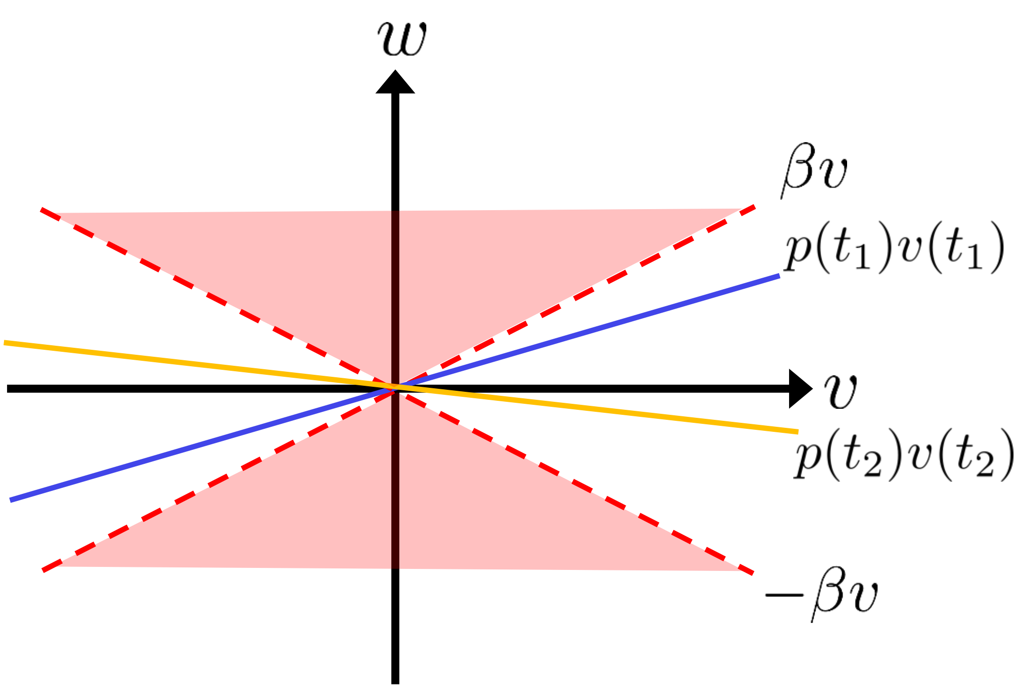

Special case:
If $\alpha = -\beta$ then $|p(t)| \leq \beta$.

$\Updownarrow$

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 2\beta^2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0$$

$\Updownarrow$

The constraint is unchanged when scaled by $\frac{1}{2}$ (or any positive constant).

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \underbrace{\begin{bmatrix} \beta^2 & 0 \\ 0 & -1 \end{bmatrix}}_{:=J} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0$$

General Static QCs

A QC with $J \in \mathbb{R}^{2\times 2}$ defines a set of input/output pairs:

$$\mathcal{S} := \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}^2 \ : \ \begin{bmatrix} v \\ w \end{bmatrix}^\top J \begin{bmatrix} v \\ w \end{bmatrix} \geq 0 \right\}$$

A) $J_{22} \geq 0$: For each v the QC is satisfied as $w \to \pm\infty$, e.g. $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ corresponds to functions in the first/third quadrant.

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A) $J_{22} \geq 0$: For each v the QC is satisfied as $w \to \pm\infty$, e.g. $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ corresponds to functions in the first/third quadrant.

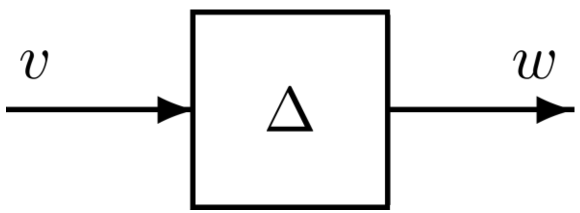
B) $J_{22} < 0$: We can scale $J_{22} = -2$ without loss of generality. It can also be shown the set S is non-empty if and only if $J_{12}^2 + 2J_{11} \geq 0$. If this condition holds then define:

$$\beta := \tfrac{1}{2}\left(J_{12} + \sqrt{J_{12}^2 + 2J_{11}}\right) \text{ and } \alpha := \tfrac{1}{2}\left(J_{12} - \sqrt{J_{12}^2 + 2J_{11}}\right)$$

