International Graduate School on Control: Lesson 5

1. **Sums of Squares Polynomials**

   (a) Every polynomial can be expressed in the Gram matrix form $Z(x)^T Q Z(x)$ where Q is a symmetric matrix and $Z(x)$ is a vector of monomials. What monomials must be included in $Z(x)$ to represent a generic polynomial of degree 4 in 2 variables?

   (b) Consider the following degree 4 polynomial in two variables:

   $$p(x_1, x_2) := 25x_1^4 - 90x_1^3x_2 + 90x_1^2x_2^2 + 9x_2^4 + 12x_1^2x_2 - 12x_1x_2^2 + 8x_1^2$$

   We would like to determine if p is a SOS, i.e. if p can be represented as $Z(x)^T Q Z(x)$ for some $Q \geq 0$. Equate the coefficients of p and $Z(x)^T Q Z(x)$ to find a collection of linear equality constraints on the entries of Q. Use these equations to find matrices Q_0 and Q_1 such that all solutions to $p = Z(x)^T Q Z(x)$ can be expressed as $Q_0 + \lambda Q_1$. [Hint: It is possible to use the properties of semidefinite matrices to argue that certain monomials need not be included in $Z(x)$.]

   (c) Plot the minimum eigenvalue of $Q_0 + \lambda Q_1$ versus λ. For what values of λ is $Q \geq 0$?

   (d) Pick a value of λ for which $Q \geq 0$. Use the Cholesky decomposition of Q to construct polynomials $\{f_1, \ldots, f_N\}$ such that $p = \sum_{k=1}^{N} f_k^2$. How can you choose λ to minimize the number of terms N in the SOS decomposition?

2. **Lyapunov Stability**

   Download both SOSTOOLs and Sedumi (or other SDP solver that is compliant with SOSTOOLs). Add both toolboxes (and the necessary subfolders) to your Matlab Path. Run `sosdemo1` and `sosdemo2` to verify that your installation is working properly.

   Consider the following third-order nonlinear system:

   $$\dot{x} = A_1 Z_1(x) + A_2 Z_2(x) + A_3 Z_3(x)$$

   where:

   $$Z_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} x_1^3 \\ x_1^2x_2 \\ x_1x_2^2 \\ x_2^3 \end{bmatrix}$$

   and

   $$A_1 = \begin{bmatrix} -4 & 5 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \frac{1}{4}\begin{bmatrix} 3 & 6 & 3 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_3 = \frac{1}{8}\begin{bmatrix} -1 & 0 & -9 & 6 \\ 0 & -3 & 6 & -7 \end{bmatrix}$$

   (a) What is the linearization of this system at $x = 0$? Verify that this linearization is stable.

   (b) If there exists a V such that $V(0) = 0$, $V \geq x_1^2 + x_2^2$ and $\dot{V} \leq -rV$ then $x = 0$ is a globally exponentially stable and all trajectories converge to the origin like e^{-rt}. This is a slight modification of the Lyapunov Theorem presented in class and is similar to an LMI condition we derived earlier. Use SOSTOOLs to find the largest value of r (to within an accuracy of 0.1) for which there exists a quadratic Lyapunov function which satisfies these conditions. How does r_{max} compare with the natural frequency of the poles of the linearized system?

   (c) Simulate the nonlinear system from several initial conditions and plot them on a single figure. On the same figure, plot several contours of the Lyapunov function computed by SOSTOOLs. Comment on the graphical interpretation of $\dot{V} = \nabla V \cdot \dot{x} < 0$.

3. **Input-Output Gain Analysis**

   Consider the following third-order nonlinear system:

   $$\begin{aligned} \dot{x} &= A_1 Z_1(x) + A_2 Z_2(x) + A_3 Z_3(x) + Bu \\ y &= Cx \end{aligned}$$

   where $\{A_k\}_{k=1}^3$ and $\{Z_k(x)\}_{k=1}^3$ are as defined in the previous problem. The input and output matrices are:

   $$B = \begin{bmatrix} 10 & 2 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

(a) What is the linearization, $G(s)$, of this system at $x = 0$? Compute the H_∞ norm of this linearization.

(b) Construct an input signal which approximately achieves $\|G\|_\infty$. Specifically, construct $u_{wc}(t)$ for $t \in [0, 100]$ such that the response to this input, y_{wc}, satisfies $\frac{\|y_{wc}\|_2}{\|u_{wc}\|_2} \approx \|G\|_\infty$. Simulate the linear system G with the input u_{wc} and zero initial conditions. Compute both $\|y_{wc}\|_2$, $\|u_{wc}\|_2$ and verify the ratio is approximately $\|G\|_\infty$.

(c) Simulate the nonlinear system response y_{nl} due to the input u_{wc} constructed in the previous part. Use this response to compute a lower bound for the L_2-L_2 gain of the nonlinear system.

(d) Use SOSTOOLs to compute an upper bound on the L_2-L_2 input-output gain of the nonlinear system. How does this upper bound compare to $\|G\|_\infty$ and the lower bound computed in the previous part? How would you reduce the gap between the upper and lower bounds?