Sum Capacity of DS-CDMA with Colored Noise

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Abstract — **We consider a Direct Sequence Code Division Multiple Access (DS-CDMA) channel in colored additive Gaussian noise and focus on the sum capacity of this channel. Sum Capacity is the maximum sum of rates at which users can jointly reliably transmit, in an information theoretic sense. We completely characterize optimum sum capacity, which is obtained by choosing the signature sequences of the users appropriately. Our characterization is constructive in that we provide a combinatorial algorithm to generate the optimum signature sequences as a function of the covariance of the additive background noise and power constraints of the users. The characterization also allows us to identify a saddle property of the optimum sum capacity: convexity in the covariance matrix of the additive noise and concavity in the vector of user power constraints.**

I. Introduction and Problem Statement

A discrete time baseband no fading DS-CDMA channel (with short signature sequences) is the following:

$$\mathbf{y}(n) = \sum_{i=1}^{K} x_i(n)\mathbf{s}_i(n) + \mathbf{w}(n) \ .$$

Here K denotes the number of users and n the channel use instant. The user symbols are denoted by x_i and $\mathbf{y}(n)$ is the signal (thought of as a N dimensional vector, N being the processing gain or number of chips per symbol) at the receiver at time instant n. Here $\mathbf{w}(n)$ is an additive Gaussian noise with covariance matrix Σ. Each user i is subject to a time averaged power constraint of p_i. We denote D to be the diagonal matrix of the user power constraints.

Our focus will be on sum capacity: sum of rates at which users jointly reliably communicate. These rates are time averaged with the power constraint on the users also averaged in time. A generalization of the results in [2] to the colored noise case allows us to write the following expression for sum capacity of the DS-CDMA channel with signature sequences $S \stackrel{\text{def}}{=} [\mathbf{s}_1 \ldots \mathbf{s}_K]$.

$$C_{sum}(S, D, \Sigma) = \frac{1}{2} \log \det \left(I + \Sigma^{-1} SDS^t \right) \ .$$

Our main focus in this paper is to characterize the maximum sum capacity:

$$C_{opt}\left(D, \Sigma\right) \stackrel{\text{def}}{=} \max_{S \in \mathcal{S}} C_{sum}\left(S, D, \Sigma\right)$$

where $\mathcal{S}$ is the set of all $N \times K$ real matrices with all columns having l_2 norm equal to 1. Observe that C_sum is a continuous function defined on a compact set $\mathcal{S}$ and thus the use of max in above is justified.

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II. Main Results

Our main result is a complete characterization of C_{opt} as a function of D and Σ. This characterization is constructive in the sense that we develop a *combinatorial* algorithm to generate the optimum signature sequences (these achieve the maximum sum capacity). The details of this result are available in [3]. In this summary, we briefly describe a qualitative property of the optimum sum capacity that emerges out of our characterization. Our first result is a saddle property of the optimum sum capacity:

Theorem 1 *For every fixed Σ, $C_{opt}(D, \Sigma)$ is a concave function in D and a convex function in Σ for every fixed D.*

We can strengthen this result using the partial order of Schur majorization on vectors in $\mathbb{R}^N$. We say that a vector $\mathbf{a}$ majorizes another vector $\mathbf{b}$ if their components have the same sum and the components of $\mathbf{a}$ are "more spread out" than those of $\mathbf{b}$. For example, every vector in $\mathbb{R}^N$ with sum N *majorizes* the vector with all components unity. An exhaustive resource for results on this partial order is [1]. We show that the optimum sum capacity is a Schur-saddle function in the following sense. Below we have denoted the vector of eigenvalues of Σ by $(\sigma_1^2, \ldots, \sigma_N^2)$.

Theorem 2 *1. For every fixed D, $C_{opt}(D, \Sigma) > C_{opt}(D, \tilde{\Sigma})$ for every $\Sigma \neq \tilde{\Sigma}$ such that $(\sigma_1^2, \ldots, \sigma_N^2)$ majorizes $(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_N^2)$.*

2. For every fixed Σ and for every $D \neq \tilde{D}$ such that $(p_1, \ldots, p_K)$ majorizes $(\tilde{p}_1, \ldots, \tilde{p}_K)$ we have $C_{opt}(\tilde{D}, \Sigma) > C_{opt}(D, \Sigma)$.

References

[1] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its applications*, Academic Press, 1979.

[2] P. Viswanath and V. Anantharam, "Optimal Sequences and Sum Capacity of Synchronous CDMA Systems", *IEEE Transactions on Information Theory*, vol. 45(6), Sept. 1999, pp. 1984-1991.

[3] P. Viswanath and V. Anantharam, "Total Capacity of Vector Multiple Access Channels", UCB/ERL Memorandum. M99/47.