International Graduate School on Control, Stuttgart, May 2024

Dissipation Inequalities and Quadratic Constraints for Control, Optimization, and Learning

Lesson 1: Introduction to Dissipation Inequalities

Murat Arcak[1] and Peter Seiler[2]

[1] University of California, Berkeley

[2] University of Michigan, Ann Arbor

Learning Objectives

In this lesson, we will:

- Recall state space models, and equilibrium and stability concepts
- Learn about the essence of dissipation inequalities through Lyapunov functions
- Learn the fundamental notions of dissipativity, storage functions, and supply rates
- See how dissipativity can be used together with constraints on system uncertainty to establish robust stability and performance

Outline

1. State space models
2. Equilibria and stability
3. Lyapunov functions
4. Special case: linear systems
5. Dissipativity
6. Constructing storage functions
7. Robust stability and performance

State Space Models

Differential equation model for a nonlinear dynamical system:

$$\dot{x}(t) = f(x(t))$$

$x(t) \in \mathbb{R}^n$: vector of state variables, e.g., position and velocity

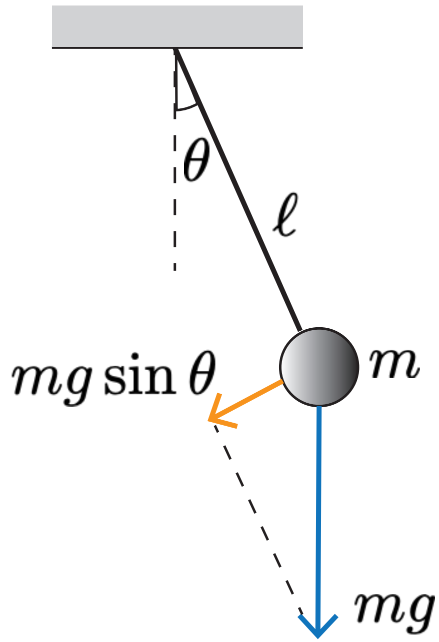
$\dot{x}(t)$: shorthand for time derivative, $\frac{d}{dt}x(t)$

$f : \mathbb{R}^n \to \mathbb{R}^n$: function describing the evolution of the states, typically derived from physical laws

n : number of state variables needed to describe the dynamics (”system order”)

State Space Models

Example: Pendulum



State variables: angle and angular velocity

$$x(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$

Dynamical model:

$$\dot{x}_1(t) = x_2(t)$$

from definition of state variables

$$\dot{x}_2(t) = -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t)$$

from: mass x acceleration = force

$$m\,\ell\ddot{\theta} = mg\sin\theta - k\dot{\theta}$$

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix}$$

State Space Models

Linear systems: special case where $f(x)$ has no nonlinear terms

$$f(x) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix} \quad \rightarrow \quad \begin{matrix} f(x) = Ax \\ A \in \mathbb{R}^{n \times n} \end{matrix}$$

Systems above are time-invariant, also called *autonomous* systems.

Time-varying (*nonautonomous*) systems: dynamics change in time, e.g., rocket with reducing mass due to fuel consumption

$$\dot{x}(t) = f(\mathbf{t}, x(t))$$

Time-varying linear systems: $\dot{x}(t) = A(\mathbf{t})x(t)$

Going forward, time-invariant models unless otherwise stated.

State Space Models

Systems with inputs and outputs:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

$x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$

Input, u: variables we can manipulate ("control") or exogeneous variables that affect the dynamics ("disturbance")

Output, y: variables of particular interest, e.g., attitude of satellite we would like to control

Linear case: $f(x,u) = Ax + Bu,\ h(x,u) = Cx + Du$

A, B, C, D appropriately dimensioned matrices

State Space Models

History's Mysteries

Why letter 'u' for input?

Possibly from Russian "Upravlenie" for "control."

Before Sputnik, control theory in the East was driven by mechanics and used the state space language. In the West it emerged from circuit theory, dominated by input-output language: transfer functions, frequency domain...

Equilibria and Stability

"Thermodynamicists get very excited when nothing happens."

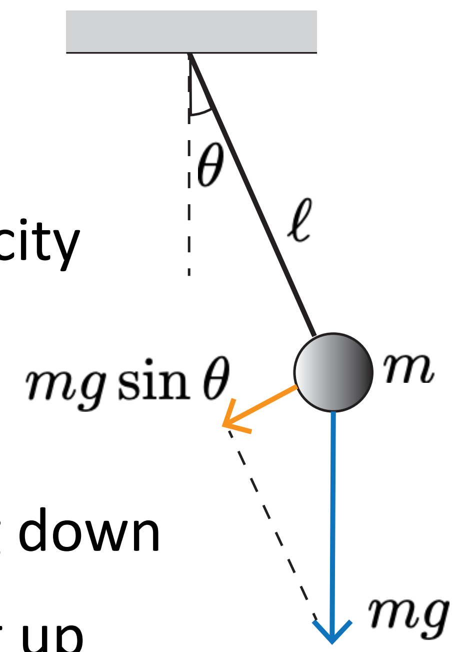
Peter Atkins, chemist at Oxford

Equilibria and Stability

An **equilibrium** (or **rest point**) of a dynamical system $\dot{x} = f(x)$ is a point x^* such that $f(x^*) = 0$

If the state vector starts at x^* it remains there because the time derivative is zero: $x(0) = x^* \quad \Rightarrow \quad x(t) = x^*,\, t \geq 0$

Example: Pendulum



$$f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix}$$

x_1 : angle

x_2 : angular velocity

$$f(x) = 0 \;\Rightarrow\; x_2 = 0,\; \sin x_1 = 0$$

Two equilibrium points: $(x_1, x_2) = (0, 0)$ pointing down

$(x_1, x_2) = (\pi, 0)$ pointing up

Equilibria and Stability

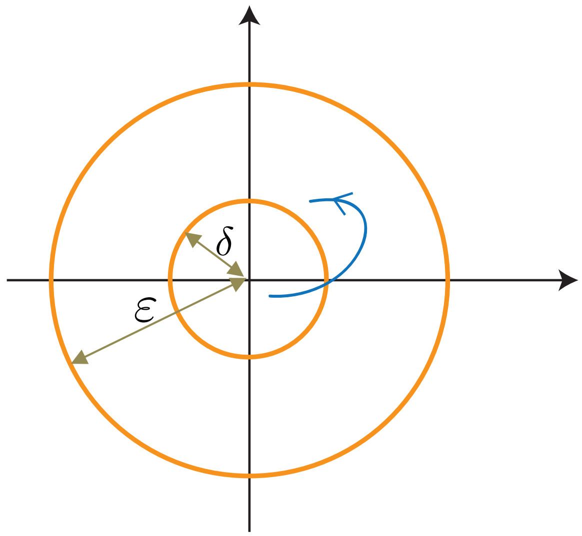
For stability definitions we assume equilibrium at the origin:

$$f(0) = 0$$

No loss of generality in this assumption: for nonzero equilibrium x^* define shifted state $\tilde{x} = x - x^*$ so the equilibrium is now $\tilde{x} = 0$.

The equilibrium $x = 0$ is called **stable** if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x(0)| \leq \delta \;\Rightarrow\; |x(t)| \leq \varepsilon \quad t \geq 0$$



i.e., if trajectory starts close to the equilibrium, it remains close.