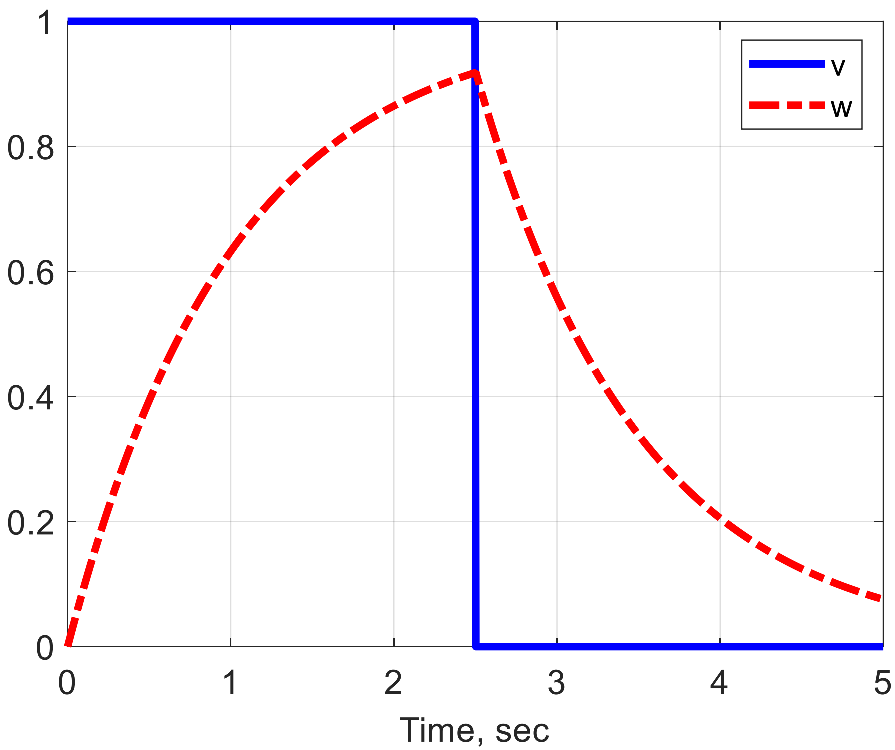
Then $-\infty < \alpha \leq \beta < \infty$ and $J = \begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix}$. Every nontrivial QC with $J_{22} < 0$ corresponds to a finite sector bound.

Motivation: Integral Quadratic Constraints (IQCs)

Static QCs are useful to bound the I/O behavior pointwise in time. However, if a system has memory then the output w at time t typically cannot be bounded by the input v at time t.



Example: $\Delta(s) = \frac{1}{s+1}$



Motivation: Integral Quadratic Constraints (IQCs)

Static QCs are useful to bound the I/O behavior pointwise in time. However, if a system has memory then the output w at time t typically cannot be bounded by the input v at time t.

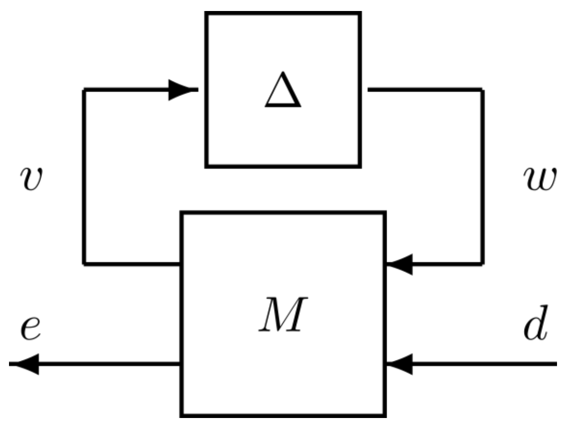
This motivates the more general constraint defined below.

Definition: Δ satisfies the IQC defined by a matrix $J = J^\top \in \mathbb{R}^{(n_v+n_w)\times(n_v+n_w)}$ if every $v \in L_2$ and $w = \Delta(v)$ satisfies:

$$\int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top J \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \quad \forall T \geq 0$$

Here L_2 denotes the set of signals with bounded L_2 norm, i.e. $v \in L_2$ if

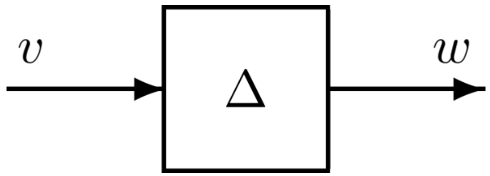
$$\|v\|_2 := \sqrt{\int_0^{+\infty} v^\top(t)v(t)dt} < \infty$$



Example: Passive Systems

A system Δ is *passive* if every $v \in L_2$ and $w = \Delta(v)$ satisfies

$$\int_0^T w(t)^\top v(t)\, dt \geq 0 \ \ \forall T \geq 0$$



The system can possibly be nonlinear and time-varying.

This is equivalent to:

$$\int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \ \ \forall T \geq 0$$

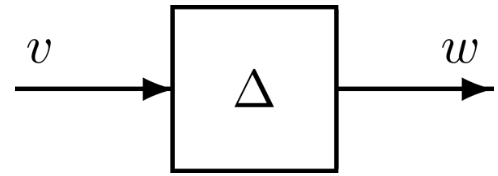
A system is passive if and only if it satisfies the IQC defined by

$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

Example: L_2 Bounded Systems

A system Δ has L_2 gain $\leq \beta$ if every $v \in L_2$ and $w = \Delta(v)$ satisfies

$$\int_0^T w(t)^\top w(t)\,dt \leq \beta^2 \int_0^T v(t)^\top v(t)\,dt \;\; \forall T \geq 0$$



Again, the system can possibly be nonlinear and time-varying.

This is equivalent to:

$$\int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \begin{bmatrix} \beta^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \;\; \forall T \geq 0$$

