4.1 Mathematical Objects in Robotic Motion

- Time: \( t \in \mathbb{R}, t \in [0, T], q \in C \subset \mathbb{R}^D \) where \( D \) is the number of degrees of freedom of the robot.

- Trajectory: \( \xi : [0, T] \rightarrow C \), mapping \( t \mapsto q \).
  
  - At each time \( t \), the robot’s configuration is \( [q_1(t), q_2(t), ..., q_D(t)]^\top \in C \).
  
  - In discrete time, a trajectory can be written \( \xi = [q_1^\top, q_2^\top, ..., q_N^\top]^\top \) which is of dimension \( DN \times 1 \). This approximation converges to the actual trajectory as \( N \rightarrow \infty \).
  
  - We denote by \( \Xi \) the space of possible (continuous) time trajectories \( \xi \) through the configuration space \( C \).

- Cost \( U : \Xi \rightarrow \mathbb{R}^+ \), which is a functional over the space of trajectories.

4.2 Scalar Functions

4.2.1 Derivative

For \( f(x) : \mathbb{R} \rightarrow \mathbb{R} \)

\[
f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon)-f(x)}{\varepsilon}.
\]

Geometrically, \( f'(x) \) is the slope of the graph of \( f \) at point \( x \).

For example: \( f(x) = 2x; f'(x) = \lim_{\varepsilon \to 0} \frac{2(x+\varepsilon)-2x}{\varepsilon} = 2 \)

4.2.2 Chain Rule

Let \( F(x) = f(g(x)) \). Then \( F'(x) = f'(g(x)) \cdot g'(x) \).

Leibniz notation: let \( y = f(x) \) and \( z = f(y) \), then \( \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \).
4.3 Multivariate and Vector-Valued Functions

4.3.1 Partial Derivative

A derivative taken with respect to one variable while keeping the others constant. For $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f(x, y)}{\partial x} = \lim_{\epsilon \to 0} \frac{f(x + \epsilon, y) - f(x, y)}{\epsilon}.$$  

For example: $f(x, y) = 2x + y + xy$,  

$$\frac{\partial f(x, y)}{\partial x} = \lim_{\epsilon \to 0} \frac{(2x + \epsilon) + y + (x + \epsilon)y - 2x - y - xy}{\epsilon} = 2 + y.$$  

4.3.2 Gradient of a multivariate scalar function

Defined for functions $f : \mathbb{R}^m \to \mathbb{R}$ with $m \geq 1$. Shown for $m = 2$.

$$\nabla f(x, y) = \begin{bmatrix} \frac{\partial f(x, y)}{\partial x} \\ \frac{\partial f(x, y)}{\partial y} \end{bmatrix} \equiv \text{Gradient}$$

The gradient gives the direction of greatest rate of increase, and the magnitude is the slope in that direction.

For example: $f(x) = 2x + y + xy$,  

$$\nabla f(x, y) = \begin{bmatrix} 2 + y \\ 1 + x \end{bmatrix}.$$  

4.3.3 Ordinary derivative of a univariate vector-valued function

Defined for functions $f : \mathbb{R} \to \mathbb{R}^n$ with $n \geq 1$. Shown for $n = 2$.

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \quad f'(x) = \begin{bmatrix} \frac{df_1(x)}{dx} \\ \frac{df_2(x)}{dx} \end{bmatrix}.$$  

For example: $f(x) = \begin{bmatrix} x + 2 \\ 2x \end{bmatrix}$,  

$$f'(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$  

4.3.4 Jacobian of a multivariate, vector-valued function

Defined for functions $f : \mathbb{R}^m \to \mathbb{R}^n$ with $m, n \geq 1$. Shown for $m = n = 2$.

$$f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}, \quad \frac{\partial f(x, y)}{\partial (x, y)} = \begin{bmatrix} \frac{\partial f_1(x, y)}{\partial x} & \frac{\partial f_1(x, y)}{\partial y} \\ \frac{\partial f_2(x, y)}{\partial x} & \frac{\partial f_2(x, y)}{\partial y} \end{bmatrix} \equiv \text{Jacobian matrix}$$

