Homework 6: Due in class on Thursday 4/19

1. We are given a set of $k$ parallel independent additive Gaussian noise channels with noise variances $N_1, \ldots, N_k$ respectively. A single transmitter is permitted to communicate to a single receiver over this set of channels. The transmitter is power constrained to $P$. Find the capacity of the system (in bits per use) in each of the following scenarios:

(a) The transmitter can distribute its available power among the $k$ channels in any way it likes and can choose the inputs to each channel in any way it likes (as a function of the message it wants to send) subject to the power constraints determined by the way it distributes power over the channels. The receiver receives information from each of the $k$ channels separately.

(b) The transmitter is constrained to use exactly the same input in each of the $k$ channels (as a function of the message it wants to send). The receiver receives information from each of the $k$ channels separately.

(c) The transmitter can distribute its available power among the $k$ channels in any way it likes. The inputs to each channel have to be scaled versions of a single input (as a function of the message the transmitter wishes to send) and subject to the individual power constraints determined by the way the transmitter distributes power over the channels. The receiver receives information from each of the $k$ channels separately.

(d) The transmitter is constrained to use exactly the same input in each of the $k$ channels (as a function of the message it wants to send). The receiver, however, only sees the sum of the outputs of the $k$ channels.

2. Problem 10.7 on pg. 338 of the text.

3. Problem 10.8 on pp. 338-339 of the text.

4. Successive refinement of information

Consider a sequence $X_1, X_2, \ldots$ of i.i.d. random variables drawn from a finite alphabet $\mathcal{X}$ with marginal distribution $p(x)$. Let $\hat{\mathcal{X}}$ be a finite (reproduction) alphabet and let $d: \mathcal{X} \times \hat{\mathcal{X}} \mapsto \mathbb{R}_+$ be a distortion measure. Let $D_1 \geq D_2$ be two distortion levels. Let $R_1$ denote $R(D_1)$ and $R_2$ denote $R(D_2)$, where $R(D)$ is the rate-distortion function of the source.

We say that successive refinement from distortion $D_1$ to distortion $D_2$ is achievable if there exists a sequence of encoding functions

$$i_n: \mathcal{X}^n \mapsto \{1, \ldots, 2^{nR_1}\}$$

and

$$j_n: \mathcal{X}^n \mapsto \{1, \ldots, 2^{n(R_2-R_1)}\},$$


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and reconstruction functions

\[ g_{1n} : \{1, \ldots, 2^{nR_1}\} \mapsto \hat{X}^n \]

and

\[ g_{2n} : \{1, \ldots, 2^{nR_1}\} \times \{1, \ldots, 2^{n(R_2-R_1)}\} \mapsto \hat{X}^n, \]

such that

\[
\limsup_{n \to \infty} E[d(X^n, g_{1n}(i_n(X^n)))] \leq D_1,
\]

and

\[
\limsup_{n \to \infty} E[d(X^n, g_{2n}(i_n(X^n), j_n(X^n)))] \leq D_2.
\]

Equitz and Cover proved that successive refinement from distortion \(D_1\) to distortion \(D_2\) is achievable if and only if there exists a conditional distribution \(p(\hat{x}_1, \hat{x}_2 | x)\) from \(X\) to \(\hat{X} \times \hat{X}\) with

\[
E[d(X, \hat{X}_1)] \leq D_1, \quad E[d(X, \hat{X}_2)] \leq D_2,
\]

and such that

\[
I(X; \hat{X}_1) = R(D_1), \quad I(X; \hat{X}_2) = R(D_2),
\]

and such that

\[
p(\hat{x}_1, \hat{x}_2 | x) = p(\hat{x}_2 | x)p(\hat{x}_1 | \hat{x}_2).
\]

(a) Consider the binary source with binary reproduction alphabet and Hamming distortion, with arbitrary marginal distribution for the source. Show that, for every \(D_1 \geq D_2\), successive refinement from distortion \(D_1\) to distortion \(D_2\) is achievable.

(b) Fix \(d > 2\). Let \(X = \{1, 2, \ldots, d\} = \hat{X}\) with Hamming distortion and with the source having uniform distribution. Show that, for every \(D_1 \geq D_2\), successive refinement from distortion \(D_1\) to distortion \(D_2\) is achievable.

The result for the second part of the problem is known to be true even when the source has arbitrary marginal distribution, but this is harder to show.


5. Problem 10.16 on pp. 341 -344 of the text.

6. Problem 10.18 on pg. 344 of the text.

7. Problem 13.5 on pg. 458 of the text.