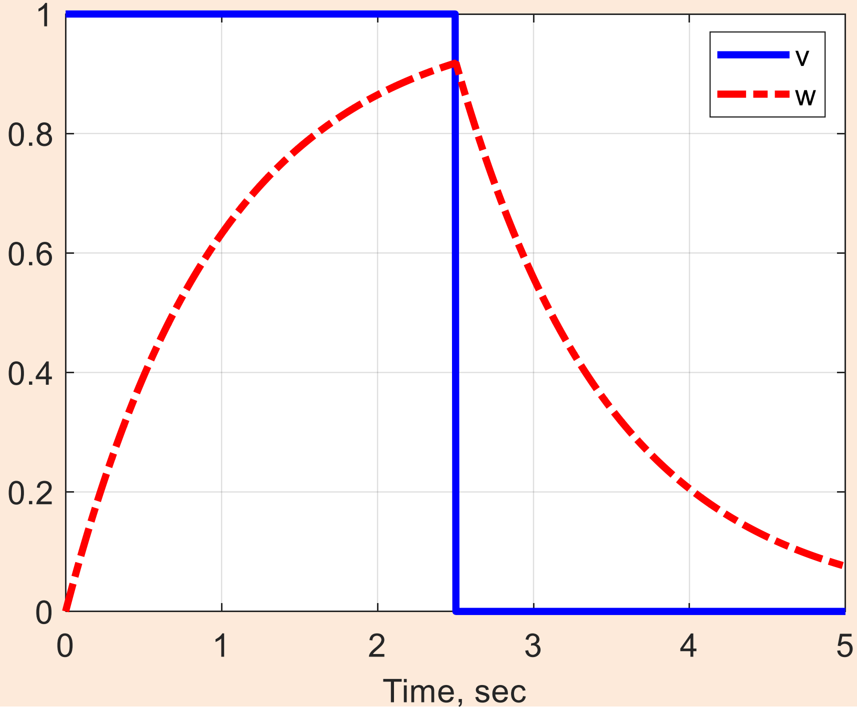
A system has L_2 gain $\leq \beta$ if and only if it satisfies the IQC defined by $J = \begin{bmatrix} \beta^2 I & 0 \\ 0 & -I \end{bmatrix}$.

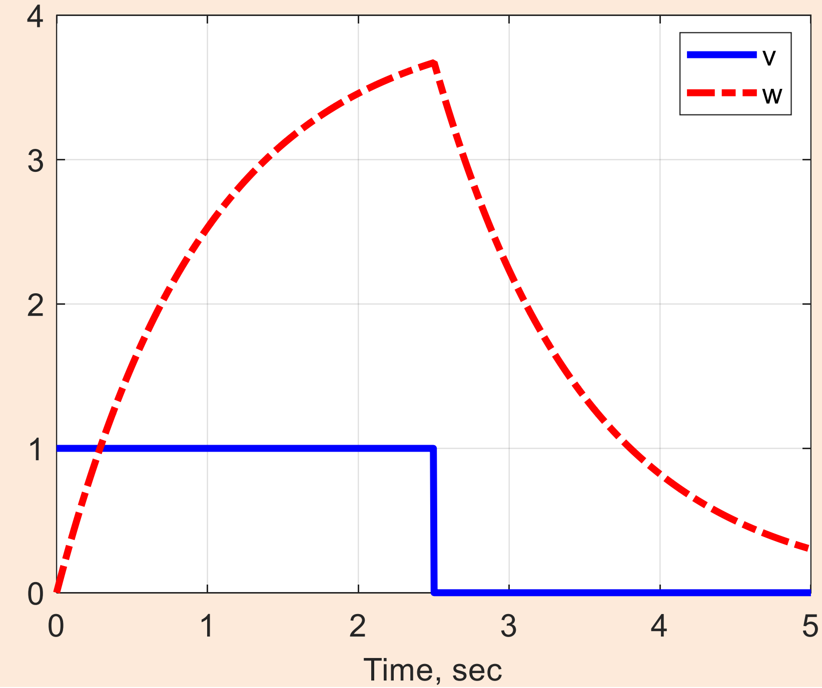
Integral Quadratic Constraints

Wake-up Problems

1) True or False? A system $w = \Delta(v)$ generated the input/output pair below on the left. The system could be passive.

2) True or False? A system $w = \Delta(v)$ generated the input/output pair below on the right. The system could have L_2 gain ≤ 1.



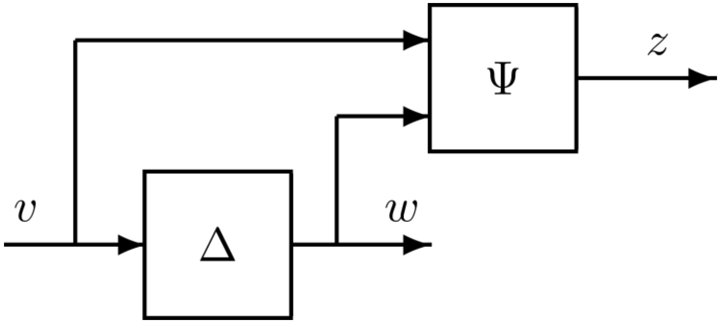


Dynamic IQCs

We can further enlarge the class of IQCs by allowing constraints on filtered input/output signals. This allows the IQC to capture additional system properties, e.g. time-invariance.

Definition: Δ satisfies the IQC defined by a stable filter Ψ and a matrix $J = J^\top \in \mathbb{R}^{(n_v+n_w)\times(n_v+n_w)}$ if every $v \in L_2$ and $w = \Delta(v)$ satisfies:

$$\int_0^T z(t)^\top J z(t)\, dt \geq 0 \qquad \forall T \geq 0$$

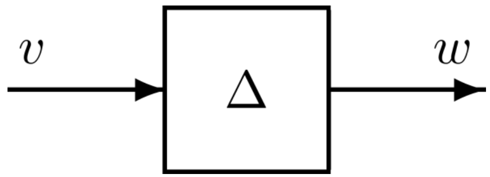


If $\Psi = I$ then $z = \begin{bmatrix} v \\ w \end{bmatrix}$ and this simplifies to our previous IQC definition.

