CS 294-115 Algorithmic Human-Robot Interaction **Fall 2015**

Lecture 4: A Review of Multivariate and Variational Calculus

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4.1 Mathematical Objects in Robotic Motion

- Time: $t \in \mathbb{R}$, $t \in [0, T]$, $q \in C \subset R^D$ where D is the number of degrees of freedom of the robot.
- Trajectory: $\xi : [0, T] \to C$, mapping $t \mapsto q$.
  - At each time t, the robot's configuration is $[q_1(t), q_2(t), ..., q_D(t)]^\top \in C$.
  - In discrete time, a trajectory can be written $\xi = [q_1^\top, q_2^\top, ..., q_N^\top]^\top$ which is of dimension $DN \times 1$. This approximation converges to the actual trajectory as $N \to \infty$.
  - We denote by Ξ the space of possible (continuous) time trajectories ξ through the configuration space C.
- Cost $\mathcal{U} : \Xi \to \mathbb{R}^+$, which is a functional over the space of trajectories.

4.2 Scalar Functions

4.2.1 Derivative

For $f(x) : \mathbb{R} \to \mathbb{R}$

$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$. Geometrically, $f'(x)$ is the slope of the graph of f at point x.

For example: $f(x) = 2x; f'(x) = \lim_{\varepsilon \to 0} \frac{2(x+\varepsilon) - 2x}{\varepsilon} = 2$

4.2.2 Chain Rule

Let $F(x) = f\big(g(x)\big)$. Then $F'(x) = f'\big(g(x)\big) \cdot g'(x)$.

Leibniz notation: let $y = f(x)$ and $z = f(y)$, then $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$.

4.3 Multivariate and Vector-Valued Functions

4.3.1 Partial Derivative

A derivative taken with respect to one variable while keeping the others constant. For $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon, y) - f(x,y)}{\varepsilon}.$$

For example: $f(x,y) = 2x + y + xy$, $\frac{\partial f(x,y)}{\partial x} = \lim_{\varepsilon \to 0} \frac{2(x+\varepsilon)+y+(x+\varepsilon)y-2x-y-xy}{\varepsilon} = 2 + y$.

4.3.2 Gradient of a multivariate scalar function

Defined for functions $f : \mathbb{R}^m \to \mathbb{R}$ with $m \geq 1$. Shown for $m = 2$.

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} \end{bmatrix} \equiv \text{Gradient}$$

The gradient gives the direction of greatest rate of increase, and the magnitude is the slope in that direction.

For example: $f(x) = 2x + y + xy$, $\nabla f(x,y) = \begin{bmatrix} 2+y \\ 1+x \end{bmatrix}$.

4.3.3 Ordinary derivative of a univariate vector-valued function

Defined for functions $f : \mathbb{R} \to \mathbb{R}^n$ with $n \geq 1$. Shown for $n = 2$.

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \qquad f'(x) = \begin{bmatrix} \frac{df_1(x)}{dx} \\ \frac{df_2(x)}{dy} \end{bmatrix}.$$

For example: $f(x) = \begin{bmatrix} x+2 \\ 2x \end{bmatrix}$ $f'(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

4.3.4 Jacobian of a multivariate, vector-valued function

Defined for functions $f : \mathbb{R}^m \to \mathbb{R}^n$ with $m, n \geq 1$. Shown for $m = n = 2$.

$$f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}, \qquad \frac{\partial f(x,y)}{\partial (x,y)} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \equiv \text{Jacobian matrix}$$

Example 1: $f(x,y) = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ y \end{bmatrix}$, $\frac{\partial f(x,y)}{\partial (x,y)} = A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ (f is a linear function).

Example 2: $g(v) = v^\top A v$, $\frac{\partial g(v)}{\partial v} = \frac{\partial v^\top A v}{\partial v} = v^\top (A^\top + A)$.

4.3.5 Multivariate Chain Rule

In Leibniz notation: let $F = F\big(x(\varepsilon), y(\varepsilon)\big)$, then $\dfrac{dF}{d\varepsilon} = \dfrac{\partial F}{\partial x} \cdot \dfrac{dx}{d\varepsilon} + \dfrac{\partial F}{\partial y} \cdot \dfrac{dy}{d\varepsilon}$.

4.4 Integration by parts

Given two functions $f, g : \mathbb{R} \to \mathbb{R}$, we have:

$$\int_a^b f \frac{dg}{dx} dx = fg\Big|_a^b - \int_a^b \frac{df}{dx} g dx$$

4.5 Functionals

4.5.1 Inner Product

Given two trajectories $\xi_1, \xi_2 : [0, T] \to C$ in the trajectory space Ξ, their Euclidean inner product is:

$$\langle \xi_1, \xi_2 \rangle = \int_0^T \xi_1(t)^\top \xi_2(t) dt.$$

4.5.2 Gradient Formula

Consider a functional $\mathcal{U} : \Xi \to \mathbb{R}^+$ mapping $\xi(\cdot) \mapsto \mathcal{U}[\xi]$, in the form of an integral cost given by:

$$\mathcal{U} = \int_0^T F\big(t, \xi(t), \xi'(t)\big) dt.$$

Then,

$$\nabla_\xi \mathcal{U}(t) = \frac{\partial F}{\partial \xi} - \frac{d}{dt} \frac{\partial F}{\partial \xi'}.$$

Note that the above gradient is itself a function in Ξ.

