Quantum union bound

Anurag Anshu

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In a recent work [OV21], O'Donnell and Venkateswaran obtained a remarkably simple proof of Gao's quantum union bound [Gao15]. In this note, we highlight some connection of their argument with Sen's quantum union bound [Sen11], which also has a remarkably simple proof but a quadratically weaker result than Gao's.

TL;DR: Fidelity wins over Euclidean distance.

Set-up: Given projectors $\Pi_1, \Pi_2, \ldots \Pi_k$ and a state $|v\rangle$, suppose $\|\Pi_t|v\rangle\|^2 = 1 - \varepsilon_t$ (for all $t \in \{1, 2, \ldots k\}$). Here, $\|.\|$ is the Euclidean norm. Define $|\tilde{w}_0\rangle := |v\rangle$, $|\tilde{w}_t\rangle = \Pi_t \ldots \Pi_2 \Pi_1 |v\rangle$ and $|w_t\rangle = \frac{|\tilde{w}_t\rangle}{\||\tilde{w}_t\rangle\|}$. We also define $\bar{\Pi}_t := \mathrm{I} - \Pi_t$, which means $\varepsilon_t = \|\bar{\Pi}_t|v\rangle\|^2$. Quantum union bounds show that $\||\tilde{w}_t\rangle\|^2$ is large and $|w_t\rangle$ is close to $|v\rangle$, if $\sum_i \varepsilon_i$ is small.

Closeness measures: It will be convenient to work with pure 'sub-normalized' states, which have norm less or equal to 1. We abbreviate $v := |v\rangle\langle v|$, $\tilde{w}_t := |\tilde{w}_t\rangle\langle \tilde{w}_t|$ and $w_t := |w_t\rangle\langle w_t|$. Given $|\rho\rangle, |\sigma\rangle$, we consider two notions of 'closeness': $\mathrm{F}(\rho, \sigma) = |\langle \rho||\sigma\rangle|^2$ and $\Delta(\rho, \sigma) = 1 - \||\rho\rangle - |\sigma\rangle\|^2$.

O'Donnell-Venkateswaran argument: Based on the following inequality [OV21, Lemma 2.1], for a projector Π and pure sub-normalized states $|\rho\rangle, |\sigma\rangle$.

$$\sqrt{\mathrm{F}(\rho, \sigma)} \leq \sqrt{\mathrm{F}(\rho, \Pi\sigma\Pi)} + \|\bar{\Pi}|\rho\rangle\| \|\bar{\Pi}|\sigma\rangle\|. \tag{1}$$

Using it recursively starting with $\rho, \sigma = v$, we find

$$\begin{aligned} 1 - \sqrt{\mathrm{F}(v, \tilde{w}_k)} &\leq \sum_{t=1}^{k} \|\bar{\Pi}_t|\rho\rangle\| \|\bar{\Pi}_t|\tilde{w}_{t-1}\rangle\| \leq \sqrt{\left(\sum_{t=1}^{k} \|\bar{\Pi}_t|\rho\rangle\|^2\right)\left(\sum_{t=1}^{k} \|\bar{\Pi}_t|\tilde{w}_{t-1}\rangle\|^2\right)} \\ &= \sqrt{\left(\sum_{t=1}^{k} \varepsilon_t\right)(1 - \||\tilde{w}_k\rangle\|^2)}. \end{aligned}$$

Since $1 - \sqrt{\mathrm{F}(v, \tilde{w}_k)} \geq 1 - \||\tilde{w}_k\rangle\|$, [OV21] find that

$$\frac{1 - \||\tilde{w}_k\rangle\|}{1 + \||\tilde{w}_k\rangle\|} \leq \sum_{t=1}^{k} \varepsilon_t \implies \||\tilde{w}_k\rangle\| \geq \frac{1 - \sum_{t=1}^{k} \varepsilon_t}{1 + \sum_{t=1}^{k} \varepsilon_t}.$$

This is used to also lower bound $\mathrm{F}(v, w_k)$.

Sen argument: Based on the following inequality [Sen11, Lemma 2], for a projector Π and pure sub-normalized states $|\rho\rangle, |\sigma\rangle$.

$$\Delta(\rho, \sigma) \leq \Delta(\rho, \Pi\sigma\Pi) + \|\bar{\Pi}|\rho\rangle\|^2. \tag{2}$$

Using it recursively starting with $\rho, \sigma = v$, we find

$$1 - \Delta(v, \tilde{w}_k) \leq \sum_{t=1}^{k} \|\bar{\Pi}_t|\rho\rangle\|^2 = \sum_{t=1}^{k} \varepsilon_t.$$

Since

$$\begin{aligned} 1 - \Delta(v, \tilde{w}_k) &= \||v\rangle - |\tilde{w}_k\rangle\|^2 \geq \||v\rangle\|^2 + \||\tilde{w}_k\rangle\|^2 - 2|\langle v||\tilde{w}_k\rangle| \\ &= 1 + \||\tilde{w}_k\rangle\|^2 - 2\||\tilde{w}_k\rangle\|\sqrt{\mathrm{F}(v, w_k)} \\ &\geq (1 - \||\tilde{w}_k\rangle\|)^2, \end{aligned}$$

[Sen11] finds that $\||\tilde{w}_k\rangle\| \geq 1 - \sqrt{\sum_{t=1}^{k} \varepsilon_t}$. This can also be used to lower bound $\mathrm{F}(v, w_k)$:

$$\sqrt{\mathrm{F}(v, w_k)} \geq \frac{1 + \||\tilde{w}_k\rangle\|^2 - \sum_{t=1}^{k} \varepsilon_t}{2\||\tilde{w}_k\rangle\|} \geq \frac{2 - 2\sqrt{\sum_{t=1}^{k} \varepsilon_t}}{2} = 1 - \sqrt{\sum_{t=1}^{k} \varepsilon_t}.$$

References

[Gao15] Jingliang Gao. Quantum union bounds for sequential projective measurements. *Phys. Rev. A*, 92:052331, Nov 2015.

[OV21] Ryan O'Donnell and Ramgopal Venkateswaran. The quantum union bound made easy, 2021. arXiv:2103.07827.

[Sen11] Pranab Sen. Achieving the han-kobayashi inner bound for the quantum interference channel by sequential decoding, 2011. arXiv:1109.0802.