Example: Norm-bounded LTI Uncertainty

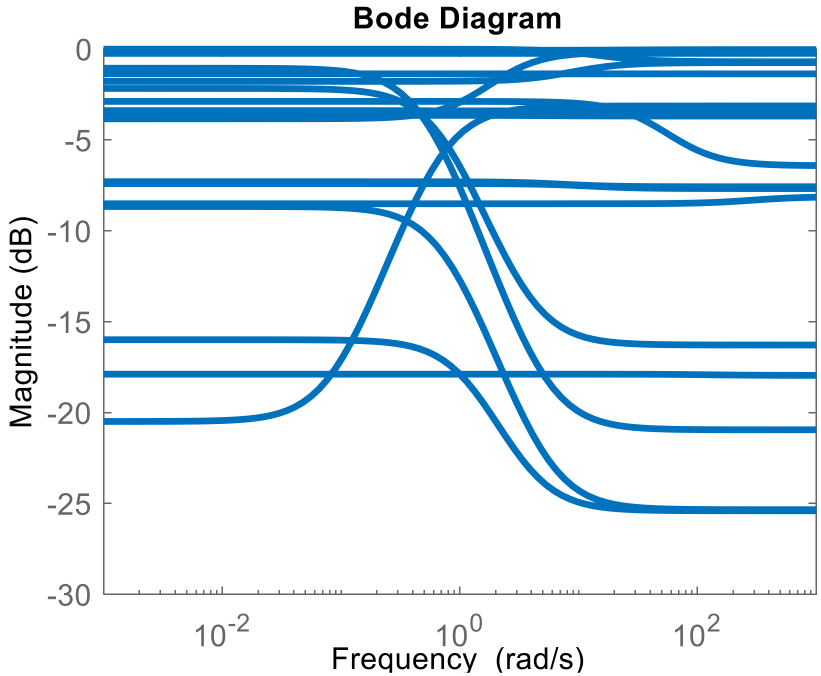
Consider a stable, causal, SISO, LTI system Δ with

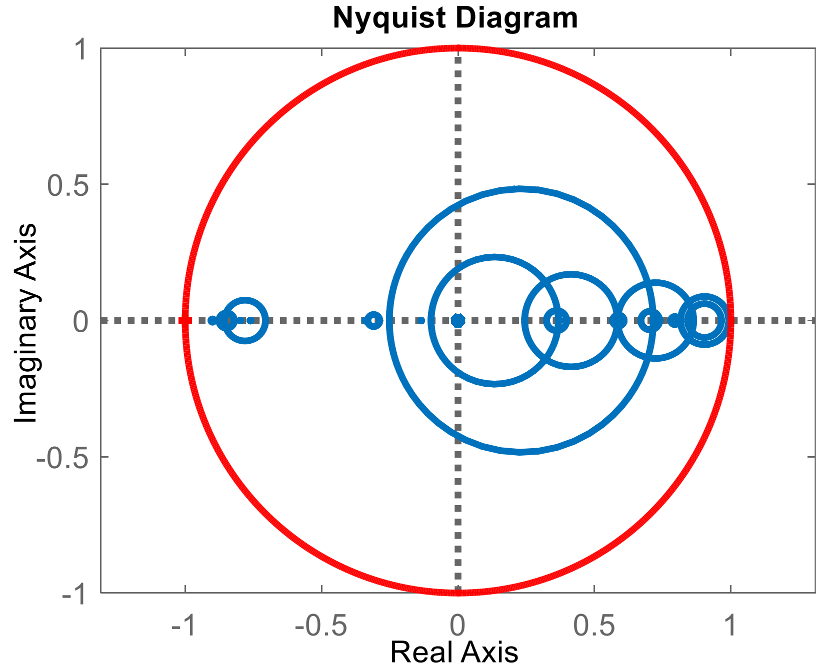
$$\|\Delta\|_\infty := \sup_\omega |\Delta(j\omega)| \leq 1$$



Here $||\Delta||_\infty$ is the H_∞ norm. The peak on a Bode magnitude plot is ≤ 1 and the Nyquist plot lies within the unit circle.

This was used previously to capture unmodeled dynamics / non-parametric uncertainty.





Example: Norm-bounded LTI Uncertainty

We need to recall a few additional facts to express the bound $||\Delta||_\infty \leq 1$ as a time-domain IQC.

1. *Fourier Transforms (FTs):* If $v \in L_2$ then we can define its FT and Inverse Fourier Transform (IFT):

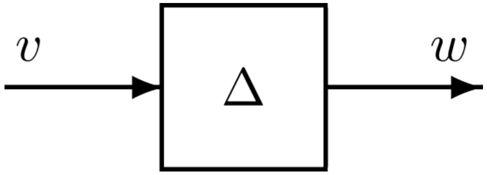
$$\hat{v}(j\omega) = \int_0^{+\infty} v(t) e^{-j\omega t} dt \text{ and } v(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{v}(j\omega) e^{+j\omega t} d\omega$$

2. *Parseval's (Plancheral's) theorem:* Assume $u, v \in L_2$ and let $\hat{u}$, $\hat{v}$ be their Fourier Transforms. The time and frequency domain inner products are equal:

$$\int_0^{+\infty} u(t) v(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}^*(\omega) \hat{v}(\omega)\, d\omega$$

3. *Linear system Response:* Assume Δ is a stable, causal LTI system. Let $v \in L_2$ be an input with (zero IC) output $w = \Delta\, v$. The FTs are related by:

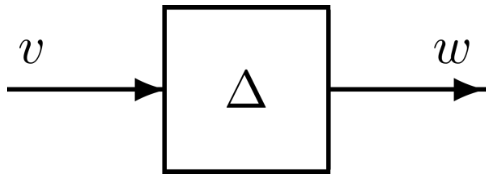
$$\hat{w}(j\omega) = \Delta(j\omega) \hat{v}(j\omega)$$



Example: Norm-bounded LTI Uncertainty

Consider a stable, causal, SISO, LTI system Δ with

$$\|\Delta\|_\infty := \sup_\omega |\Delta(j\omega)| \leq 1$$



Every $v \in L_2$ and $w = \Delta(v)$ satisfies

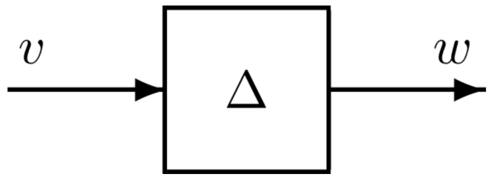
$$|\hat{w}(\omega)| \leq |\hat{v}(\omega)| \; \forall \omega \quad \Longleftrightarrow \quad \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} \geq 0 \; \forall \omega$$

This quadratic constraint holds pointwise in frequency.

Example: Norm-bounded LTI Uncertainty

Consider a stable, causal, SISO, LTI system Δ with

$$\|\Delta\|_\infty := \sup_\omega |\Delta(j\omega)| \le 1$$



Every $v \in L_2$ and $w = \Delta(v)$ satisfies

$$|\hat{w}(\omega)| \le |\hat{v}(\omega)| \; \forall \omega \quad \Leftrightarrow \quad \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} \ge 0 \; \forall \omega$$

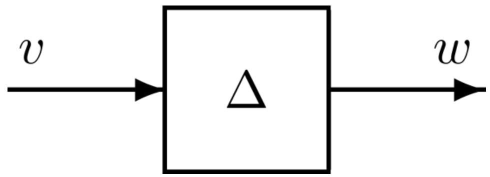
$$\Rightarrow \quad \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} d\omega \ge 0$$

Integrate over frequency to obtain a frequency domain IQC.

Example: Norm-bounded LTI Uncertainty

Consider a stable, causal, SISO, LTI system Δ with

$$\|\Delta\|_\infty := \sup_\omega |\Delta(j\omega)| \leq 1$$



Every $v \in L_2$ and $w = \Delta(v)$ satisfies

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$$\Longrightarrow \quad \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} d\omega \geq 0$$

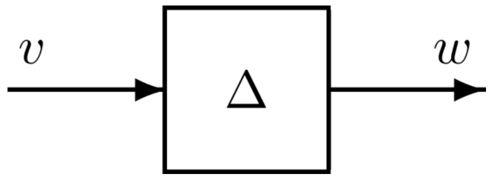
$$\Longleftrightarrow \quad \int_0^{\infty} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0$$

Use Parseval’s theorem to convert to an infinite-horizon, time-domain IQC. This is sometimes called a “soft” IQC.

Example: Norm-bounded LTI Uncertainty

Consider a stable, causal, SISO, LTI system Δ with

$$\|\Delta\|_\infty := \sup_\omega |\Delta(j\omega)| \le 1$$



Every $v \in L_2$ and $w = \Delta(v)$ satisfies

$$|\hat{w}(\omega)| \le |\hat{v}(\omega)| \; \forall \omega \quad \iff \quad \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} \ge 0 \; \forall \omega$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} d\omega \ge 0$$

$$\iff \quad \int_0^{\infty} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \ge 0$$

$$\iff \quad \int_0^{T} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \ge 0 \;\; \forall T \ge 0$$

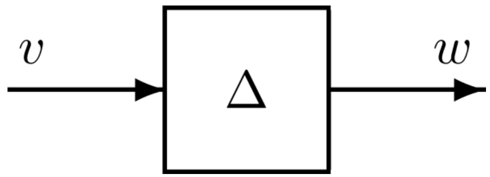
By causality, the constraint holds over all finite-time horizons.

Thus, Δ satisfies the ("hard") IQC defined by $\Psi = I$ and $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Example: Norm-bounded LTI Uncertainty

Consider a stable, causal, SISO, LTI system Δ with

$$\|\Delta\|_\infty := \sup_\omega |\Delta(j\omega)| \leq 1$$



Every $v \in L_2$ and $w = \Delta(v)$ satisfies

$$\int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \;\; \forall T \geq 0$$

This is just the IQC for a norm-bounded uncertainty.

Next, we exploit the fact that Δ is LTI.

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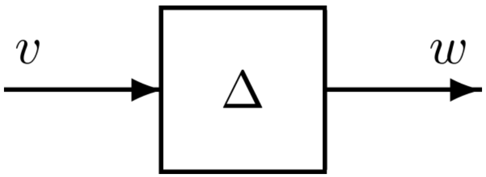

Let $D(s)$ be a bi-proper LTI system with all poles/zeros in the LHP, i.e. both $D(s)$ and $D^{-1}(s)$ are stable and proper.

Example: Norm-bounded LTI Uncertainty

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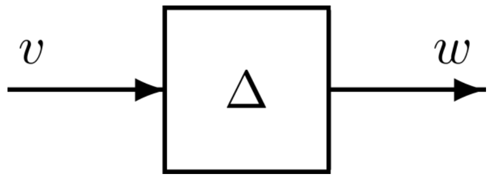


Let $D(s)$ be a bi-proper LTI system with all poles/zeros in the LHP.