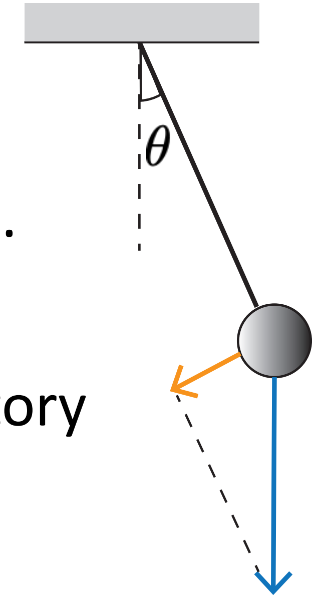
Called **unstable** if not stable.

Equilibria and Stability

The equilibrium $x = 0$ is called **asymptotically stable** if it is stable and $x(t) \to 0$ from initial conditions close to the origin.

Globally asymptotically stable if convergence guaranteed from *all* initial conditions.

Example: Pendulum



Downward equilibrium is stable even without friction: small perturbation leads to small amplitude oscillation.

If there is friction, then asymptotically stable, but not *globally*: there are initial conditions from which trajectory doesn't converge to origin (e.g., upward equilibrium).

Upward equilibrium: unstable.

Equilibria and Stability

When is the origin (asymptotically) stable for linear system $\dot{x} = Ax$?

Eigenvalue Test: Asymptotically stable if and only if all eigenvalues of A have negative real parts. For linear systems asymptotic stability is always global.

If at least one eigenvalue has positive real part, then unstable.

If no eigenvalue has positive real part, but some have zero real parts: stable if and only if all eigenvalues with zero real part have Jordan blocks of order one (trivially satisfied if no repeated eigenvalues).

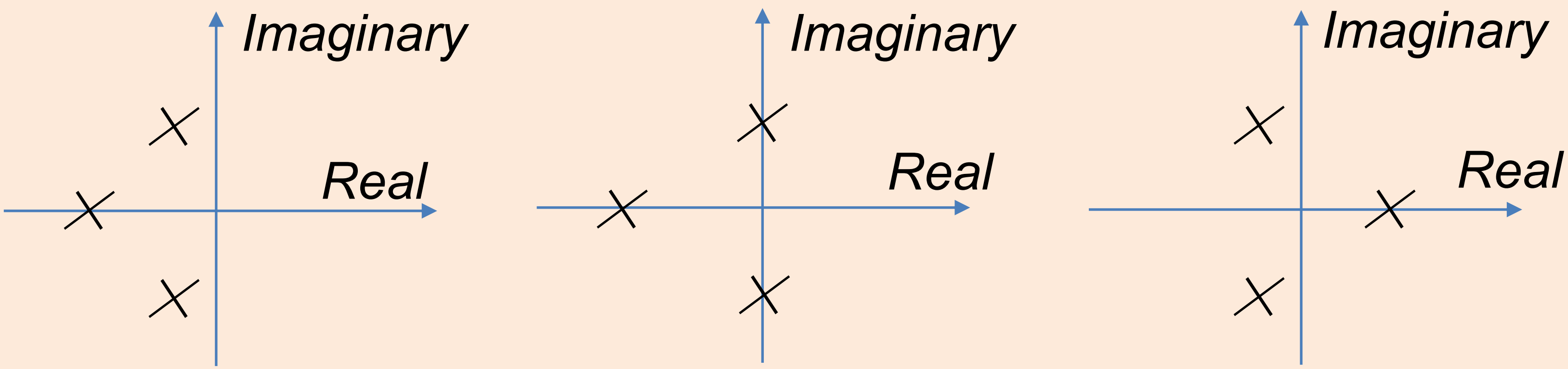
Recall: $\lambda \in \mathbb{C}$ is an eigenvalue of square matrix A if

$$\det(\lambda I - A) = 0$$

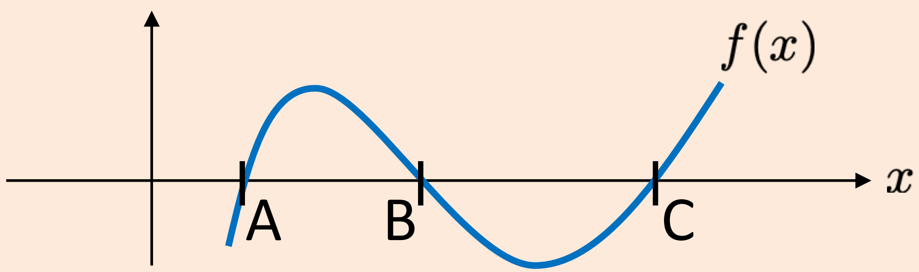
Equilibria and Stability

Wake-up Problems

1) Which of the following eigenvalue configurations for a linear system indicates (non-asymptotic) stability?



2) Given scalar system $\dot{x} = f(x)$ where f is as shown below, which equilibrium is asymptotically stable?



Lyapunov Functions

Solutions of general nonlinear systems not known explicitly. How to establish stability of equilibria?

A. M. Lyapunov, The General Problem of the Stability of Motion, 1892:

If we can find a function:

- zero at equilibrium, positive elsewhere
- whose value decreases along the trajectories of the system,

then the equilibrium is stable.

We can show the function is decreasing along the trajectory without knowing the trajectory with a *dissipation inequality.*

Alexandr Mikhailovich Lyapunov (1857-1918)

Lyapunov Functions

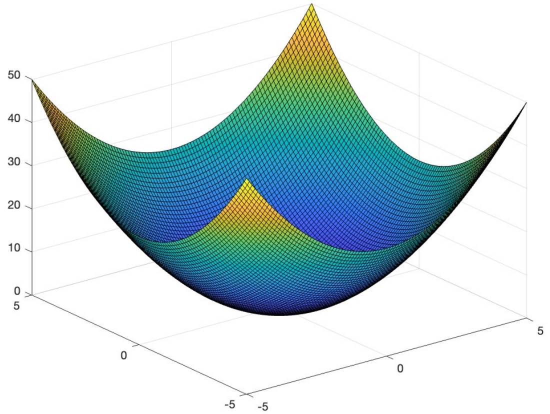
A scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$ that is zero at zero is called **positive semidefinite** if nonnegative everywhere:

$$V(x) \geq 0 \quad \forall x \in \mathbb{R}^n$$

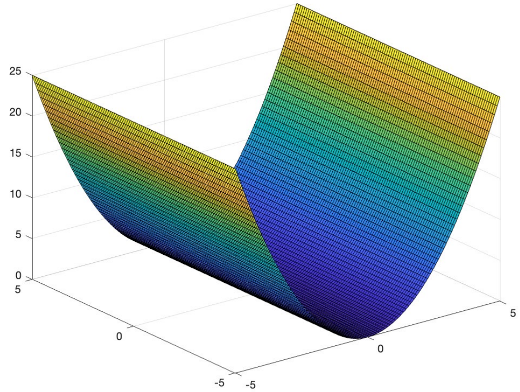
positive definite if strictly positive except at zero:

$$V(x) > 0 \quad \forall x \neq 0$$

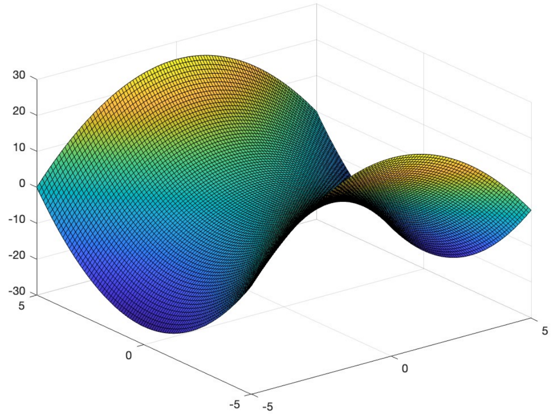
and **negative (semi)definite** if $-V$ is positive (semi)definite.