Proof. We will use the calculus of variations. Introduce a smooth deformation $\eta \in \Xi$ s.t. $\eta(0) = \eta(T) = 0$ and some arbitrarily small $\varepsilon \in \mathbb{R}$. We can then write the first-order Taylor series expansion of $\mathcal{U}$ around ξ:

$$\mathcal{U}[\xi + \varepsilon\eta] \simeq \mathcal{U}[\xi] + \varepsilon \cdot \langle \nabla_\xi \mathcal{U}, \eta \rangle.$$

In the limit, we have:

$$\langle \nabla_\xi \mathcal{U}, \eta \rangle = \lim_{\varepsilon \to 0} \frac{\mathcal{U}[\xi + \varepsilon\eta] - \mathcal{U}[\xi]}{\varepsilon}$$

Let $\phi(\varepsilon) = \mathcal{U}[\xi + \varepsilon\eta]$. Then from above we have $\langle \nabla_\xi \mathcal{U}, \eta \rangle = \lim\limits_{\varepsilon \to 0} \dfrac{\phi(\varepsilon) - \phi(0)}{\varepsilon} = \phi'(0)$.

We have

$$\frac{d\phi}{d\varepsilon}(0) = \frac{d}{d\varepsilon} \int_0^T F\big(t, \xi(t) + \varepsilon\eta(t), \xi'(t) + \varepsilon\eta'(t)\big) dt \ (0),$$

reordering differentiation and integration gives

$$\frac{d\phi}{d\varepsilon}(0) = \int_0^T \frac{d}{d\varepsilon} F\big(t, \xi(t) + \varepsilon\eta(t), \xi'(t) + \varepsilon\eta'(t)\big) dt\ (0),$$

and, by the chain rule, this becomes

$$\frac{d\phi}{d\varepsilon}(0) = \int_0^T \left(\frac{\partial F\big(t, \xi(t) + \varepsilon\eta(t), \xi'(t) + \varepsilon\eta'(t)\big)}{\partial[\xi(t) + \varepsilon\eta(t)]} \right)^\top \eta(t) + \left(\frac{\partial F\big(t, \xi(t) + \varepsilon\eta(t), \xi'(t) + \varepsilon\eta'(t)\big)}{\partial[\xi'(t) + \varepsilon\eta'(t)]} \right)^\top \eta'(t) dt\ (0)$$

Evaluating this expression at $\varepsilon = 0$, we can more compactly write

$$\frac{d\phi}{d\varepsilon}(0) = \int_0^T \frac{\partial F\big(t, \xi(t), \xi'(t)\big)}{\partial \xi(t)}^\top \eta(t) + \frac{\partial F\big(t, \xi(t), \xi'(t)\big)}{\partial \xi'(t)}^\top \eta'(t) dt.$$

Integrating by parts the second term — with $\frac{\partial F}{\partial \xi'(t)}$ and $\eta'(t)dt$—, we have that

$$\int_0^T \frac{\partial F}{\partial \xi'(t)}^\top \eta'(t) dt = \frac{\partial F}{\partial \xi'(t)}^\top \eta(t)\Big|_0^T - \int_0^T \eta(t)^\top \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)} dt,$$

and the first term is in fact equal to zero, because $\eta(0) = \eta(T) = 0$ from the definition of a variation above. We therefore conclude:

$$\frac{d\phi}{d\varepsilon}(0) = \int_0^T \left[\frac{\partial F}{\partial \xi(t)} - \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)} \right]^\top \eta(t) dt,$$

and so, writing $\phi'(0)$ again as $\langle \nabla_\xi \mathcal{U}, \eta \rangle$, we have

$$\int_0^T \nabla_\xi \mathcal{U}(t)^\top \eta(t) dt = \int_0^T \left[\frac{\partial F}{\partial \xi(t)} - \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)} \right]^\top \eta(t) dt.$$

Since the above is true for any arbitrary smooth deformation η, it can only be that

$$\nabla_\xi \mathcal{U}(t) = \frac{\partial F}{\partial \xi(t)} - \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)}$$

which concludes the proof.

