Notation from the last lecture:

\[ K_N = Q_N \]
\[ K_k = A_k^T K_{k+1} A_k - \Gamma_k + Q_k \]

where \( \Gamma_k = A_k^T K_{k+1} B_k (B_k^T K_{k+1} B_k + R_k)^{-1} B_k^T K_{k+1} \).

Also, \( L_k = -(R_k + B_k^T K_{k+1} B_k)^{-1} B_k^T K_{k+1} A_k \).

1 Fully Observed LQ Problem

Recall the fully observed LQ problem. The linear system evolves as:

\[ x_{k+1} = A_k x_k + B_k u_k + w_k \quad k = 0, 1, \ldots, N - 1, \]

and the objective (informally) is to minimize the quadratic cost:

\[
\min_E \left[ \sum_{k=0}^{N-1} (X_k^T Q_k X_k + U_k^T R_k U_k) + X_N^T Q_N X_N \right]
\]

where \( Q_k \) is a positive semidefinite matrix for \( k = 0, \ldots, N \), and \( R_k \) is a positive definite matrix for \( k = 0, \ldots, N - 1 \). The minimization is over all causal strategies where the controller has access to the states. The optimal cost-to-go is given by:

\[ J_k(x) = x^T K_k x + \sum_{l=k}^{N-1} w_l^T K_{l+1} w_l \]

and the optimal control at time \( k \) is \( u_k = L_k x_k \).

2 Partially Observed LQ Problem

We have the same dynamics:

\[ x_{k+1} = A_k x_k + B_k u_k + w_k \quad k = 0, 1, \ldots, N - 1. \]
The controller has causal access to observations given by:

\[ y_k = C_k x_k + v_k \quad k = 0, 1, \ldots, N. \]

The objective (informally) is:

\[
\min E \left[ \sum_{k=0}^{N-1} (X_k^T Q_k X_k + U_k^T R_k U_k) + X_N^T Q_N X_N \right]
\]

where the minimization is now over a different set of strategies than in the fully observed case. We know from the general theory that we should be writing a DP recursion backwards in time for functions \( J_k(\lambda) \), where \( \lambda \) ranges over probability distributions on \( \mathbb{R}^n \), starting with \( J_N(\lambda) \) given by:

\[
J_N(\lambda) = \text{expected final cost if the conditional law of } X_N \text{ given } (Y_0, \ldots, Y_N, U_0, \ldots, U_{N-1}) \text{ is } \lambda
\]

\[
= E_\lambda [X^T Q_N X]
\]

\[
= \int_{\mathbb{R}^n} x^T Q_N x \lambda(dx).
\]

Let \( m = E_\lambda[X] = \int_{\mathbb{R}^n} x \lambda(dx) \). Note that:

\[
E_\lambda^0 [X^T Q_N X] = \int_{\mathbb{R}^n} x^T Q_N x \lambda^0(dx)
\]

\[= \int_{\mathbb{R}^n} (x - m)^T Q_N (x - m) \lambda(dx)
\]

\[= \int_{\mathbb{R}^n} x^T Q_N x \lambda(dx) - m^T Q_N m
\]

\[= E_\lambda [X^T Q_N X] - (E_\lambda[X])^T Q_N (E_\lambda[X])
\]

where \( \lambda^0 \) is the centered probability distribution corresponding to \( \lambda \), i.e. the translate of \( \lambda \) that results in a distribution with mean the zero vector in \( \mathbb{R}^n \). Thus we may write:

\[
J_N(\lambda) = E_\lambda^0 [X^T Q_N X] + (E_\lambda[X])^T Q_N (E_\lambda[X])
\]

