Today’s Outline

• Problem Statement

• Functional Gradient Descent

• CHOMP: Covariant Hamiltonian Optimization for Motion Planning

• Next Lecture: Non-Euclidean Inner Product

6.1 Problem Statement

• Trajectory (function) $\xi : [0, T] \rightarrow \mathbb{C}$

• Cost (functional) $U : \Xi \rightarrow \mathbb{R}^+$

• Optimization Problem:

$$\xi^* = \arg\min_{\xi \in \Xi} U(\xi)$$

subject to

$$\xi(0) = q_s,$$

$$\xi(T) = q_g$$

• Update Equation (functional gradient descent):

$$\xi_{i+1} \leftarrow \xi_i - \frac{1}{\alpha} \nabla_{\xi} U(\xi_i)$$

$\nabla_{\xi} U(\xi_i)$: also a function of time
• $\Xi$ is a Hilbert space, a complete vector space with an inner product.

• Inner product:
  For this lecture, we assume a particular Hilbert space specified by the Euclidean inner product. Given two trajectories $\xi_1, \xi_2 \in \Xi$, the Euclidean inner product is
  \[ <\xi_1, \xi_2> = \int_0^T \xi_1(t)^T \xi_2(t) dt \]

In discrete time, $\xi = [q_1, ..., q_N]^T$, thus $<\xi_1, \xi_2> = \xi_1^T \xi_2$.

Properties of inner products:
  
  - **Symmetry:** $<\xi_1, \xi_2> = <\xi_2, \xi_1>$
  - **Positive definite:** $\forall \xi, <\xi, \xi> \geq 0; <\xi, \xi> = 0 \iff \xi = 0$
    
    ($\xi = 0$ is the zero trajectory that always maps time to the zero configuration.)
  - **Linearity in the first argument:**
    
    $<\xi_1 + \xi_2, \xi_3> = <\xi_1, \xi_3> + <\xi_2, \xi_3>$
    
    (The same holds true for the second argument by symmetry.)

### 6.2 Functional Gradient Descent

We use calculus of variation in computing the derivatives of a functional.

• **Euler-Lagrange Equation:**
  
  If
  \[ <\xi_1, \xi_2> = \int_0^T \xi_1(t)^T \xi_2(t) dt \]
(i.e. $Ξ$ is a Hilbert space with the Euclidean inner product) and

$$U[ξ] = \int_0^T F(t, ξ(t), ξ'(t)) dt$$

then

$$\nabla_ξ U(t) = \frac{∂F}{∂ξ(t)}(t) - \frac{d}{dt} \frac{∂F}{∂ξ'(t)}(t)$$

Note that $\nabla_ξ U(t) \in Ξ$

• Example:
  Consider the example where you minimize the squared norm of velocity in trajectory subject to starting at $q_s$ and ending at $q_g$:

$$U[ξ] = \frac{1}{2} \int_0^T \|ξ'(t)\|^2 dt$$

The optimal trajectory has

Shape: straight line. Intuitively, in the same amount time, a trajectory traversing a longer path needs a faster velocity, thus has higher cost.

Timing: constant velocity. Intuitively, in discrete time with $T$ time steps, $U[ξ^*] < U[ξ]$
Apply Euler-Lagrange equation,

\[ \nabla_{\xi} U(t) = 0 - \frac{d}{dt} \xi'(t) = -\xi''(t) \]

Since \( U \) is quadratic/convex, to find \( \xi \) that minimizes cost \( U \), we set the gradient to 0, and solve for \( \xi^* \), which is global minimum.

\[ \xi''(t) = 0 \]
\[ \xi'(t) = a \]

shows that optimal trajectory has constant velocity.

\[ \xi(t) = at + b \]

shows that optimal trajectory is a straight line.

Then to solve for \( a \) and \( b \) in \( \xi^* \), we use constraints \( \xi(0) = q_s \) and \( \xi(T) = q_g \).

• Proof:
  First order Taylor series expansion (relates \( f \) to \( f' \))
  \[ f : \mathbb{R} \rightarrow \mathbb{R}, \]
  \[ f(x + \epsilon) \approx f(x) + \epsilon f'(x) \]
  \[ f'(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon) - f(x)}{\epsilon} \]

  \( U : \Xi \rightarrow \mathbb{R}^+, \)
  \[ U[\xi + \epsilon \eta] \approx U[\xi] + \epsilon < \nabla_{\xi} U, \eta > \]
  smooth disturbance \( \eta \in \Xi \) s.t. \( \eta(0) = \eta(T) = 0 \)

  arbitrary small \( \epsilon \in \mathbb{R} \)

  \[ < \nabla_{\xi} U, \eta > = \lim_{\epsilon \to 0} \frac{U[\xi + \epsilon \eta] - U[\xi]}{\epsilon} \]
  \[ (1) \]
  \[ < \nabla_{\xi} U, \eta > = \int_0^T \nabla_{\xi} U(t)^T \eta(t)dt \]
  \[ (2) \]

We are going to massage equation (1) to equation (2) and term match to find \( \nabla_{\xi} U \).