Example 1: $f(x, y) = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ y \end{bmatrix}$,  

$$\frac{\partial f(x, y)}{\partial (x, y)} = A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} (f \text{ is a linear function}).$$  

Example 2: $g(v) = v^\top Av$,  

$$\frac{\partial g(v)}{\partial v} = \frac{\partial v^\top Av}{\partial v} = v^\top (A^\top + A).$$
4.3.5 Multivariate Chain Rule

In Leibniz notation: let \( F = F(x(\epsilon), y(\epsilon)) \), then \( \frac{dF}{d\epsilon} = \frac{\partial F}{\partial x} \frac{dx}{d\epsilon} + \frac{\partial F}{\partial y} \frac{dy}{d\epsilon} \).

4.4 Integration by parts

Given two functions \( f, g : \mathbb{R} \to \mathbb{R} \), we have:
\[
\int_a^b f \frac{dg}{dx} \, dx = \left. fg \right|_a^b - \int_a^b \frac{df}{dx} g \, dx.
\]

4.5 Functionals

4.5.1 Inner Product

Given two trajectories \( \xi_1, \xi_2 : [0, T] \to \mathbb{C} \) in the trajectory space \( \Xi \), their Euclidean inner product is:
\[
\langle \xi_1, \xi_2 \rangle = \int_0^T \xi_1(t)^\top \xi_2(t) \, dt.
\]

4.5.2 Gradient Formula

Consider a functional \( \mathcal{U} : \Xi \to \mathbb{R}^+ \) mapping \( \xi(\cdot) \mapsto \mathcal{U}[\xi] \), in the form of an integral cost given by:
\[
\mathcal{U}[\xi] = \int_0^T F(t, \xi(t), \xi'(t)) \, dt.
\]
Then,
\[
\nabla_\xi \mathcal{U}(t) = \frac{\partial F}{\partial \xi} - \frac{d}{dt} \frac{\partial F}{\partial \xi'}.
\]
Note that the above gradient is itself a function in \( \Xi \).

Proof. We will use the calculus of variations. Introduce a smooth deformation \( \eta \in \Xi \) s.t. \( \eta(0) = \eta(T) = 0 \) and some arbitrarily small \( \epsilon \in \mathbb{R} \). We can then write the first-order Taylor series expansion of \( \mathcal{U} \) around \( \xi \):
\[
\mathcal{U}[\xi + \epsilon \eta] \simeq \mathcal{U}[\xi] + \epsilon \cdot \langle \nabla_\xi \mathcal{U}, \eta \rangle.
\]
In the limit, we have:
\[
\langle \nabla_\xi \mathcal{U}, \eta \rangle = \lim_{\epsilon \to 0} \frac{\mathcal{U}[\xi + \epsilon \eta] - \mathcal{U}[\xi]}{\epsilon}.
\]
Let \( \phi(\epsilon) = \mathcal{U}[\xi + \epsilon \eta] \). Then from above we have \( \langle \nabla_\xi \mathcal{U}, \eta \rangle = \lim_{\epsilon \to 0} \frac{\phi(\epsilon) - \phi(0)}{\epsilon} = \phi'(0) \).
We have
\[
\frac{d\phi}{d\epsilon}(0) = \frac{d}{d\epsilon} \int_0^T F(t, \xi(t) + \epsilon \eta(t), \xi'(t) + \epsilon \eta'(t)) \, dt \big|_{\epsilon = 0}.
\]
reordering differentiation and integration gives

\[ \frac{d\phi}{d\epsilon}(0) = \int_0^T \frac{d}{d\epsilon} F(t, \xi(t) + \epsilon \eta(t), \xi'(t) + \epsilon \eta'(t)) dt \ (0), \]

and, by the chain rule, this becomes

\[ \frac{d\phi}{d\epsilon}(0) = \int_0^T \left( \frac{\partial F(t, \xi(t) + \epsilon \eta(t), \xi'(t) + \epsilon \eta'(t))}{\partial \xi(t) + \epsilon \eta(t)} \right) \eta(t) + \left( \frac{\partial F(t, \xi(t) + \epsilon \eta(t), \xi'(t) + \epsilon \eta'(t))}{\partial \xi'(t) + \epsilon \eta'(t)} \right) \eta'(t) dt \ (0) \]

