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Corrigendum

A correction and some additional remarks on: Stationary solutions of stochastic recursions describing discrete event systems

[Stochastic Process. Appl. 68 (1997) 181–194][1]

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1. Preliminaries

In this note, we report and correct an error in Theorem 1 of Anantharam and Konstantopoulos (1997). This theorem is not true without additional assumptions. In what follows, we propose a correction based on Theorem A stated and proved in the Appendix below. Theorem A may be of independent interest and is thus stated separately. Our blunder is in the sequence of displayed equations preceding the statement of Theorem 1 on p. 185 of Anantharam and Konstantopoulos (1997). The first equation there holds if $\Theta^{-1}C$ is a Q-continuity set, while the last equation holds if C is a Q-continuity set. Thus the equation there should be read as $Q = Q \circ \Theta^{-1}$ on the class of sets $\mathscr{C} := \{C \in \mathscr{F} \otimes \mathscr{E} : C$ and $\Theta^{-1}C$ are Q-continuity sets$\}$. This class may not, in general, generate $\mathscr{F} \otimes \mathscr{E}$; hence, the conclusion that $Q = Q \circ \Theta^{-1}$ on $\mathscr{F} \otimes \mathscr{E}$ (i.e., that Q is Θ-invariant) may be false.

Recall that Q was extracted as a weak subsequential limit of $\bar{Q}_n$. The calculation on p. 185 of Anantharam and Konstantopoulos (1997) shows that

$$|\bar{Q}_n \circ \Theta^{-1}(C) - \bar{Q}_n(C)| \leqslant 2/n \quad \text{for all } C \in \mathscr{F} \otimes \mathscr{E}, \quad n \geqslant 1. \tag{1}$$

hence $\bar{Q}_n \circ \Theta^{-1}$ has Q as a weak subsequential limit also. To actually conclude that $Q = Q \circ \Theta^{-1}$, we need that one of the following hypotheses be added to the statement of the first paragraph of Theorem 1 of Anantharam and Konstantopoulos (1997):
(A1) Θ is continuous.

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(A2) There is a subsequential weak limit Q of $\{\bar{Q}_n,\ n \geqslant 1\}$ under which the discontinuity set of Θ is Q-null.

(A3) There exist a sequence $\{\tilde{\Theta}_\ell,\ \ell \geqslant 1\}$ of continuous transformations of $\Omega \times E$, and a sequence of open sets $\{U_\ell,\ \ell \geqslant 1\}$ in $\Omega \times E$, such that $\tilde{\Theta}_\ell = \Theta$ outside U_ℓ, for all $\ell \geqslant 1$, and

$$\lim_{\ell \to \infty} \liminf_{n \to \infty} \bar{Q}_n(U_\ell) = 0. \tag{2}$$

Let us prove that, under *any* of the above three additional assumptions, Theorem 1 of Anantharam and Konstantopoulos (1997) is valid. First note that (A1) is a convenient special case of (A2). Under (A2), there is a subsequence $\bar{Q}_{n_k}$ of $\bar{Q}_n$ such that $\bar{Q}_{n_k} \Rightarrow Q$, as $k \to \infty$, and the set of discontinuities of Θ is Q-null. Hence, from Billingsley (1971), (Corollary 2, p. 9), we have $\bar{Q}_{n_k} \circ \Theta^{-1} \Rightarrow Q \circ \Theta^{-1}$. On the other hand, from Eq. (1), $\bar{Q}_{n_k} \circ \Theta^{-1} \Rightarrow Q$, and hence $Q = Q \circ \Theta^{-1}$. Finally, under (A3) and Eq. (1), the assumptions of Theorem A of the appendix are fulfilled (see also Remark 1 at the end of the proof of this theorem) and we again conclude that $Q = Q \circ \Theta^{-1}$.

Let us now show that the theorem, in its modified form, constructs a stationary solution for the examples we considered in Anantharam and Konstantopoulos (1997), which were representative of the type of applications we had in mind.