Let \( \phi(\epsilon) = U[\xi + \epsilon \eta], \)

\[ < \nabla_{\xi} U, \eta > = \lim_{\epsilon \to 0} \frac{\phi(\epsilon) - \phi(0)}{\epsilon} \]
\[
\frac{d\phi}{d\epsilon}
\]

\[
= \frac{d}{d\epsilon} \int_0^T F[t, \xi(t) + \epsilon \eta(t), \xi'(t) + \epsilon \eta'(t)] dt \bigg|_{\epsilon = 0}
\]

Exchange differentiation with integration,

\[
= \int_0^T \frac{d}{d\epsilon} F[t, \xi(t) + \epsilon \eta(t), \xi'(t) + \epsilon \eta'(t)] dt \bigg|_{\epsilon = 0}
\]

Change of variables, denote \( x(\epsilon) = \xi(t) + \epsilon \eta(t) \) and \( y(\epsilon) = \xi'(t) + \epsilon \eta'(t) \), then apply chain rule,

\[
= \int_0^T \left( \frac{\partial F[t, x(\epsilon), y(\epsilon)]}{\partial x} \right) T \frac{dx}{d\epsilon} + \left( \frac{\partial F[t, x(\epsilon), y(\epsilon)]}{\partial y} \right) T \frac{dy}{d\epsilon} dt \bigg|_{\epsilon = 0}
\]

Evaluate at \( \epsilon = 0 \),

\[
= \int_0^T \left( \frac{\partial F[t, \xi(t), \xi'(t)]}{\partial \xi(t)} \right) T \eta(t) + \left( \frac{\partial F[t, \xi(t), \xi'(t)]}{\partial \xi'(t)} \right) T \eta'(t) dt
\]

Write in compact form,

\[
= \int_0^T \left( \frac{\partial F}{\partial \xi(t)} \right) T \eta(t) + \left( \frac{\partial F}{\partial \xi'(t)} \right) T \eta'(t) dt
\]

Apply integration by parts to solve \( \int_0^T \left( \frac{\partial F}{\partial \xi'(t)} \right) T \eta'(t) dt \),

\[
\int_0^T \left( \frac{\partial F}{\partial \xi'(t)} \right) T \eta'(t) dt \bigg|_{0}^{T} - \int_0^T \frac{d}{dt} \left( \frac{\partial F}{\partial \xi'(t)} \right) T \eta(t) dt
\]

By definition, \( \eta(0) = \eta(T) = 0 \),

\[
= - \int_0^T \frac{d}{dt} \left( \frac{\partial F}{\partial \xi'(t)} \right) T \eta(t) dt
\]

Thus,

\[
< \nabla_{\xi} U, \eta >= \int_0^T \left( \frac{\partial F}{\partial \xi(t)} - \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)} \right) T \eta(t) dt = \int_0^T \nabla_{\xi} U(t) T \eta(t) dt
\]

for every \( \eta \)

Therefore,

\[
\nabla_{\xi} U(t) = \frac{\partial F}{\partial \xi(t)}(t) - \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)}(t)
\]

\( \square \)
6.3 CHOMP: Covariant Hamiltonian Optimization for Motion Planning

CHOMP instantiates functional gradient descent for cost

$$U[\xi] = U_{\text{smooth}}[\xi] + \lambda U_{\text{obs}}[\xi]$$

Smoothness cost is defined as in our example

$$U_{\text{smooth}}[\xi] = \frac{1}{2} \int_0^T \| \xi'(t) \|^2 dt$$

Obstacle cost is defined as

$$U_{\text{obs}}[\xi] = \int \int \int \int c(\phi_u(\xi(t))) : \left\| \frac{d}{dt} \phi_u(\xi(t)) \right\| du dt$$

Understanding $U_{\text{obs}}[\xi]$:

- Define a cost function in $W$, $c : W \rightarrow \mathbb{R}$ that uses a signed distance field to compute distance to the closest obstacle, and returns a higher cost the closer the point is.
- Then for each time point along the trajectory (thus the integral over time), look at the configuration $\xi(t)$.
- For each body point on the robot $u$ (thus the integral over body points), apply for forward kinematics mapping $\phi_u$ to get the xyz locations of the points when the robot is in configuration $\xi(t)$.
- For each body point location, compute the cost $c$.

The second term in the integral (the norm of the velocity) is there to create a path integral formulation.