positive definite



positive semidefinite



sign indefinite

Lyapunov Functions

Consider the nonlinear system:

$$\dot{x} = f(x), \;\; x \in \mathbb{R}^n$$

and assume the origin is an equilibrium: $f(0) = 0$

Given $V : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ define notation:

$$L_f V(x) := \nabla V(x)^\top f(x)$$

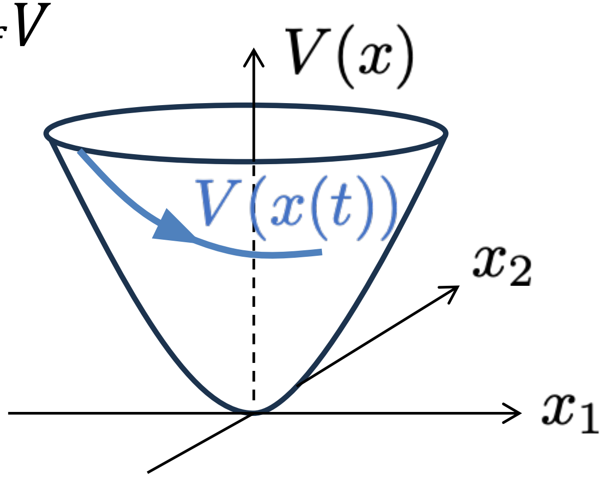
Theorem (Lyapunov): If there exists positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ s.t. $L_f V$ is negative semidefinite, then the origin is **stable**. If $L_f V$ is negative definite, then **asymptotically stable**.

If, further, $V(x) \to \infty$ as $\|x\| \to \infty$, then **globally asymp. stable**.

Lyapunov Functions

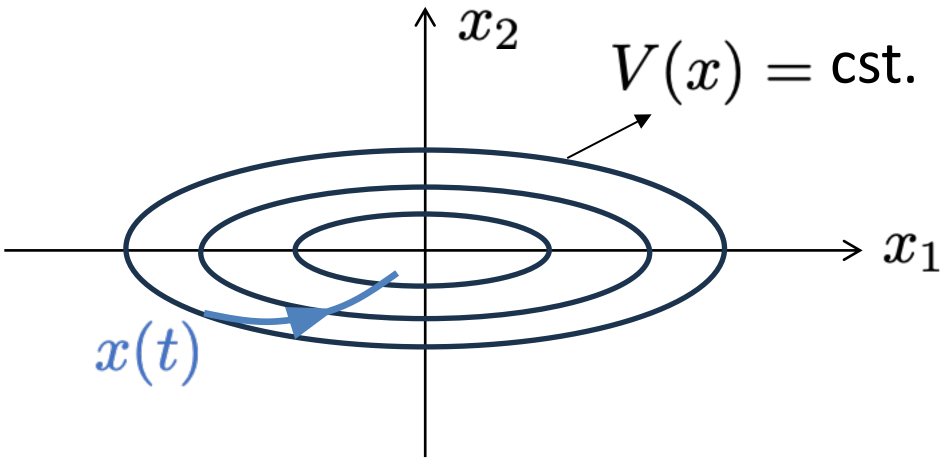
Proof idea: By the chain rule, negativity of $L_f V$ implies V is decreasing along trajectories:

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t))^\top f(x(t))$$
$$= L_f V(x(t)) \leq 0$$



Because of the decreasing property above, sublevel sets of V trap trajectories:

If $x(0) \in \{x : V(x) \leq c\}$ then $x(t) \in \{x : V(x) \leq c\}\ \forall t \geq 0$

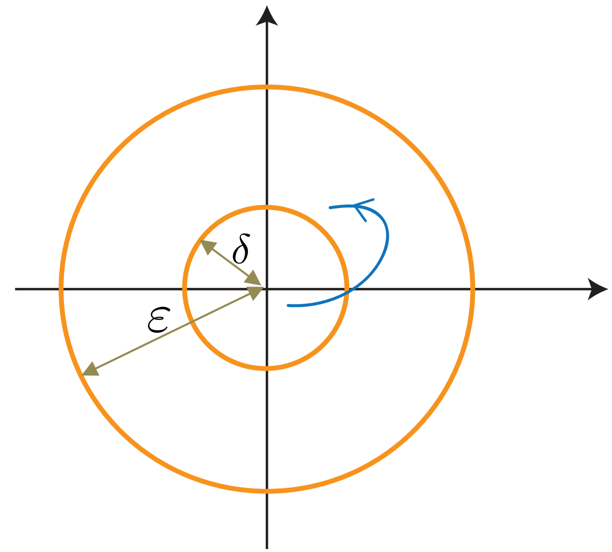


Lyapunov Functions

Recall the stability definition:

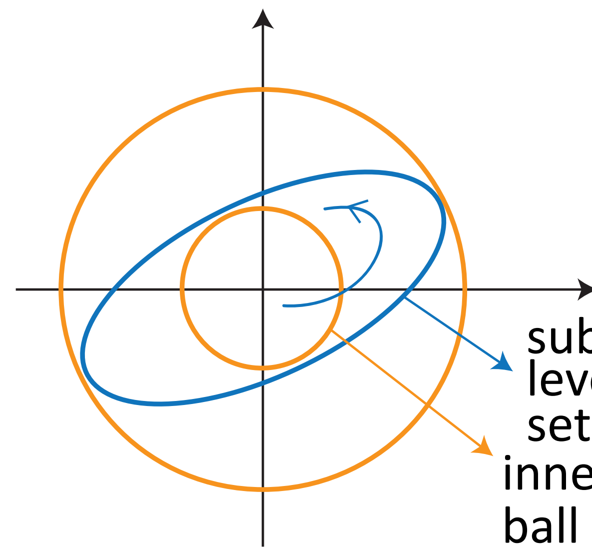
The equilibrium $x = 0$ is **stable** if, for every $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$|x(0)| \leq \delta \;\Rightarrow\; |x(t)| \leq \varepsilon \quad t \geq 0$$



How does a Lyapunov function ensure this?

- We can find $c > 0$ such that the sublevel set $\{x : V(x) \leq c\}$ fits into the outer ball in the stability definition above.
- Trajectories starting in this sublevel set are trapped there.
- Into this sublevel set we can fit an inner ball, thus satisfying the stability definition.



Lyapunov Functions

Key takeaways:

- Existence of a positive definite function V decreasing along trajectories guarantees stability of the origin.
- We don't need to know the trajectories to check the decreasing property. Instead, verify the dissipation inequality:

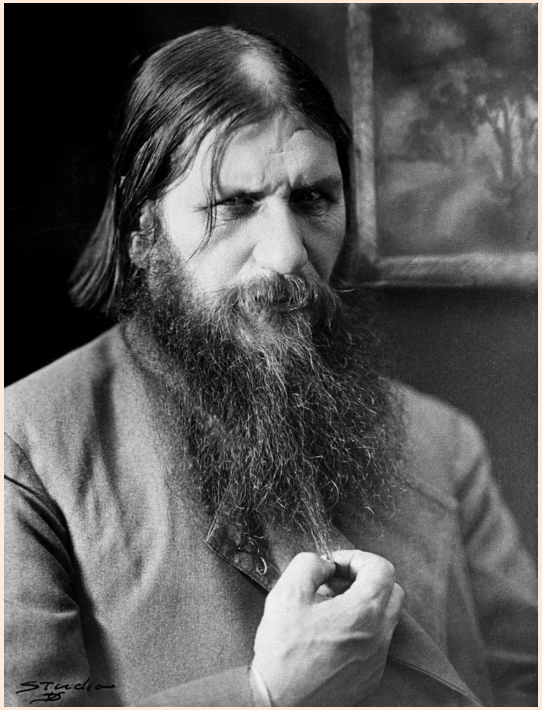
$$L_f V(x) := \nabla V(x)^\top f(x) \leq 0 \; \forall x \in \mathbb{R}^n$$

  which involves points in the state space, not trajectories.
- In the rest of the course, we will discuss other dissipation inequalities to certify properties besides stability.