2. Examples

2.1. The G/G/1 queue

Here $E = \mathbb{R}_+$, and the problem is to construct a stationary solution of the E-valued recursion

$$W_{n+1} = (W_n + \xi_n)^+, \tag{3}$$

where $\{\xi_n,\ n \in \mathbb{Z}\}$ is a stationary ergodic sequence of real valued random variables, with $E[\xi_0] < 0$. We may take $\Omega = \mathbb{R}^Z$ with the product topology, $\mathscr{F}$ to be the corresponding Borel σ-field, θ to be the left shift on Ω, and P the probability measure on $(\Omega, \mathscr{F})$ under which the sequence of coordinates has the distribution of $\{\xi_n,\ n \in \mathbb{Z}\}$. We demonstrated in Anantharam and Konstantopoulos (1997) that with $Q_0 := P \otimes \delta_0$ and $Q_n := Q_0 \circ \Theta^{-n}$, the sequence $\{Q_n,\ n \geqslant 0\}$ is tight. Here one can check that condition (A1) applies, so (the corrected) Theorem 1 of Anantharam and Konstantopoulos (1997) proves the existence of a stationary weak solution.

2.2. The non-monotone recursion 3.2 of Anantharam and Konstantopoulos (1997)

Again $E = \mathbb{R}_+$, and the problem is to construct a stationary solution to the E-valued recursion

$$\tilde{W}_{n+1} = g(\tilde{W}_n + \xi_n), \tag{4}$$

where g is the function defined by

$$g(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leqslant x < 1 \\ 2(x-1), & 1 \leqslant x < 2 \\ x, & 2 \leqslant x. \end{cases}$$

and $\{\xi_n,\ n \in \mathbb{Z}\}$ is a stationary ergodic sequence of real valued random variables, with $E[\xi_0] < 0$. We may take $(\Omega, \mathscr{F}, P)$, and θ as in the preceding paragraph. We may take $Q_0 := P \otimes \delta_0$ and $Q_n := Q_0 \circ \Theta^{-n}$, where Θ now denotes the skew product appropriate to the recursion (4), viz.,

$$\Theta(\omega, x) := (\theta\omega, g(x + \xi_0(\omega))).$$

In Anantharam and Konstantopoulos (1997), we demonstrated the tightness of $\{Q_n,\ n \geqslant 0\}$, using only the observation that $g(x) \leqslant x^+$ and the tightness proof for the preceding problem. Let Q be a subsequential weak limit of $\{\bar{Q}_n,\ n \geqslant 1\}$, say $Q = \lim_{k\to\infty} \bar{Q}_{n_k}$.

If g were continuous, condition (A1) would apply. However, the function g is discontinuous. All the same, condition (A2) can be shown to apply under additional restrictions on the distribution on $\{\xi_n,\ n \in \mathbb{Z}\}$, for instance that ξ_0 has a distribution admitting a bounded density. But even condition (A2) is violated in general. However, we can show that condition (A3) always holds.

For $\ell \geqslant 1$, let $\delta_\ell := 2^{-\ell}$, choose $g_\ell : \mathbb{R} \to \mathbb{R}$ a continuous function that equals g outside the interval $(1 - \delta_\ell, 1)$, define the continuous map $\tilde{\Theta}_\ell : \Omega \times E \to \Omega \times E$ by

$$\tilde{\Theta}_\ell(\omega, x) := (\theta\omega, g_\ell(x + \xi_0(\omega))),$$

and consider the open set

$$U_\ell := \{(\omega, x):\ 1 - \delta_\ell < x + \xi_0(\omega) < 1\}.$$

With these definitions, it is immediate that $\Theta = \tilde{\Theta}_\ell$ outside U_ℓ. It remains to verify Eq. (2).

For $m \leqslant n$, $m, n \in \mathbb{Z}$, let $\tilde{W}_{m,n}$ be the result at time n of recursion (4) started with $\tilde{W}_m = 0$ and $W_{m,n}$ the result at time n of recursion (3) started with $W_m = 0$. From $g(x) \leqslant x^+$, we see that $\tilde{W}_{m,n} \leqslant W_{m,n}$ for all $m \leqslant n,\ m, n \in \mathbb{Z}$. The law of large numbers and the assumption $E[\xi_0] < 0$ imply that there is a P-a.s. finite $\tau \leqslant 0$ such that, for all $i \leqslant \tau$ we have $W_{i,\tau} = 0$, so that $W_{i,0} = W_{\tau,0}$ for all $i \leqslant \tau$. But then also $\tilde{W}_{i,\tau} = 0$ for all $i \leqslant \tau$, so that $\tilde{W}_{i,0} = \tilde{W}_{\tau,0}$ for all $i \leqslant \tau$. Thus, we have

$$\begin{aligned} \bar{Q}_n(U_\ell) &= \frac{1}{n}(Q_0 + \cdots + Q_{n-1})(U_\ell) = \frac{1}{n}\sum_{i=-n}^{-1} P(\tilde{W}_{i,0} + \xi_0 \in (1 - \delta_\ell, 1)) \\ &\to P(\tilde{W}_{\tau,0} + \xi_0 \in (1 - \delta_\ell, 1)), \quad \text{as } n \to \infty. \end{aligned}$$