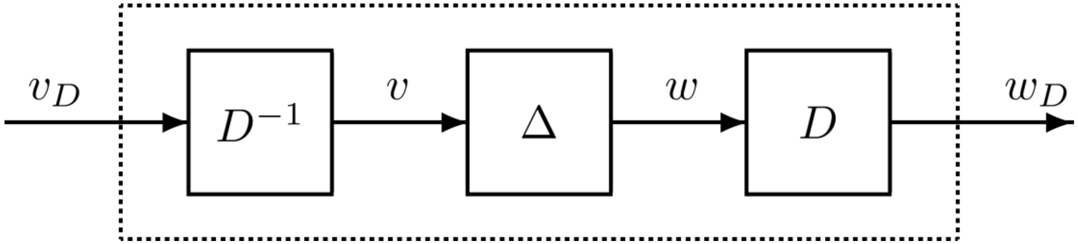
SISO, LTI systems commute:

$$D(s)\Delta(s) = \Delta(s)D(s) \implies \Delta(s) = D(s)\Delta(s)D^{-1}(s)$$

Thus $||\Delta||_\infty \leq 1$ implies $||\mathrm{D}\Delta\mathrm{D}^{-1}||_\infty \leq 1$ for any such D.

Every $v_D \in L_2$ and $w_D = (\mathrm{D}\Delta\mathrm{D}^{-1})(v_D)$ satisfies

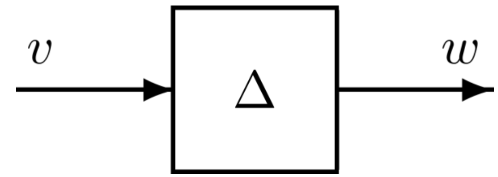
$$\int_0^T \begin{bmatrix} v_D(t) \\ w_D(t) \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_D(t) \\ w_D(t) \end{bmatrix} dt \geq 0 \;\; \forall T \geq 0$$



Example: Norm-bounded LTI Uncertainty

Consider a stable, causal, SISO, LTI system Δ with

$$\|\Delta\|_\infty := \sup_\omega |\Delta(j\omega)| \leq 1$$

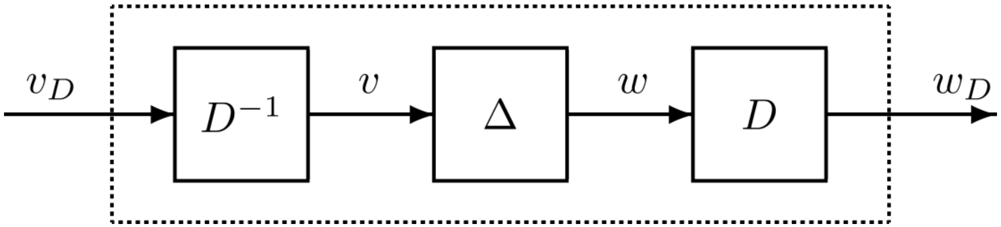


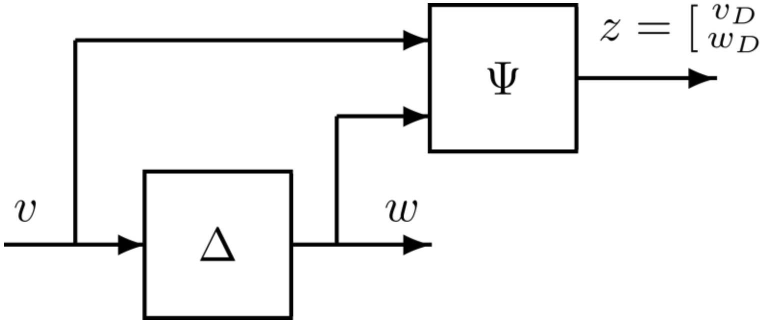
Let $D(s)$ be a bi-proper LTI system with all poles/zeros in the LHP.
Every $v_D \in L_2$ and $w_D = (\mathrm{D}\Delta\mathrm{D}^{-1})(v_D)$ satisfies

$$\int_0^T \begin{bmatrix} v_D(t) \\ w_D(t) \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_D(t) \\ w_D(t) \end{bmatrix} dt \geq 0 \;\; \forall T \geq 0$$

Note that $w_D = D\, w$ and $v_D = D\, v$.

Δ satisfies the IQC defined by $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\Psi = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$ for any such D. This is called a "D-scale" in the μ/SSV literature.





Partial Dictionary of IQCs [1]

Uncertainty/Nonlinearity	QC/IQC
1. Sector-bounded $[\alpha, \beta]$	$\begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix}$
2. Passive system	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$
3. L_2 gain-bounded by β	$\begin{bmatrix} \beta^2 I & 0 \\ 0 & -I \end{bmatrix}$
4. Stable, LTI with $\|\|\Delta\|\|_\infty \leq \beta$	$\begin{bmatrix} \beta^2 I & 0 \\ 0 & -I \end{bmatrix}, \ \Psi = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$ D is LTI with D, D^{-1} stable.

There are many more IQCs in the literature for delays, slope-restricted nonlinearities, etc.