Lyapunov Functions

Wake-up Problems

1) Which one is Lyapunov?



Lyapunov Functions

Wake-up Problems

2) Which one is a suitable Lyapunov function for the system below?

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -x_1^3 - x_2$$

A) $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

B) $V(x) = x_1^2 - x_2^2$

C) $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$

D) $V(x) = (x_1 + x_2)^2$

Special Case: Linear Systems

When $f(x) = Ax$ the following statements are equivalent:

1. The origin is (globally) asymptotically stable
2. All eigenvalues of A have negative real parts
3. We can find a quadratic positive definite Lyapunov function

$$V(x) = x^\top P x$$

such that $L_f V$ is negative definite.

Note: we write quadratic functions like $x^\top P x$ with the convention that P is symmetric: $P^\top = P$. No loss of generality: if $\tilde{P}^\top \neq \tilde{P}$ $x^\top \tilde{P} x = x^\top P x$ where $P = 0.5(\tilde{P}^\top + \tilde{P})$ is symmetric.

Thus, quadratic Lyapunov functions are enough for linear systems. In addition, when V is quadratic, so is $L_f V$:

$$L_f V(x) = \nabla V(x)^\top A x = (2Px)^\top A x = 2x^\top P A x = x^\top (PA + A^\top P) x$$

Special Case: Linear Systems

Easy to determine sign definiteness of a quadratic function

$$x^\top Qx, \; Q = Q^\top$$

Compute eigenvalues of Q, which are real since Q is symmetric.

If all eigenvalues are **positive**, then $x^\top Qx$ is		**positive definite**
	nonnegative	**positive semidefinite**
	negative	**negative definite**
	nonpositive	**negative semidefinite**
	of mixed signs	**sign indefinite**.

We say that a symmetric matrix is positive/negative (semi)definite if the corresponding quadratic function is such.

Notation: $Q \succ 0, \; Q \succeq 0, \; Q \prec 0, \; Q \preceq 0$

Special Case: Linear Systems

Examples:

	evalues:		
$x_1^2 + 2x_1x_2 + x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$0, 2$	$\rightarrow$	pos. semidef.
$x_1^2 + 4x_1x_2 + x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$-1, 3$	$\rightarrow$	sign indefinite
$x_1^2 + 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$1, 2$	$\rightarrow$	positive def.

Back to Lyapunov functions:

Suppose all eigenvalues of A have negative real parts. Then, for any $Q = Q^\top \succ 0$ there exists $P = P^\top \succ 0$ such that

$$PA + A^\top P = -Q \qquad \text{(Lyapunov Equation)}$$

MATLAB command $lyap(A', Q)$ returns P.

Thus, we can choose Q and find V that gives $L_f V(x) = -x^\top Q x$

Special Case: Linear Systems

Example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_2 - bx_1 \quad a > 0, b > 0$$

A simple choice for Lyapunov function:

$$V(x) = \frac{b}{2}x_1^2 + \frac{1}{2}x_2^2 \quad \Rightarrow \quad L_f V(x) = -ax_2^2$$ (negative semidef.)

Let's look for another Lyapunov function that makes $L_f V$ strictly negative definite. We know we can find one because eigenvalues of

$$A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

are roots of characteristic polynomial $\det(\lambda I - A) = \lambda^2 + b\lambda + a$ which have negative real parts when $a > 0, b > 0$.

Special Case: Linear Systems

Pick $Q = \begin{bmatrix} \epsilon & 0 \\ 0 & a \end{bmatrix}$, $\epsilon > 0$ so $L_f V(x) = -x^\top Q x = -\epsilon x_1^2 - a x_2^2$

(negative definite)

Look for $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ satisfying $PA + A^\top P = -Q$

Substituting $A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$ we get

$$A^\top P + PA = \begin{bmatrix} -2bp_2 & p_1 - ap_2 - bp_3 \\ \star & 2p_2 - 2ap_3 \end{bmatrix}$$

Matching this to $-Q$ we get three equations for three unknowns:

$$-2bp_2 = -\epsilon, \quad p_1 - ap_2 - bp_3 = 0, \quad 2p_2 - 2ap_3 = -a$$

$$\Rightarrow \; P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} \frac{b}{2} + \frac{\epsilon}{2}\left(\frac{a}{b} + \frac{1}{a}\right) & \frac{\epsilon}{2b} \\ \frac{\epsilon}{2b} & \frac{1}{2} + \frac{\epsilon}{2ab} \end{bmatrix}$$

Special Case: Linear Systems

Instead of specifying Q and solving the Lyapunov equation

$$PA + A^\top P = -Q$$

can use semidefinite programming (SDP) solvers to find $P \succ 0$ s.t.

$$A^\top P + PA \preceq 0 \qquad \textbf{(★)}$$

(or $A^\top P + PA + \varepsilon I \preceq 0,\ \varepsilon > 0$ to ensure $A^\top P + PA \prec 0$).

(★) is a Linear Matrix Inequality (LMI) in P.

General form of a LMI: $\max_{x \in \mathbb{R}^q} \quad e^\top x$

$$s.t. \quad \sum_{i=1}^{q} x_i F_i - G \preceq 0$$

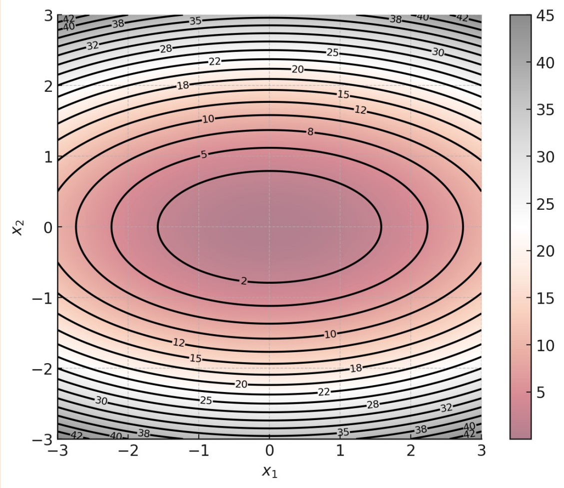
(★) is feasibility problem (nothing to maximize) and x consists of $n(n+1)/2$ independent entries of $P = P^\top \in \mathbb{R}^{n \times n}$

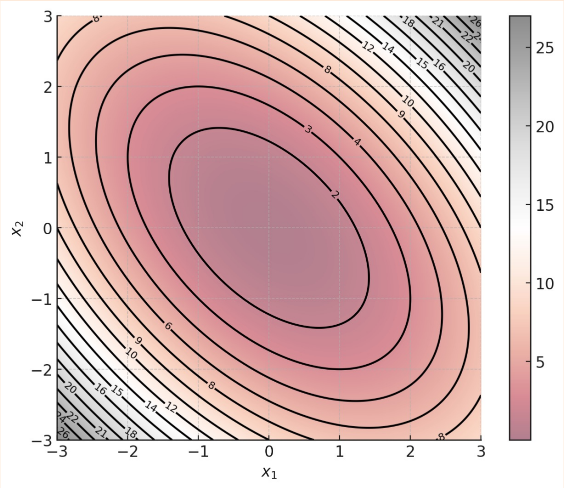
Special Case: Linear Systems

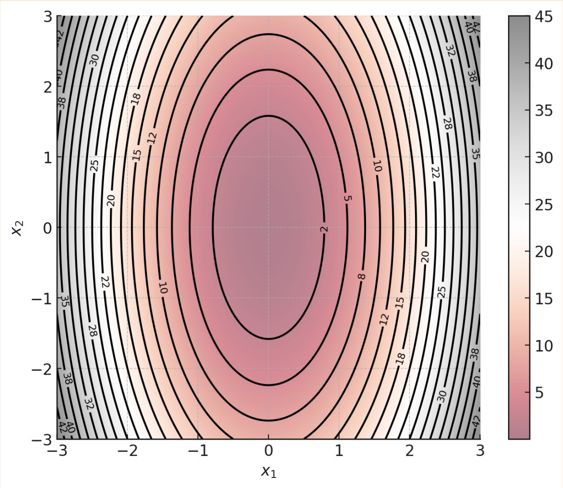
Wake-up Problem

Match the quadratic Lyapunov functions below to the sublevel sets shown at the bottom.