The reason for the latter convergence is that $\lim_{i\to-\infty} P(\tilde{W}_{i,0} + \xi_0 \in (1 - \delta_\ell, 1)) = P(\tilde{W}_{\tau,0} + \xi_0 \in (1 - \delta_\ell, 1))$, which holds from the remarks above. From this, Eq. (2) follows, completing the proof of the existence of a weak stationary solution for recursion (4), given only that $\{\xi_n,\ n \in \mathbb{Z}\}$ is a stationary ergodic sequence of real valued random variables, with $E[\xi_0] < 0$.

2.3. The $G/G/1/0$ system

Here $E = \mathbb{R}_+$, and the problem is to construct a stationary solution to the E-valued recursion

$$W_{n+1} = (W_n + \sigma_n 1(W_n = 0) - \tau_n)^+, \tag{5}$$

where $\{(\sigma_n, \tau_n),\ n \in \mathbb{Z}\}$ is a stationary ergodic sequence of nonnegative real valued random pairs, with $E[\sigma_0] < \infty$ and $0 < E[\tau_0] < \infty$. We may take Ω to be $(\mathbb{R}_+^2)^Z$, $\mathscr{F}$ to be the corresponding Borel σ-field, θ to be the left shift on Ω, and P the probability distribution on $(\Omega, \mathscr{F})$ under which the sequence of coordinate pairs has the distribution of $\{(\sigma_n, \tau_n),\ n \in \mathbb{Z}\}$. We demonstrated in Anantharam and Konstantopoulos (1997) that with $Q_0 := P \otimes \delta_0$ and $Q_n := Q_0 \circ \Theta^{-n}$, the sequence $\{Q_n,\ n \geqslant 0\}$ is tight. We now show that condition (A3) holds.

For $\ell \geqslant 1$, let $\delta_\ell := 2^{-\ell}$, let $\tilde{\Theta}_\ell : \Omega \times E \to \Omega \times E$ be given by

$$\tilde{\Theta}_\ell(\omega, x) := (x + (1 - 2x\delta_\ell^{-1})^+ \sigma_0(\omega) - \tau_0(\omega))^+.$$

and let

$$U_\ell := \{(\omega, x):\ 0 < x < \delta_\ell\}.$$

With these definitions, it is immediate that $\tilde{\Theta}_\ell$ is continuous, U_ℓ is open, and $\Theta = \tilde{\Theta}_\ell$ outside U_ℓ. It remains to verify Eq. (2).

For $m \leqslant n$, $m, n \in \mathbb{Z}$ let $W_{m,n}$ be the result at time n of recursion (5) started with $W_m = 0$. Since $Q_i = Q_0 \circ \Theta^{-i} = (P \otimes \delta_0) \circ \Theta^{-i}$, we have $Q_i(U_\ell) = P(W_{0,i} \in (0, \delta_\ell))$, hence, using the θ-invariance of P,

$$\bar{Q}_n(U_\ell) = \frac{1}{n} \sum_{i=0}^{n-1} Q_i(U_l) = \frac{1}{n} \sum_{i=0}^{n-1} P(W_{0,i} \in (0, \delta_\ell)) = \frac{1}{n} \sum_{i=-n}^{-1} P(W_{i,0} \in (0, \delta_\ell)).$$

For $i \leqslant -1$, let $R_i(\omega) := \{j \in [i, -1]:\ \sigma_j(\omega) - A_j(\omega) > 0\}$, where $A_j(\omega) := \tau_j(\omega) + \cdots + \tau_{-1}(\omega)$. The interpretation is that $R_i(\omega)$ contains the indices of those customers that can be present in the queue at time 0, if the queue started empty with customer i. Note that $R(\omega) := \bigcup_{i \leqslant -1} R_i(\omega)$ has a finite number of elements for P-a.e. $\omega \in \Omega$. Indeed, from the law of large numbers, and the assumptions $E[\sigma_0] < \infty$, $0 < E[\tau_0] < \infty$, the sequence $\sigma_i - A_i$ converges, as $i \to -\infty$, to $-\infty$, P-a.s. Let $N(\omega) := \min R(\omega)$ be the minimum integer in $R(\omega)$. Clearly, $P(N > -\infty) = 1$. With these definitions we have, for $i \leqslant -1$,

$$P(W_{i,0} \in (0, \delta_\ell)) \leqslant P(N \leqslant i) + P\left(\min_{j \in R}(\sigma_j - A_j) \leqslant \delta_\ell\right).$$