Reference

[1] Megretski & Rantzer, System analysis via IQCs, TAC, 1997. [IQCs derived based on much prior literature]

Integral Quadratic Constraints

Wake-up Problems

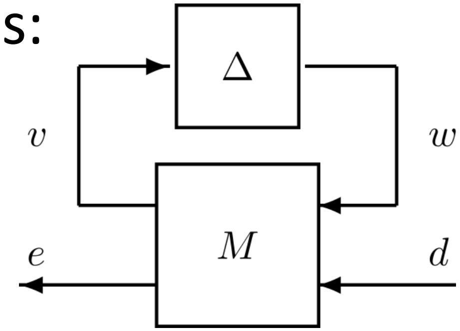
Consider a stable, SISO, system $w = \Delta(v)$. We defined the system to be passive if:

$$\int_0^T w(t)v(t)\,dt \geq 0 \;\; \forall T \geq 0$$

a) If $v, w \in L_2$ then their FTs exist. Use Parseval's theorem to express the passivity constraint as a frequency domain constraint on $\hat{v}, \hat{w}$

b) Suppose Δ is LTI in addition to being stable, SISO and passive. Use part a) to specify a constraint on the system transfer function $\Delta(j\omega)$.

Constructing storage functions using IQCs



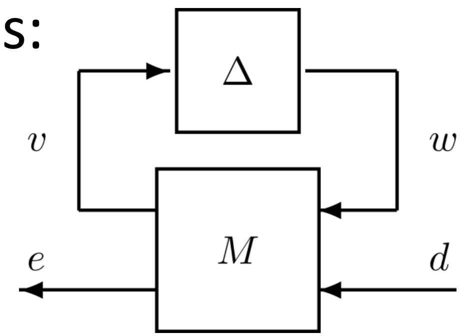
The analysis procedure consists of the following steps:

1. Express the uncertain system as an LFT $F_U(M, \Delta)$ with the uncertainty/nonlinearity in Δ.

2. Specify an IQC (J, Ψ) for Δ. This bounds the Input/output characteristics of Δ.

Constructing storage functions using IQCs

The analysis procedure consists of the following steps:



1. Express the uncertain system as an LFT $F_U(M, \Delta)$ with the uncertainty/nonlinearity in Δ.

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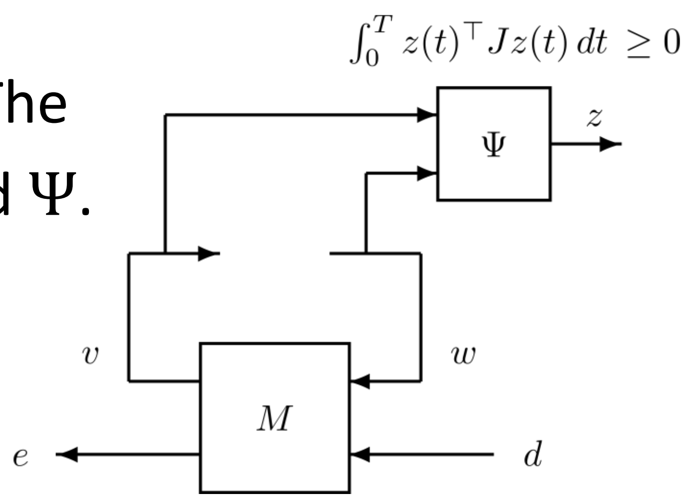

3. Append the IQC dynamics to the system. The appended system has the dynamics of M and Ψ.

$$\begin{bmatrix} \dot{x}_e(t) \\ z(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} x_e(t) \\ w(t) \\ d(t) \end{bmatrix}$$

4. Write a dissipation inequality on the appended system exploiting the IQC. (See next slide.)

Note: Multiple uncertainties/nonlinearities can be combined into $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_n)$ and each block can have multiple IQCs.

Constructing storage functions using IQCs

The appended system has the form:
$$\begin{bmatrix} \dot{x}_e(t) \\ z(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} x_e(t) \\ w(t) \\ d(t) \end{bmatrix}$$

Suppose there is a storage function $V(x_e) = x_e^\top P\, x_e$ with $P \geq 0$ such that the dissipation inequality (DI) holds along trajectories:

$$\frac{d}{dt} V(x_e(t)) + \begin{bmatrix} e(t) \\ d(t) \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e(t) \\ d(t) \end{bmatrix} + z(t)^\top J z(t) \leq 0$$

Constructing storage functions using IQCs

The appended system has the form: $\begin{bmatrix} \dot{x}_e(t) \\ z(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} x_e(t) \\ w(t) \\ d(t) \end{bmatrix}$

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$$\frac{d}{dt}V(x_e(t)) + \begin{bmatrix} e(t) \\ d(t) \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e(t) \\ d(t) \end{bmatrix} + z(t)^\top J z(t) \leq 0$$

Integrating from $t = 0$ to $t = T$ yields:

$$\underbrace{V(x_e(T))}_{\geq 0} - V(x_e(0)) + \int_0^T e(t)^\top e(t)dt + \underbrace{\int_0^T z(t)^\top J z(t)dt}_{\geq 0} \leq \gamma^2 \int_0^T d(t)^\top d(t)dt$$

If $x_e(0) = 0$, $d \in L_2$ then we can let $T \to \infty$ to obtain $||e||_2 \leq \gamma ||d||_2$.

The DI + IQC verifies the uncertain system $F_U(M, \Delta)$ has L_2 gain $\leq \gamma$.

With a few additional technical details, we can prove $x_e(t) \to 0$.