$$V_1(x) = 4x_1^2 + x_2^2, \quad V_2(x) = x_1^2 + 4x_2^2, \quad V_3(x) = x_1^2 + x_1 x_2 + x_2^2$$







Dissipativity

Jan Willems (1939-2013) was instrumental in bridging the gap between input-output and state space languages. His *dissipativity* theory generalized Lyapunov functions to establish state-space characterization of input-output properties, such as L_2 gain and passivity.

Dissipative Dynamical Systems
Part I: General Theory

JAN C. WILLEMS

Communicated by C. TRUESDELL

Contents

Archive for Rational Mechanics and Analysis, pp. 321-392, 1972

Dissipative Dynamical Systems
Part II: Linear Systems with Quadratic Supply Rates

JAN C. WILLEMS

Communicated by C. TRUESDELL

Contents

Dissipativity

From "closed" to "open" systems with inputs and outputs:

$$y \longleftarrow \boxed{\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}} \longleftarrow u$$

The system above is said to be dissipative with supply rate $s(u, y)$ if there exists pos. semidef. ``storage function'' $V : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\nabla V(x)^\top f(x,u) \leq s(u, h(x,u)) \quad \forall x, u.$$

Implication for trajectories: $\dfrac{d}{dt} V(x(t)) \leq s(u(t), y(t))$

Thus, for any $T >$ 0 in the interval of existence of solutions,

$$-V(x(0)) \leq V(x(T)) - V(x(0)) \leq \int_0^T s(u(t), y(t))dt$$

Dissipativity

For zero initial conditions: $\int_0^T s(u(t), y(t))dt \geq 0$

Common supply rates:

- Passivity: $s(u,y) = u^\top y \quad \Rightarrow \quad \langle u(\cdot), y(\cdot)\rangle_{L_2} \geq 0$

  Output strict passivity:

  $s(u,y) = u^\top y - \varepsilon y^\top y, \; \varepsilon > 0 \quad \Rightarrow \quad \langle u(\cdot), y(\cdot)\rangle_{L_2} \geq \varepsilon \|y(\cdot)\|_{L_2}^2$

- L_2 gain: $s(u,y) = \gamma^2 u^\top u - y^\top y \; \Rightarrow \; \|y(\cdot)\|_{L_2} \leq \gamma \|u(\cdot)\|_{L_2}$

These are quadratic supply rates:

$$s(u,y) = \begin{bmatrix} u \\ y \end{bmatrix}^\top X \begin{bmatrix} u \\ y \end{bmatrix} \qquad X = \begin{bmatrix} 0 & \frac{1}{2}I \\ \frac{1}{2}I & -\varepsilon I \end{bmatrix} \qquad X = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$$

- Input-to-state stability: $s(u,x) = -\alpha(|x|) + \sigma(|u|), \; \alpha, \sigma \in \mathcal{K}_\infty$

Dissipativity

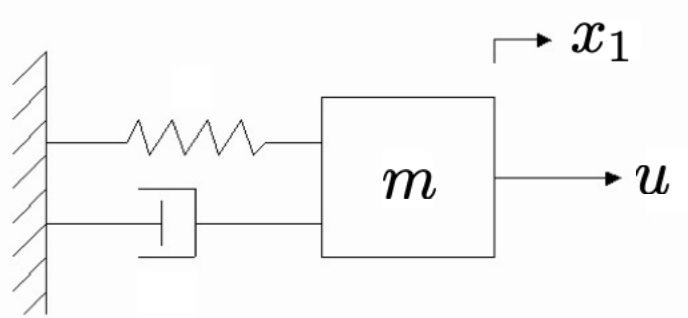
Example 1: $\dot{x} = u, \; y = x$

Take $V(x) = \frac{1}{2}x^\top x$

$$\nabla V(x)^\top f(x,u) = x^\top u = y^\top u$$

Therefore, passive (in fact "lossless" because of equality).

Example 2:
$$\dot{x}_1 = x_2$$
$$m\dot{x}_2 = -\varepsilon x_2 - \phi'(x_1) + u$$
$$y = x_2$$



$\phi(x_1)$: potential energy of the spring

Passivity established with energy function $V(x) = \phi(x_1) + \frac{m}{2}x_2^2$

$$\nabla V(x)^\top f(x,u) = \phi'(x_1)x_2 - \varepsilon x_2^2 - x_2\phi'(x_1) + x_2 u = -\varepsilon y^2 + yu$$

Dissipativity

Does dissipativity imply stability when $u = 0$?

Suppose the supply rate is such that $s(0, y) \leq 0 \ \forall y$, *e.g.*,

$$s(u, y) = u^\top y - \varepsilon y^\top y, \varepsilon \geq 0, \ \ s(u, y) = \gamma^2 u^\top u - y^\top y$$

If, in addition, the storage function is positive definite, then we can use it as a Lyapunov function and conclude stability of the origin.

Other observations:

- If a dynamical system is dissipative with supply rate s_1 and $s_1(u, y) \leq s_2(u, y) \ \forall u, y$ then it is also dissipative with rate s_2
- If a dynamical system is dissipative with supply rate s then it is also dissipative with supply rate αs, $\alpha \geq 0$
- Output strict passivity implies finite L_2 gain $\gamma = 1/\varepsilon$, because

$$u^\top y - \varepsilon y^\top y \leq \frac{1}{2\varepsilon} u^\top u - \frac{\varepsilon}{2} y^\top y = \frac{\varepsilon}{2} \left(\frac{1}{\varepsilon^2} u^\top u - y^\top y \right)$$

Dissipativity

Wake-up Problems

True of false?

1) If a dynamical system is dissipative with supply rates s_1 and s_2 then is also dissipative with supply rate

$$s(u,y) = \alpha s_1(u,y) + \beta s_2(u,y), \;\; \alpha \geq 0, \beta \geq 0$$

2) A passive dynamical system must satisfy:

$$u(t)^\top y(t) \geq 0 \quad \forall t$$

Constructing Storage Functions

Case 1: Linear systems

$$f(x,u) = Ax + Bu,\ h(x,u) = Cx + Du$$

Quadratic storage function: $V(x) = x^\top P x$

$$\nabla V(x)^\top (Ax + Bu) = 2x^\top P(Ax + Bu) = \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

Supply rate: **(1)**

$$s(u,y) = \begin{bmatrix} u \\ y \end{bmatrix}^\top X \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top X \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$ **(2)**

$(1) \leq (2)\ \forall x, u$ means:

$$\begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top X \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \preceq 0$$ **(LMI)**

Constructing Storage Functions

For **passivity** substitute: $X = \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$

Then LMI becomes:

$$\begin{bmatrix} A^\top P + PA & PB - \frac{1}{2}C^\top \\ B^\top P - \frac{1}{2}C & -\frac{1}{2}(D + D^\top) \end{bmatrix} \preceq 0$$