As $i \to -\infty$, the first term converges to zero. Hence $\limsup_{n\to\infty} \bar{Q}_n(U_\ell) \leqslant P(\min_{j \in R} (\sigma_j - A_j) \leqslant \delta_\ell)$. Letting $\ell \to \infty$, we get zero since the random variables $\sigma_j - A_j$, as j ranges over the random set R, are, by definition, positive, a.s. Hence Eq. (2) holds, completing the proof of the existence of a weak stationary solution for the recursion (5).

2.4. Compact state space

For Theorem 2 of Anantharam and Konstantopoulos (1997) to be true, we also need conditions that give $Q = Q \circ \Theta^{-1}$. Any one of conditions (A1), (A2), or (A3) will suffice for this purpose.

To apply this theorem, one would typically be given a stationary ergodic process $\{\tilde{\varphi}_n,\ n \in \mathbb{Z}\}$ of measurable maps from $(E, \mathscr{E})$ into itself, and would like to construct an appropriate Polish sample space $(\Omega, \mathscr{F})$ supporting a measurable shift θ, a random variable φ_0 that takes values in the space of measurable maps from $(E, \mathscr{E})$ into itself, and a θ-invariant probability distribution P on $(\Omega, \mathscr{F})$ such that the sequence $\{\varphi_n,\ n \in \mathbb{Z}\}$ given by $\varphi_n(\omega) = \varphi_0(\theta^n \omega)$ has the same distribution as $\{\tilde{\varphi}_n,\ n \in \mathbb{Z}\}$, and then hope to apply the theorem to prove the existence of a weak solution on $\Omega \times E$. We now remark that if $\tilde{\varphi}_0$ takes values in $C(E, E)$, the space of continuous functions from E to E, and is measurable with respect to the Borel σ-field of the topology of uniform convergence in $C(E, E)$, then this can always be done. Indeed, for E compact, $C(E, E)$ is a Polish space in the topology of uniform convergence, see Appendix B, so we may take Ω to be $C(E, E)^Z$ with the product topology, $\mathscr{F}$ to be the Borel σ-field of Ω, θ to be the left shift on Ω and P to be the probability distribution under which the coordinate sequence has the distribution of $\{\tilde{\varphi}_n,\ n \in \mathbb{Z}\}$. The skew product Θ on $\Omega \times E$ is then easily seen to be continuous.

2.5. The need for the additional assumptions

Note that even a deterministic discontinuous recursion on a compact Polish space may not admit a weak stationary solution in our sense, without the additional assumptions introduced above. For example, take the unit interval $E = [0, 1]$ and the map $\varphi : E \mapsto E$ given by

$$\varphi(x) = \begin{cases} x/2 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

One can check that it is impossible to construct a weak stationary solution for this recursion (Meyn, 1997). It is also not hard to check that for any attempt to construct a stationary weak solution by the skew product construction, even though tightness of $\{\bar{Q}_n,\ n \geqslant 1\}$ is automatic, none of conditions (A1), (A2), or (A3) holds for any subsequential weak limit of this sequence.

Appendix A. A weak convergence criterion

Let $\{P_n,\ n \geqslant 1\}$ be a sequence of probability measures on a Polish space S, converging weakly to a probability measure P. Let h be a measurable mapping from S into a Polish space S'. We address the issue of weak convergence of $P_n \circ h^{-1}$ to $P \circ h^{-1}$. It seems that the most widely known sufficient condition for this type of result is that the set D_h of discontinuities of h have $P(D_h) = 0$; see e.g., Billingsley (1971), (Corollary 2, p. 9). However, roughly speaking, it is sufficient to assume only that the probability measures P_n converge to P in such a way as to "avoid the discontinuity". This is clarified below:

Theorem A. *Suppose that $P_n \Rightarrow P$ on a Polish space S, and let $h: S \to S'$ be a measurable mapping from S into a Polish space S'. Suppose that there exists a sequence of continuous mappings $h_\ell: S \to S'$, $\ell \geqslant 1$, and a sequence of open sets U_ℓ, $\ell \geqslant 1$, contained in S, such that, for all ℓ, $h_\ell = h$ outside U_ℓ, and such that*

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} P_n(U_\ell) = 0. \tag{6}$$

Then $P_n \circ h^{-1} \Rightarrow P \circ h^{-1}$.