Evaluating this expression at $\epsilon = 0$, we can more compactly write

\[ \frac{d\phi}{d\epsilon}(0) = \int_0^T \frac{\partial F(t, \xi(t), \xi'(t))}{\partial \xi(t)} \eta(t) + \frac{\partial F(t, \xi(t), \xi'(t))}{\partial \xi'(t)} \eta'(t) dt. \]

Integrating by parts the second term — with $\frac{\partial F}{\partial \xi'(t)}$ and $\eta'(t)dt$—, we have that

\[ \int_0^T \frac{\partial F}{\partial \xi'(t)} \eta'(t) dt = \frac{\partial F}{\partial \xi'(t)} \eta(t) \bigg|_0^T - \int_0^T \eta(t) \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)} dt, \]

and the first term is in fact equal to zero, because $\eta(0) = \eta(T) = 0$ from the definition of a variation above. We therefore conclude:

\[ \frac{d\phi}{d\epsilon}(0) = \int_0^T \left[ \frac{\partial F}{\partial \xi(t)} - \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)} \right] \eta(t) dt, \]

and so, writing $\phi'(0)$ again as $\langle \nabla_\xi \mathcal{U}, \eta \rangle$, we have

\[ \int_0^T \nabla_\xi \mathcal{U}(t) \top \eta(t) dt = \int_0^T \left[ \frac{\partial F}{\partial \xi(t)} - \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)} \right] \eta(t) dt. \]

Since the above is true for any arbitrary smooth deformation $\eta$, it can only be that

\[ \nabla_\xi \mathcal{U}(t) = \frac{\partial F}{\partial \xi(t)} - \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)} \]

which concludes the proof.
4.6 Discrete-Time Trajectory: Costs, Gradients and Inner Products

4.6.1 Gradient of the quadratic velocity cost

Consider a discrete time trajectory $\xi = [q_1^T, q_2^T, ..., q_N^T]^T$ representing $N$ intermediate points in $C \subset \mathbb{R}^D$ between fixed initial and terminal configurations $q_0 \equiv S$ and $q_{N+1} \equiv G$ respectively. We will consider a particular cost $U : C^N \rightarrow \mathbb{R}_+$ given as:

$$U = \frac{1}{2} \sum_{i=0}^{N} ||q_{i+1} - q_i||^2.$$

Note that this cost function penalizes large differences between any two consecutive configurations. The gradient of this “velocity cost” with respect to the $i$th intermediate configuration $q_i$ (which will be a vector in $\mathbb{R}^D$) can then be written as:

$$\nabla_{\xi} U(i) = -(q_{i+1} - q_i) + (q_i - q_{i-1}) = 2q_i - q_{i+1} - q_{i-1},$$

noting the two special cases:

$$i = 1 : \quad \nabla_{\xi} U(1) = 2q_1 - q_2 - q_0,$$
$$i = N : \quad \nabla_{\xi} U(N) = 2q_N - q_{N+1} - q_{N-1}.$$

We can then write $\nabla_{\xi} U = A_{\xi} + b$, where

$$A = \begin{bmatrix} 2I & -I & 0 & \ldots & 0 \\ -I & 2I & -I & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2I & -I \\ 0 & \ldots & -I & 2I & -I \\ \end{bmatrix},$$

with $I$ the identity matrix of size $D \times D$.