Constructing storage functions using IQCs

The appended system has the form: $\begin{bmatrix} \dot{x}_e(t) \\ z(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} x_e(t) \\ w(t) \\ d(t) \end{bmatrix}$

Suppose there is a storage function $V(x_e) = x_e^\top P\, x_e$ with $P \geq 0$ such that the dissipation inequality (DI) holds along trajectories:

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This DI can be expressed as an LMI:

$$\begin{bmatrix} \mathcal{A}^\top P + P\mathcal{A} & P\mathcal{B}_1 & P\mathcal{B}_2 \\ \mathcal{B}_1^\top P & 0 & 0 \\ \mathcal{B}_2^\top P & 0 & 0 \end{bmatrix} + (\cdot)^\top \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \mathcal{C}_2 & \mathcal{D}_{22} & \mathcal{D}_{22} \\ 0 & 0 & I \end{bmatrix}$$

$$+ (\cdot)^\top J \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \end{bmatrix} \preceq 0$$

We'll revisit the numerical aspects of this LMI in a later lesson.

Summary

In this lesson:

- We used the linear fractional transformation (LFT) to model systems with uncertainties and nonlinearities.
- We introduced static quadratic constraints (QCs) defined by a symmetric matrix J:

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^\top J \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0 \quad \forall t \geq 0$$

  These are pointwise-in-time constraints on the I/O behavior of Δ.
- We defined integral quadratic constraints (IQCs). These bound the I/O behavior of of Δ when integrated over any finite time horizon.
- We combined dissipation inequalities with QCs/IQCs to assess the stability and performance of uncertain systems.

Next lesson: Applications to networks and differential algebraic equations (DAEs).

Further Reading

Uncertainty Modeling:

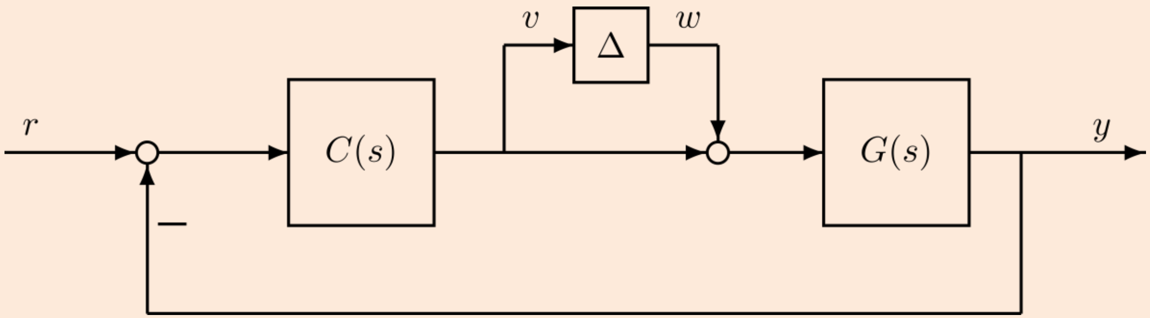
- Zhou, Doyle, Glover, Robust and Optimal Control, 1995.
- Skogestad, Postlethwaite, Multivariable Feedback Control, 2005.
- Dullerud, Paganini, A Course in Robust Control Theory, 2010.

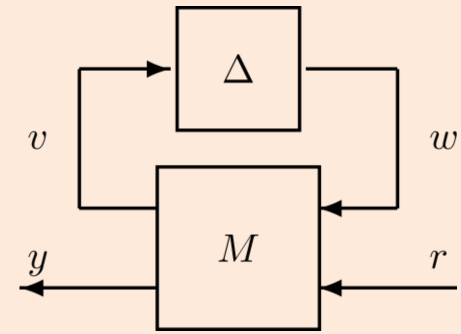
Integral Quadratic Constraints (IQCs):

- Yakubovich, S-procedure in nonlinear control theory, VLU, 1971.
- Megretski, Rantzer, System analysis via IQCs, TAC, 1997.
- Seiler, Stability Analysis with Dissipation Inequalities and Integral Quadratic Constraints, TAC, 2015.
- Veenman, Scherer, Köroğlu. Robust stability and performance analysis based on IQCs, EJC, 2016.
- Hu, Lacerda, Seiler, Robustness Analysis of Uncertain Discrete-Time System with … IQCs, IJRNC, 2016.
- Scherer, Dissipativity and Integral Quadratic Constraints: Tailored Computational Robustness Tests for Complex Interconnections, CSM, 2022.

Self-Study Problems

1) Express the SISO feedback diagram below as an LFT $F_U(M, \Delta)$.





2) Consider the following state-space system:

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

Show that the transfer function $G(s) = C(sI - A)^{-1}B + D$ can be expressed as an LFT $F_U(M, \frac{1}{s}I)$ where:

$$\mathrm{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Self-Study Problems

3) A stable, SISO, LTI system Δ satisfies the IQC defined by $J = \mathrm{diag}(1, -1)$, and $\Psi = \mathrm{diag}(W, 1)$ where $W(s) = \frac{2s+0.1}{s+1}$. What is the uncertainty bound at $\omega = 0$? What is the uncertainty bound as $\omega \to \infty$?

4) True or False? A SISO system Δ satisfies the IQC defined by $J = \mathrm{diag}(1, -1)$, and Ψ =I. The system $-\Delta$ satisfies the IQC defined by $J = \mathrm{diag}(-1,1)$, and Ψ =I.

5) True or False? A system Δ satisfies the IQCs defined by (J_1, Ψ_1) and (J_2, Ψ_2). For any non-negative λ_1, λ_2, Δ satisfies the IQC with:

$$J = \begin{bmatrix} \lambda_1 J_1 & 0 \\ 0 & \lambda_2 J_2 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$$