Proof. We first observe that the assumptions imply that the sequence $\{P_n \circ h^{-1},\ n \geqslant 1\}$ is tight. To see this, given $\varepsilon > 0$, we first find a compact $K \subset S$ with

$$P_n(K) > 1 - \varepsilon \quad \text{for all } n, \tag{7}$$

which exists because the sequence of probability measures P_n converges weakly on S. Next, we find an integer $L > 0$ such that

$$\limsup_{n\to\infty} P_n(U_L) < \varepsilon. \tag{8}$$

This exists by Eq. (6). Now, $K \cap U_L^c$ is compact, because K is compact and U_L is open, and $P_n(K \cap U_L^c) > 1 - 2\varepsilon$ for all sufficiently large n, by Eqs. (7) and (8). Since $h = h_L$ off U_L, by assumption, we have $h(K \cap U_L^c) = h_L(K \cap U_L^c)$, and this is a compact subset of S', because h_L is continuous. Now,

$$P_n \circ h^{-1}(h(K \cap U_L^c)) \geqslant P_n(K \cap U_L^c) > 1 - 2\varepsilon,$$

for all sufficiently large n, so by taking the union of $K \cap U_L^c$ with a finite number of other compact sets, if necessary, to deal with the initial values of n, we have found, for each $\varepsilon > 0$, a compact subset K' of S' such that $P_n \circ h^{-1}(K') > 1 - 2\varepsilon$ for all n.

Let Q be a subsequential limit of $\{P_n \circ h^{-1}\}$, say $P_{n_j} \circ h^{-1} \Rightarrow Q$. By assumption, for any nonnegative continuous function f on S', bounded by K, say, we have, for each $\ell \geqslant 1$,

$$f \circ h \leqslant f \circ h_\ell + K 1_{U_\ell}.$$

Hence, we have

$$P_{n_j}(f \circ h) \leqslant P_{n_j}(f \circ h_\ell) + K P_{n_j}(U_\ell).$$

But $P_{n_j}(f \circ h) = (P_{n_j} \circ h^{-1})(f) \to Q(f)$, as $j \to \infty$. Since $P_n \Rightarrow P$, and $f \circ h_\ell$ is bounded and continuous for each l, we have $P_{n_j}(f \circ h_\ell) \to P(f \circ h_\ell)$. Hence

$$Q(f) \leqslant P(f \circ h_\ell) + K \liminf_{j\to\infty} P_{n_j}(U_\ell). \tag{9}$$

By assumption again, we have

$$f \circ h_\ell \leqslant f \circ h + K 1_{U_\ell}.$$

Integrating with respect to P, we obtain

$$P(f\circ h_\ell)\leqslant P(f\circ h)+KP(U_\ell). \tag{10}$$

Combining Eqs. (9) and (10) we have

$$Q(f)\leqslant P(f\circ h)+KP(U_\ell)+K\liminf_{j\to\infty}P_{n_j}(U_\ell). \tag{11}$$

Since $P_n\Rightarrow P$ and U_ℓ is open, $P(U_\ell)\leqslant\liminf_{j\to\infty}P_{n_j}(U_\ell)$. By Eq. (6), the ℓ-dependent terms of the right-hand side of Eq. (11) converge to zero as $\ell\to\infty$, yielding

$$Q(f)\leqslant P(f\circ h)=(P\circ h^{-1})(f).$$

Since this inequality is true for all nonnegative bounded continuous functions f, it follows that $Q=Ph^{-1}$. But Q is an arbitrary subsequential limit of $\{P_nh^{-1},\ n\geqslant 1\}$. Hence $P_nh^{-1}\Rightarrow Ph^{-1}$. □

Remark. (1) If we assume, to begin with, that $P_n\circ h^{-1}$ converges weakly to a probability measure Q on S', then Eq. (6) can be replaced by the weaker assumption $\lim_{\ell\to\infty}\liminf_{n\to\infty}P_n(U_\ell)=0$. This explains why the lim inf suffices in Eq. (2).