4.6.2 Inner products

The Euclidean inner product in the discrete trajectory space is simply

$$\langle \xi_1, \xi_2 \rangle = \xi_1^T \xi_2.$$

Consider three discrete-time 1-dimensional trajectories, each comprised by a sequence of 5 configurations in $C = \mathbb{R}$:

$$a = [0 \ 0 \ 0 \ 0 \ 0]^T,$$
$$b = [0 \ 0 \ 10 \ 0 \ 0]^T,$$
$$c = [0 \ 5 \ 10 \ 5 \ 0]^T.$$

Under the Euclidean norm $||x|| = (x^T x)^{1/2}$, we have:

$$||a - b||^2 = (a - b)^T (a - b) = 100,$$
$$||a - c||^2 = (a - c)^T (a - c) = 150.$$
Under this metric, trajectory $b$ is closer to trajectory $a$ than $c$ is. However, this can be misleading, since the cost of $b$ under $\mathcal{U}$ is in fact larger than the cost of $c$, therefore farther from the zero cost associated to $a$. An attractive idea would therefore be to define an alternative Euclidean product whose induced metric was sensitive to the cost $\mathcal{U}$ imposed on the trajectory space.

Noting that $A$ is a symmetric positive-definite matrix (often written $A \succ 0$), we can define a “weighted” inner product:

$$\langle \xi_1, \xi_2 \rangle_A = \xi_1^\top A \xi_2.$$

Using the induced metric, we now have:

$$||a - b||_A^2 = (a - b)^\top A (a - b) = 200,$$

$$||a - c||_A^2 = (a - c)^\top A (a - c) = 100.$$

Thus, under this new metric, we have that $c$ is closer to $a$ than $b$ is. The off-diagonal terms in $A$ are taking temporal variation into account, favoring smoother trajectory deformations in gradient descent approaches, which can often have the desirable effect of accelerating convergence in numerical trajectory optimization methods.

### 4.6.3 First-Order Approximation

Truncating the Taylor expansion to the first order, can locally approximate the cost of a trajectory $\bar{\xi}$ in the neighborhood of an initial trajectory $\bar{\xi}_i$ with known cost.

$$\mathcal{U}[\bar{\xi}] \simeq \mathcal{U} + \langle \nabla_{\bar{\xi}} \mathcal{U}, \bar{\xi} - \bar{\xi}_i \rangle.$$

If instead, we use the inner product space as defined by the positive-definite weighting matrix $A$, we would similarly have:

$$\mathcal{U}[\bar{\xi}] \simeq \mathcal{U} + \langle \nabla_{\bar{\xi}}^A \mathcal{U}, \bar{\xi} - \bar{\xi}_i \rangle_A.$$

We would like to determine the appropriate form of the “weighted gradient” $\nabla_{\bar{\xi}}^A \mathcal{U}$, which we can do by equating the two above expressions for all small deformations $\bar{\xi} - \bar{\xi}_i$:

$$\langle \nabla_{\bar{\xi}} \mathcal{U}, \bar{\xi} - \bar{\xi}_i \rangle = \langle \nabla_{\bar{\xi}}^A \mathcal{U}, \bar{\xi} - \bar{\xi}_i \rangle_A$$

$$\nabla_{\bar{\xi}} \mathcal{U}^\top (\bar{\xi} - \bar{\xi}_i) = \nabla_{\bar{\xi}}^A \mathcal{U}^\top A (\bar{\xi} - \bar{\xi}_i)$$

$$\nabla_{\bar{\xi}} \mathcal{U} = A \nabla_{\bar{\xi}}^A \mathcal{U}$$

$$\nabla_{\bar{\xi}}^A \mathcal{U} = A^{-1} \nabla_{\bar{\xi}} \mathcal{U}$$

In the above derivation, we have made use of the fact that $A^\top = A \succ 0$. We see that, since matrix $A$ represents an invertible linear map in the discrete trajectory space, we can arbitrarily transition between one gradient and the other.