Now let us try to compute \( J_{N-1}(\lambda) \) (for \( \lambda \) a probability distribution on \( \mathbb{R}^n \)) using the DP recursion. Think of \( X \in \mathbb{R}^n \) drawn with distribution \( \lambda \) and think of applying a control \( u \in \mathbb{R}^n \); the next state is then \( A_{N-1} X + B_{N-1} u + w_{N-1} \). We observe \( Y = C_N (A_{N-1} X + B_{N-1} u + w_{N-1}) + v_N \) and then compute the conditional law at time \( N \), i.e. \( T_{N-1,N}(\lambda, u, Y) \). Note that in this expression the distribution of \( Y \) depends on \( \lambda \). The DP equation for period \( N - 1 \) is:

\[
J_{N-1}(\lambda) = \min_u \left\{ E_\lambda [X^T Q_{N-1} X] + u^T R_{N-1} u + E[J_N(T_{N-1,N}(\lambda, u, Y))] \right\},
\]
where the expectation in the third term in the minimization is over the random variable \( Y \). Define \( \tilde{X} := A_{N-1}X + B_{N-1}u + w_{N-1} \). Note that
\[
T_{N-1,N}(\lambda, u, y)(dx) = P(\tilde{X} \in dx \mid Y = y).
\]
Thus we have:
\[
J_N(T_{N-1,N}(\lambda, u, Y)) = \int x^T Q_N x P(\tilde{X} \in dx \mid Y),
\]
so we have:
\[
E[J_N(T_{N-1,N}(\lambda, u, Y))] = \int x^T Q_N x P(\tilde{X} \in dx) = E[\tilde{X}^T Q_N \tilde{X}].
\]
But
\[
E[\tilde{X}^T Q_{N-1} \tilde{X}] = E[w_{N-1}^T Q_N w_{N-1}] + E_{\lambda^0}[(A_{N-1}X)^T Q_N (A_{N-1}X)]
+ (A_{N-1}m + B_{N-1}u)^T Q_N (A_{N-1}m + B_{N-1}u),
\]
where \( m = E_{\lambda}[X] \) and \( \lambda^0 \) is the centered distribution corresponding to \( \lambda \). Also,
\[
E_{\lambda}[X^T Q_{N-1} X] = m^T Q_{N-1} m + E_{\lambda^0}[X^T Q_{N-1} X].
\]
Substituting these into the right hand side of the expression for \( J_{N-1}(\lambda) \) we get:
\[
J_{N-1}(\lambda) = m^T Q_{N-1} m \min_u \{(A_{N-1}m + B_{N-1}u)^T Q_N (A_{N-1}m + B_{N-1}u) + u^T R_{N-1} u\}
+ E_{\lambda^0}[X^T (Q_{N-1} + A_{N-1}^T Q_N A_{N-1}) X] + E[w_{N-1}^T Q_N w_{N-1}].
\]
We see that, as in the fully observed case, the minimum occurs when \( u = L_{N-1} m \) and also that
\[
J_{N-1}(\lambda) = m^T K_{N-1} m + E[w_{N-1}^T Q_N w_{N-1}] + E_{\lambda^0}[X^T Q_{N-1} X] + E_{\lambda^0}[(A_{N-1}X)^T Q_N (A_{N-1}X)]
= E_{\lambda}[X^T K_{N-1} X] + E[w_{N-1}^T Q_N w_{N-1}] + E_{\lambda^0}[X^T (Q_{N-1} + A_{N-1}^T Q_N A_{N-1} - K_{N-1}) X].
\]
But \( Q_{N-1} - K_{N-1} = \Gamma_{N-1} - A_{N-1}^T Q_N A_{N-1} \), so the third term is
\[
E_{\lambda^0}[X^T \Gamma_{N-1} X].
\]
We therefore have:
\[
J_{N-1}(\lambda) = E_{\lambda}[X^T K_{N-1} X] + E_{\lambda^0}[X^T \Gamma_{N-1} X] + E[w_{N-1}^T Q_N w_{N-1}].
\]
Note the analogy with the fully observed case, with the appearance of a new term, \( E_{\lambda^0}[X^T \Gamma_{N-1} X] \).