Note: when $D = 0$ (no "feedthrough") this equivalent to:

$$PB = \frac{1}{2}C^\top, \quad A^\top P + PA \preceq 0$$

Positive Real (PR) Lemma

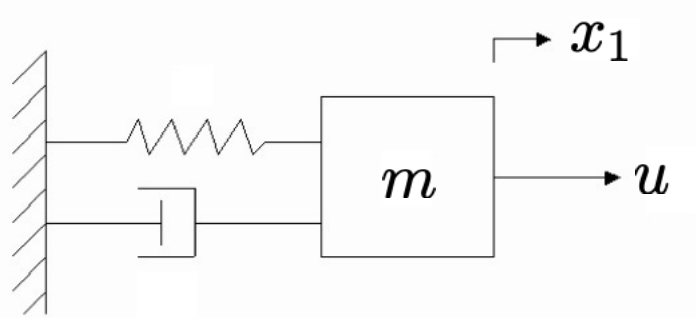
because: $\begin{bmatrix} Q_1 & Q_2 \\ Q_2^\top & 0 \end{bmatrix} \preceq 0 \;\Leftrightarrow\; Q_2 = 0,\, Q_1 \preceq 0$

Example 1: $\dot{x} = u,\; y = x \qquad A = 0, B = I, C = I, D = 0$

$$PB = \frac{1}{2}C^\top \;\Rightarrow\; P = \frac{1}{2}I$$

Constructing Storage Functions

Example 2:
$$\dot{x}_1 = x_2$$
$$m\dot{x}_2 = -\varepsilon x_2 - \phi'(x_1) + u$$
$$y = x_2$$



$$\phi(x_1) = \frac{k}{2}x_1^2 \;\Rightarrow\; \phi'(x_1) = kx_1 \;\Rightarrow\; A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\varepsilon}{m} \end{bmatrix},\; B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix},\; D = 0$$

$$PB = \frac{1}{2}C^\top \;\Rightarrow\; P = \begin{bmatrix} p_1 & 0 \\ 0 & \frac{m}{2} \end{bmatrix} \Rightarrow\; PA + A^\top P = \begin{bmatrix} 0 & p_1 - \frac{k}{2} \\ p_1 - \frac{k}{2} & -\varepsilon \end{bmatrix}$$

$PA + A^\top P \preceq 0$ implies $p_1 = \frac{k}{2} \;\Rightarrow\; x^\top P x = \frac{k}{2}x_1^2 + \frac{m}{2}x_2^2$

Consistent with $V(x) = \phi(x_1) + \frac{m}{2}x_2^2$ found earlier.

Constructing Storage Functions

For ***L_2* gain** substitute $X = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$ in:

$$\begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix}^\top - \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top X \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \preceq 0 \qquad \textbf{(LMI)}$$

Simplify: $$\begin{bmatrix} A^\top P + PA + C^\top C & PB + C^\top D \\ B^\top P + D^\top C & -\gamma^2 I + D^\top D \end{bmatrix} \preceq 0$$

Note: when $D = 0$ this is equivalent to

$$A^\top P + PA + C^\top C + \frac{1}{\gamma^2} PBB^\top P \preceq 0$$

Bounded Real (BR) Lemma

by the ***Schur Complement Lemma:*** Given symmetric $\begin{bmatrix} Q_1 & Q_2 \\ Q_2^\top & Q_3 \end{bmatrix}$ with Q_3 invertible, $\begin{bmatrix} Q_1 & Q_2 \\ Q_2^\top & Q_3 \end{bmatrix} \preceq 0 \;\Leftrightarrow\; Q_3 \prec 0,\; Q_1 - Q_2 Q_3^{-1} Q_2^\top \preceq 0$

Constructing Storage Functions

Case 2: Input-affine nonlinear systems with no feedthrough

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

To establish passivity, we need a storage function V such that:

$$\nabla V(x)^\top f(x) + \nabla V(x)^\top g(x)u \le h(x)^\top u - \varepsilon h(x)^\top h(x)$$

Rewrite as:

$$L_f V(x) + \varepsilon h(x)^\top h(x) + (L_g V(x) - h(x)^\top)u \le 0$$

which is equivalent to

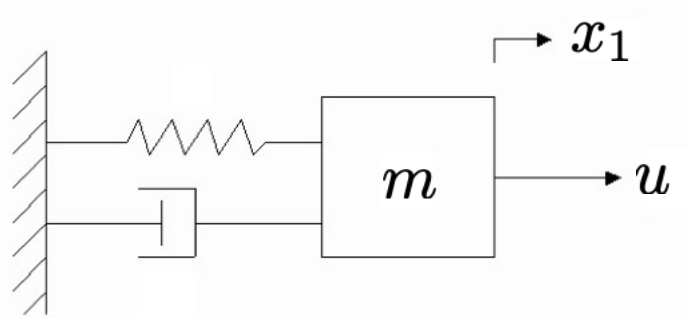
$$L_f V(x) + \varepsilon h(x)^\top h(x) \le 0, \quad L_g V(x) = h(x)^\top$$

Compare this to the PR Lemma…

Constructing Storage Functions

Example 2 revisited:

$$\dot{x}_1 = x_2$$
$$m\dot{x}_2 = -\varepsilon x_2 - \phi'(x_1) + u$$
$$y = x_2$$



$$L_g V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{1}{m}\frac{\partial V}{\partial x_2}$$ must match $h(x) = x_2$

$$\Rightarrow \quad V(x) = V_1(x_1) + \frac{m}{2}x_2^2$$

Then,
$$L_f V(x) = \begin{bmatrix} V_1'(x_1) & m x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{\varepsilon}{m}x_2 - \frac{1}{m}\phi'(x_1) \end{bmatrix}$$
$$= (V_1'(x_1) - \phi'(x_1))x_2 - \varepsilon x_2^2$$
$$\leq -\varepsilon h(x)^2$$

with the choice $V_1(x_1) = \phi(x_1)$. Thus, $V(x) = \phi(x_1) + \frac{m}{2}x_2^2$

Constructing Storage Functions

Example 3: Show passivity of the scalar affine system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

where $xf(x) \leq 0 \ \forall x \in \mathbb{R}, \ g(x) > 0 \ \forall x \in \mathbb{R}, \ xh(x) > 0 \ \forall x \neq 0$

The constraint $V'(x)g(x) = h(x)$ dictates the choice of V:

$$V(x) = \int_0^x \frac{h(s)}{g(s)} ds$$

Positive definite because $\dfrac{h(x)}{g(x)}$ has the same sign as x

In addition, $V'(x)f(x) = \dfrac{h(x)}{g(x)} f(x) \leq 0$

because $f(x)$ has the opposite sign of x

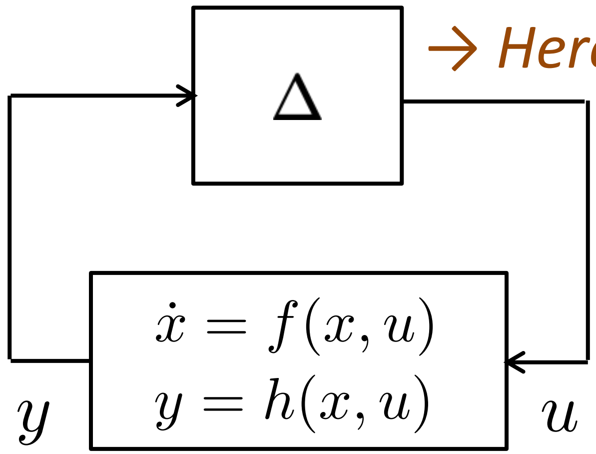
Constructing Storage Functions

Wake-up Problem

True of false? The system below is passive:

$$\dot{x} = -x^3 + (1 + x^2)u$$
$$y = \tanh(x)$$

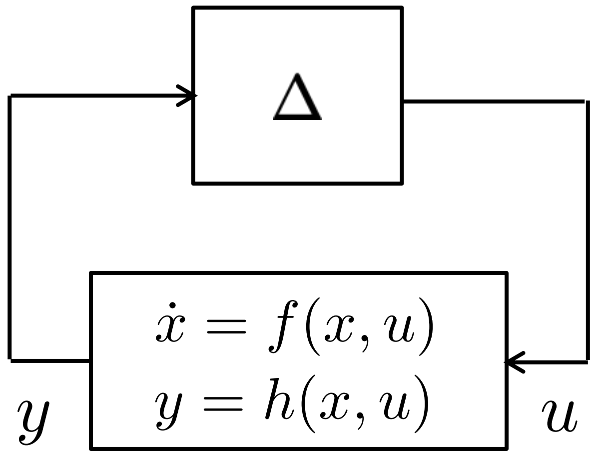
Robust Stability and Performance



→ Here be dragons

Δ: uncertain, hard-to-model, or nonlinear elements, described broadly by input-output relations, rather than by a detailed model

Robust Stability and Performance



Δ: uncertain, hard-to-model, or nonlinear elements, described broadly by input-output relations, rather than by a detailed model

Robust stability: Suppose the system $\dot{x} = f(x, u)$ $y = h(x, u)$ is dissipative with supply rate $s(u, y)$ and pos.def. storage function V. If Δ satisfies the complementary constraint

$$s(u, y) \leq 0$$

for all (u, y) such that $u = \Delta(y)$, then the origin is stable because

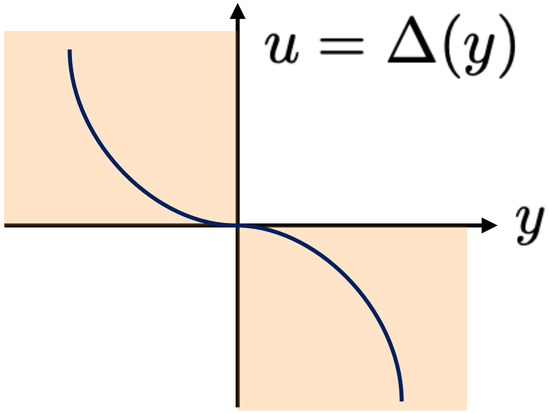
$$L_f V(x, u) \leq s(u, y) \leq 0.$$

Robust Stability and Performance

When $s(u,y)$ is quadratic as in the passivity and L_2 gain supply rates, we refer to $s(u,y) \leq 0$ as a *quadratic constraint* satisfied by Δ. Later we will also use *integral* versions of quadratic constraints:

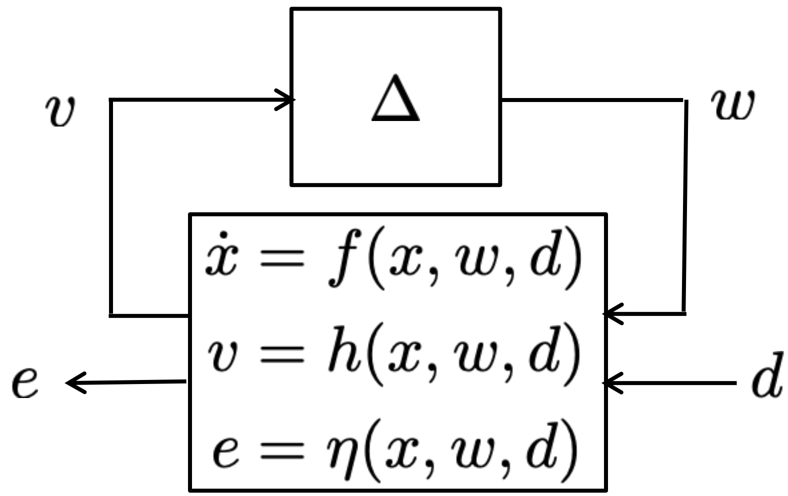
$$\int_0^T s(u(t), y(t))dt \leq 0 \quad \forall T > 0$$

Example: Passive single-input single-output system in feedback with a nonlinearity whose graph lies in the 2nd and 4th quadrants:



Asymptotic stability can be guaranteed by strengthening the dissipativity property or the constraint on Δ.

Robust Stability and Performance



"Performance" objective: dissipativity with a supply rate $\sigma(d,e)$, *e.g.* $\sigma(d,e) = \gamma^2|d|^2 - |e|^2$ for L_2 gain from disturbance d to output e.

Robust performance: If there exists storage function $x \mapsto V(x)$ s.t.

$$L_f V(x, w, d) \leq s(w, d; v, e)$$

$\forall x, w, d$ and Δ restricts (v, w) such that

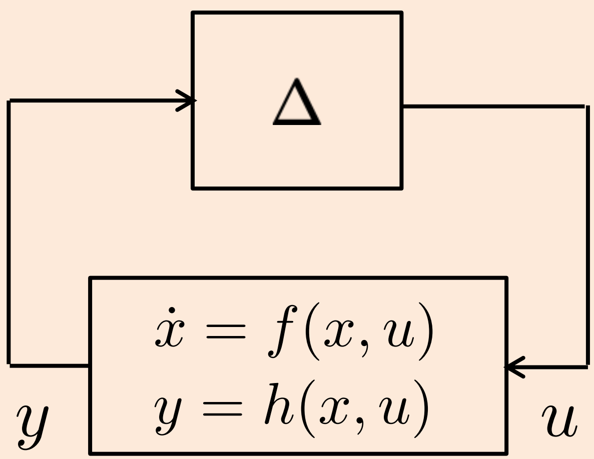
$$s(w, d; v, e) \leq \sigma(d, e)$$

then the interconnection is dissipative with supply rate $\sigma(d,e)$.

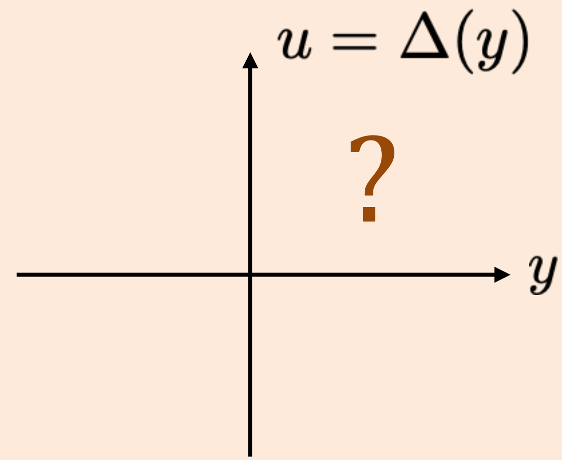
Robust Stability and Performance

Wake-up Problem

Suppose a single-input single-output system $\dot{x} = f(x,u)\;\; y = h(x,u)$ is dissipative with supply rate $s(u,y) = \gamma^2 u^2 - y^2$; that is, it has L_2 gain $\leq \gamma$, and Δ below represents a nonlinearity. Describe the region where the graph of Δ must lie for $u = \Delta(y)$ to satisfy $s(u,y) \leq 0$.



L_2 gain $\leq \gamma$



Summary

In this lesson:

- We reviewed state space models, equilibrium/stability concepts.
- We had a first glimpse of dissipation inequalities in Lyapunov analysis: if the dissipation inequality

$$\nabla V(x)^\top f(x) \leq 0$$

  holds for each point in the state space, we conclude $V(x(t))$ is nonincreasing over trajectories, *without knowledge of the trajectories*. This nonincreasing property guarantees stability.
- We introduced the notion of dissipativity, closely related to Lyapunov analysis: a storage function satisfying a dissipation inequality allows us to establish input/output relations.
- Married dissipation inequalities to complementary constraints on uncertain block Δ for robust stability and performance criteria.