□

4.6 Discrete-Time Trajectory: Costs, Gradients and Inner Products

4.6.1 Gradient of the quadratic velocity cost

Consider a discrete time trajectory $\xi = [q_1^\top, q_2^\top, ..., q_N^\top]^\top$ representing N intermediate points in $C \subset \mathcal{R}^D$ between fixed initial and terminal configurations $q_0 \equiv S$ and $q_{N+1} \equiv G$ respectively. We will consider a particular cost $\mathcal{U} : C^N \to \mathbb{R}_+$ given as:

$$\mathcal{U} = \frac{1}{2}\sum_{i=0}^{N} ||q_{i+1} - q_i||^2.$$

Note that this cost function penalizes large differences between any two consecutive configurations. The gradient of this "velocity cost" with respect to the ith intermediate configuration q_i (which will be a vector in $\mathcal{R}^D$) can then be written as:

$$\nabla_\xi \mathcal{U}_{(i)} = -(q_{i+1} - q_i) + (q_i - q_{i-1}) = 2q_i - q_{i+1} - q_{i-1},$$

noting the two special cases:

$$\begin{aligned} i = 1: &\qquad \nabla_\xi \mathcal{U}_{(1)} = 2q_i - q_{i+1} - q_{i-1}, \\ i = N: &\qquad \nabla_\xi \mathcal{U}_{(N)} = 2q_i - q_{i+1} - q_{i-1}. \end{aligned}$$

We can then write $\nabla_\xi \mathcal{U} = A\xi + b$, where

$$A = \begin{bmatrix} 2I & -I & 0 & \dots & 0 \\ -I & 2I & -I & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -I & 2I & -I \\ 0 & \dots & 0 & -I & 2I \end{bmatrix},$$

with I the identity matrix of size $D \times D$.

4.6.2 Inner products

The Euclidean inner product in the discrete trajectory space is simply

$$\langle \xi_1, \xi_2 \rangle = \xi_1^\top \xi_2.$$

Consider three discrete-time 1-dimensional trajectories, each comprised by a sequence of 5 configurations in $C = \mathbb{R}$:

$$\begin{aligned} a &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top, \\ b &= \begin{bmatrix} 0 & 0 & 10 & 0 & 0 \end{bmatrix}^\top, \\ c &= \begin{bmatrix} 0 & 5 & 10 & 5 & 0 \end{bmatrix}^\top. \end{aligned}$$

Under the Euclidean norm $||x|| = (x^\top x)^{1/2}$, we have:

$$\begin{aligned} ||a - b||^2 &= (a-b)^\top (a-b) = 100, \\ ||a - c||^2 &= (a-c)^\top (a-c) = 150. \end{aligned}$$

Under this metric, trajectory b is closer to trajectory a than c is. However, this can be misleading, since the cost of b under $\mathcal{U}$ is in fact larger than the cost of c, therefore farther from the zero cost associated to a. An attractive idea would therefore be to define an alternative Euclidean product whose induced metric was sensitive to the cost $\mathcal{U}$ imposed on the trajectory space.

Noting that A is a symmetric positive-definite matrix (often written $A \succ 0$), we can define a "weighted" inner product:

$$\langle \xi_1, \xi_2 \rangle_A = \xi_1^\top A \xi_2.$$

Using the induced metric, we now have:

$$\begin{aligned} ||a-b||_A^2 &= (a-b)^\top A(a-b) = 200, \\ ||a-c||_A^2 &= (a-c)^\top A(a-c) = 100. \end{aligned}$$

Thus, under this new metric, we have that c is closer to a than b is. The off-diagonal terms in A are taking temporal variation into account, favoring smoother trajectory deformations in gradient descent approaches, which can often have the desirable effect of accelerating convergence in numerical trajectory optimization methods.

4.6.3 First-Order Approximation

Truncating the Taylor expansion to the first order, can locally approximate the cost of a trajectory ξ in the neighborhood of an initial trajectory ξ_i with known cost.

$$\mathcal{U}[\xi] \simeq \mathcal{U} + \langle \nabla_\xi \mathcal{U}, \xi - \xi_i \rangle.$$

If instead, we use the inner product space as defined by the positive-definite weighting matrix A, we would similarly have:

$$\mathcal{U}[\xi] \simeq \mathcal{U} + \langle \nabla_\xi^A \mathcal{U}, \xi - \xi_i \rangle_A.$$

We would like to determine the appropriate form of the "weighted gradient" $\nabla_\xi^A \mathcal{U}$, which we can do by equating the two above expressions for all small deformations $\xi - \xi_i$:

$$\begin{aligned} \langle \nabla_\xi \mathcal{U}, \xi - \xi_i \rangle &= \langle \nabla_\xi^A \mathcal{U}, \xi - \xi_i \rangle_A \\ \nabla_\xi \mathcal{U}^\top (\xi - \xi_i) &= \nabla_\xi^A \mathcal{U}^\top A(\xi - \xi_i) \\ \nabla_\xi \mathcal{U} &= A \nabla_\xi^A \mathcal{U} \\ \nabla_\xi^A \mathcal{U} &= A^{-1} \nabla_\xi \mathcal{U} \end{aligned}$$

In the above derivation, we have made use of the fact that $A^\top = A \succ 0$. We see that, since matrix A represents an invertible linear map in the discrete trajectory space, we can arbitrarily transition between one gradient and the other.