(2) A simple example shows that one cannot replace the lim sup in Eq. (6) by the lim inf, in general. Take $S=S'=[0,1]$,

$$h(x)=\begin{cases}0 & \text{if } 0\leqslant x<1/2,\\ 1 & \text{if } 1/2\leqslant x\leqslant 1.\end{cases}$$

and $P_n=a_n\delta_{1/2+1/n}+(1-a_n)\delta_{1/2-1/n}$, where $\{a_n,\ n\geqslant 1\}$ is some sequence of numbers in $[0,1]$. Then $P_n\Rightarrow\delta_{1/2}$, but $P_n\circ h^{-1}$ can have several distinct subsequential weak limits.

Appendix B. A topological result

Theorem B. *Let (X,d) be a compact metric space and $(Y,\tilde d)$ a separable metric space. Then, $C(X,Y)$, endowed with the topology of uniform convergence, is separable.*

Proof[4]**.** For $m,n\in\mathbb{N}$, let $\mathscr{E}_{mn}\subset C(X,Y)$ be defined by

$$\mathscr{E}_{mn}:=\left\{f\in C(X,Y)\colon\ d(x,x')\leqslant\frac{1}{m}\Rightarrow\tilde d(f(x),f(x'))\leqslant\frac{1}{n}\right\}. \tag{12}$$

Clearly $\bigcup_{m=1}^{\infty}\mathscr{E}_{mn}=C(X,Y)$, for all $n>0$. Let $\mathscr{Y}=\{y_1,y_2,\ldots\}$ be a dense subset of Y, and for each $m\in\mathbb{N}$, let $\{x_1^m,\ldots,x_{q_m}^m\}\subset X$ be the centers of a collection of open balls of radius $1/m$ which cover X.

For each $v=(v_1,\ldots,v_{q_m})\in\mathbb{N}^{q_m}$, and $n\in\mathbb{N}$, let

$$\mathscr{F}_{mn}^{v}:=\left\{f\in\mathscr{E}_{mn}\colon\ \tilde d(f(x_k^m),y_{v_k})\leqslant\frac{1}{n},\ 1\leqslant k\leqslant q_m\right\}. \tag{13}$$

[4] The proof in this appendix is due to Arapostathis (1997), reproduced by permission.

Note that, by construction,

$$\mathscr{E}_{mn} = \bigcup_{v \in \mathbb{N}^{q_m}} \mathscr{F}_{mn}^{v}. \tag{14}$$

Let

$$\mathscr{V}_{mn} := \{v \in \mathbb{N}^{q_m}\colon\ \mathscr{F}_{mn}^{v} \neq \emptyset\}, \tag{15}$$

and form a collection $\mathscr{G}_{mn} \subset \mathscr{E}_{mn}$ by picking an arbitrary function φ_v from each member of the class $\{\mathscr{F}_{mn}^{v}\colon\ v \in \mathscr{V}_{mn}\}$.

We claim that

$$\mathscr{G} := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathscr{G}_{mn}$$

is dense in $C(X, Y)$.

Suppose $\varepsilon > 0$ is given. Select $n \in \mathbb{N}$ such that $4/n < \varepsilon$. If $f \in C(X, Y)$, then $f \in \mathscr{E}_{mn}$, for some $m \in \mathbb{N}$, and by Eqs. (14) and (15), there exists a $v \in \mathscr{V}_{mn}$ such that $f \in \mathscr{F}_{mn}^{v}$, and also some $\varphi_v \in \mathscr{F}_{mn}^{v}$. Thus, by Eq. (13), we obtain,

$$\tilde{d}(f(x_k^m), y_{v_k}) \leqslant \frac{1}{n} \quad \text{and} \quad \tilde{d}(\varphi_v(x_k^m), y_{v_k}) \leqslant \frac{1}{n} \quad \text{for all } k \in \{1, \ldots, q_m\}. \tag{16}$$

On the other hand, for each $x \in X$, we can select $k \in \{1, \ldots, q_m\}$ such that $d(x, x_k^m) < 1/m$, and hence, since $\varphi_v,\ f \in \mathscr{E}_{mn}$, Eq. (12) yields

$$\tilde{d}(f(x), f(x_k^m)) \leqslant \frac{1}{n}, \quad \tilde{d}(\varphi_v(x), \varphi_v(x_k^m)) \leqslant \frac{1}{n}. \tag{17}$$

Finally, combining Eqs. (16) and (17), $\tilde{d}(f(x), \varphi_v(x)) \leqslant 4/n < \varepsilon$. $\square$

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