Kalman-Yakubovich-Popov (KYP) Lemma

A streamlined version of [2,3,4] from [1]:

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\Gamma = \Gamma^\top \in \mathbb{R}^{(n+m)\times(n+m)}$ where $\det(j\omega I - A) \neq 0 \ \forall \omega \in \mathbb{R}$ and (A, B) controllable, the following statements are equivalent:

1) $\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Gamma \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \preceq 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}$

2) There exists $P = P^\top \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} + \Gamma \preceq 0$$

If all eigenvalues of A have negative real parts and upper left corner of Γ is positive semidefinite, then $P \succeq 0$.

[1] Rantzer, On the Kalman-Yakubovich-Popov lemma, Syst. Control Lett., 1996

[2] Kalman, Canonical structure of linear dynamical systems, 1962

[3] Yakubovich, The solution of certain matrix inequalities in automatic control theory, 1962

[4] Popov, The solution of a new stability problem for controlled systems, 1963

Kalman-Yakubovich-Popov (KYP) Lemma

Recall the LMI for dissipativity of linear system (A, B, C, D) with supply rate defined by matrix X:

$$\begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top X \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}}_{\Gamma} \preceq 0$$

From KYP, this is equivalent to:

$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^* X \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} \succeq 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\} \qquad \textbf{(1)}$$

where $H(j\omega) = C(j\omega I - A)^{-1}B + D$ is the frequency response.

Example: For passivity, i.e., $X = \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, **(1)** becomes:

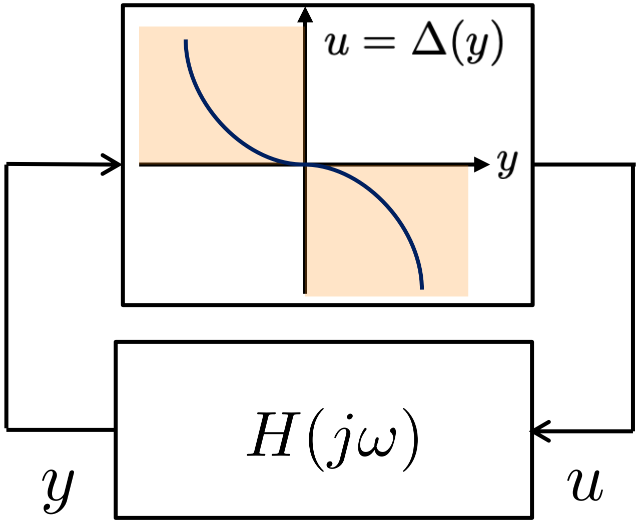
$$H(j\omega)^* + H(j\omega) \succeq 0$$

For SISO systems, this means nonnegative real part ("positive real").

Kalman-Yakubovich-Popov (KYP) Lemma

Absolute stability studies in the 1960s derived frequency domain criteria for a linear system in feedback with a nonlinearity lying in a conic sector (special case of quadratic constraints in this workshop). Prominent results include the Circle- and Popov-Criteria, and others by Zames, Falb, Sandberg, Brockett, Willems, Narendra, Tsypkin…

Example: SISO linear system in feedback with a nonlinearity whose graph lies in the 2nd and 4th quadrants:



For this sector, the Circle Criterion restricts $H(j\omega)$ to be positive real.

Kalman-Yakubovich-Popov (KYP) Lemma

Today's approach – presented in this workshop – is to leverage time-domain dissipativity properties with numerical tools, such as semidefinite programming and sum-of-square programming.

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-16, NO. 6, DECEMBER 1971 621

Least Squares Stationary Optimal Control and the Algebraic Riccati Equation

JAN C. WILLEMS, MEMBER, IEEE

Page 624: "The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms, for example."

Further Reading

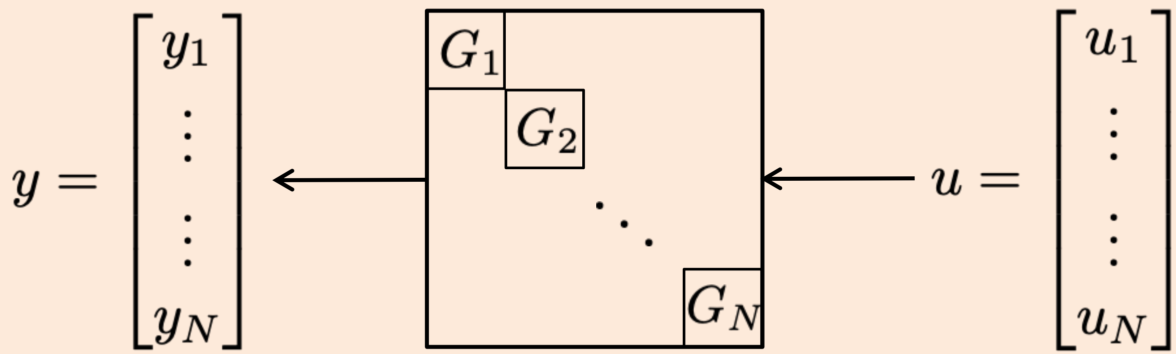
- Khalil, Nonlinear Systems, Prentice Hall, 2002
- Willems, Dissipative dynamical systems, Parts I and II, Archive for Rational Mechanics and Analysis, pp. 321-392, 1972
- Desoer and Vidyasagar, Feedback systems: input-output properties, Society for Industrial and Applied Mathematics (SIAM) Classics, 2009
- van der Schaft, L2-gain and Passivity Techniques in Nonlinear Control, Springer, 2000
- Brogliato, Lozano, Maschke, Egeland, Dissipative Systems Analysis and Control, Springer, 2007
- IEEE Control Systems Magazine, two special issues on 50 Years of Dissipativity Theory, 2022
- Rantzer, On the Kalman-Yakubovich-Popov lemma, Syst. Control Lett., 1996

Self-Study Problems: True/False

1) If a dynamical system is dissipative with respect to supply rates s_1 and s_2, then it is dissipative with respect to rate $s_1 - s_2$.
2) For a dynamical system G, let $-G$ denote the same system with the sign of the output reversed. G is dissipative with respect to s if and only if $-G$ is dissipative with respect to $-s$.
3) Define the sum of two dynamical systems G_1 and G_2 as a dynamical system whose response to u is $y = G_1(u) + G_2(u)$. If G_1 is dissipative with supply rate s_1 and G_2 with supply rate s_2, then $G_1 + G_2$ is dissipative with supply rate $s_1 + s_2$.
4) If G_i is dissipative with supply rate $u_i^\top y_i$ $i = 1,2$, then $G_1 + G_2$ is dissipative with supply rate $u^\top y$
5) If $\dot{x} = f(x,u), y = h(x,u)$ is dissipative, then so is the system $\tau\dot{x} = f(x,u), y = h(x,u)$ with the same supply rate for any $\tau > 0$.

Self-Study Problems: True/False

6) Consider N independent systems $G_i, i = 1, \dots, N$, each with input output pair (u_i, y_i), and let u and y denote the concatenations of u_i and y_i as shown below:



If G_i is dissipative with supply rate $s_i(u_i, y_i), i = 1, \dots, N$, then for any set of nonnegative weights $p_i \geq 0, i = 1, \dots, N$, the composite system is dissipative with supply rate:

$$s(u, y) = \sum_{i=1}^{N} p_i s_i(u_i, y_i)$$