3.1 Wrap up/Review: RRT & RRT*

3.1.1 Bi-Directional RRT

The bi-directional RRT algorithm grows two trees towards each other, iteratively following these two steps until a solution is found:

1. Sample a configuration in $C_{\text{free}}$.
2. Run simple planner - try to connect sample to closest node in each tree.

Pros/cons:

- (+) Scales well in high dimensions
- (-) Not optimal, but short-cutting post-processing improves result: pick two nodes at random and try to connect them
- (-) Not complete, but probabilistically complete: $\lim_{t \to \infty} P(\text{sol} \mid \exists \text{ sol}) = 1$. Note: This is a weak property.

RRT is single tree (unidirectional) version, where you sample the goal with some probability as opposed to always sampling a random configuration.

3.1.2 RRT*

A form of optimal motion planning. Differs from RRT in two ways.
• Parent selection, using cost-to-come
• Rewiring step

Pros/cons:

• (+) Asymptotically optimal
• (-) Sacrifices speed, considers optimal path to everything

3.2 Trajectory Optimization Outline

1. Problem Statement
2. (Functional) Gradient Descent
3. CHOMP [Next lecture] – See reading for a comparison between CHOMP, BiRRT, and RRT*.

3.3 Problem Statement

Notation:

• trajectory $\xi : [0, T] \rightarrow C$ - maps time to configurations, infinite dimensional
• $U : \Xi \rightarrow \mathbb{R}^+\,$, functional that maps functions to scalars
• objective $U$ can encode aspects like path length, efficiency, obstacle avoidance, legibility, uncertainty reduction (achieve goal with high probability), comfort of human

$$\begin{align*}
\xi^* &= \arg\min_{\xi \in \Xi} U(\xi) \\
\text{subject to} \quad & \xi(0) = q_s, \\
& \xi(T) = q_g
\end{align*}$$
We can solve this problem when it is convex, however it is most often non-convex (e.g. due to obstacles). At local minima, the trajectory might be inefficient, have collisions, scare people, etc.

Often: find $\xi^*$ s.t. $U(\xi^*) \leq \beta$.

### 3.4 Functional Gradient Descent

\[
\xi_{i+1} = \xi_i - \frac{1}{\alpha} \nabla \xi_i (U)
\]

**Thm:** If

\[
U[\xi] = \int_0^T F(t, \xi(t), \xi'(t)) dt
\]

and using the euclidean inner product (see following section), then you can prove that:

\[
\nabla \xi U = \frac{\partial F}{\partial \xi(t)} - \frac{d}{dt} \frac{\partial F}{\partial \xi'(t)}
\]

This has a similar derivation to the Euler-Lagrange formula from calculus of variations. We will go through the derivation in the next lecture.

**Example:** Consider the example where you minimize the squared norm of velocity subject to starting at $q_s$ and ending at $q_g$:

\[
U[\xi] = \frac{1}{2} \int_0^T \| \xi'(t) \|^2 dt
\]

Since this objective is quadratic, we can show that the optimal solution (when the gradient is 0) corresponds to a straight line path of constant velocity. Using the above theorem:

\[
\nabla \xi U = 0 - \frac{d}{dt} \xi'(t)
\]

\[
= -\xi''(t) = 0
\]
\[ \xi'(t) = a \quad (3.3) \]
\[ \xi(t) = at + b \quad (3.4) \]

To solve for \( a \) and \( b \), one can simply plug in the boundary conditions that \( \xi(0) = q_s \) and \( \xi(T) = q_g \). The resulting trajectory is the following, a straight line with constant velocity:

\[ q_s \longleftrightarrow q_g \]

### 3.4.1 Hilbert space & inner products

\[ \Xi \] is a Hilbert space, a complete vector space with an inner product.

An inner product \( \langle \xi_1, \xi_2 \rangle \) is defined to have the following properties:

- **symmetry**: \( \langle \xi_1, \xi_2 \rangle = \langle \xi_2, \xi_1 \rangle \)
- **positive definite**: \( \langle \xi_1, \xi_1 \rangle \geq 0 \), and \( \langle \xi_1, \xi_1 \rangle = 0 \iff \xi_1 = 0 \)
- **linearity** in the first argument: \( \langle a \xi_1, \xi_2 \rangle = a \langle \xi_1, \xi_2 \rangle \) and \( \langle \xi_1 + \xi_2, \xi_3 \rangle = \langle \xi_1, \xi_3 \rangle + \langle \xi_2, \xi_3 \rangle \) (the same holds for the second argument by symmetry)

The Euclidean inner product is defined as

\[ \langle \xi_1, \xi_2 \rangle = \int_0^T \xi_1(t)^T \xi_2(t) \, dt \]

In the more familiar vector form, this becomes \( \xi_1^T \xi_2 \).