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Stationary solutions of stochastic recursions describing discrete event systems

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Abstract

We consider recursions of the form $x_{n+1} = \varphi_n[x_n]$, where $\{\varphi_n, n \geq 0\}$ is a stationary ergodic sequence of maps from a Polish space $(E, \mathscr{E})$ into itself, and $\{x_n, n \geq 0\}$ are random variables taking values in $(E, \mathscr{E})$. Questions of existence and uniqueness of stationary solutions are of considerable interest in discrete event system applications.

Currently available techniques use simplifying assumptions on the statistics of $\{\varphi_n\}_n$ (such as Markov assumptions), or on the nature of these maps (such as monotonicity).

We introduce a new technique, without such simplifying assumptions, by weakening the solution concept: instead of a pathwise solution, we construct a probability measure on another sample space and families of random variables on this space whose law gives a stationary solution. The existence of a stationary solution is then translated into tightness of a sequence of probability distributions. Uniqueness questions can be addressed using techniques familiar from the ergodic theory of positive Markov operators

Keywords: Stochastic recursions; Ergodic theory; Queueing processes

1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space. We assume that $(\Omega, \mathscr{F})$ is a Polish space (i.e. Ω is a complete separable metric space and $\mathscr{F}$ is a sub-σ-algebra of the Borel σ-algebra of Ω) – see, for instance, Parthasarathy (1967) for background on Polish spaces. Let θ be a measurable map from $(\Omega, \mathscr{F})$ into itself such that P is θ-invariant: $P(\theta^{-1}(A)) = P(A)$ for all $A \in \mathscr{F}$. We assume that P is θ-ergodic: if $A \in \mathscr{F}$ is such that

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$\theta^{-1}(A) = A$ up to sets of P-probability 0, then A has P-probability 0 or 1. Our reference for basic ergodic theoretic concepts will be Krengel (1985).

Let $(E, \mathscr{E})$ be a Polish space and φ_0 a random variable defined on $(\Omega, \mathscr{F}, P)$ that takes values in the space of measurable maps from $(E, \mathscr{E})$ into itself. Let $\varphi_n(\omega) = \varphi_0(\theta^n \omega)$. Then, under P, $\{\varphi_n, n \geq 0\}$ is a stationary ergodic sequence of random maps from $(E, \mathscr{E})$ into itself.

We are interested in solutions of recursions of the form

$$x_{n+1} = \varphi_n[x_n]. \tag{1}$$

Here $\{x_n, n \geq 0\}$ are random variables taking values in $(E, \mathscr{E})$ and to avoid confusion we have used square brackets to write $\varphi_n[x_n]$ for the evaluation of the map $\varphi_n(\omega)$ at $x_n(\omega) \in E$. Given an initial condition, say x_0 at time 0, there is, of course, no problem in constructing a pathwise solution to (1) for $n \geq 0$. In view of the stationarity and ergodicity of $\{\varphi_n\}_n$, one expects that, if some stability condition holds, the solutions constructed in this manner converge in some suitable sense to a stationary solution. By a stationary solution of the above recursion we mean a stationary process $\{x_n, n \geq 0\}$ that satisfies the recursion.

The question of when such stationary solutions exist, whether they are unique, and whether there is convergence to a stationary solution starting from arbitrary initial conditions, is of considerable interest in discrete event system applications. This is because discrete event systems subject to random influences can often be described by recursions of the form (1). At the present time it is often the case in recursions of practical interest that stability conditions can be derived – and answers to the existence, uniqueness, and convergence questions can be given – only if one is willing to make strong simplifying assumptions about the statistics of $\{\varphi_n\}_n$. This situation is far from ideal, since such simplifying assumptions are typically not met in practice. Thus, there is an important role for the study of recursions of the form (1) in the general stationary ergodic context.

The canonical example of such a recursion is

$$x_{n+1} = \varphi_n(x_n, \xi_n), \tag{2}$$

where $\{\xi_n, n \geq 0\}$ is a stationary and ergodic sequence of random variables and φ is a deterministic function. If one assumes that φ is continuous in both variables and increasing in the first one, an idea due to Loynes (1962) can be used in certain cases to identify stability conditions and to explicitly construct stationary solutions. This technique has been developed to deal with a number of applications; see Baccelli and Brémaud (1993), Bambos and Walrand (1990), Konstantopoulos and Baccelli (1991), and Walrand (1988), for several examples. In many of these examples uniqueness conditions and convergence theorems can also be derived. However, the assumed monotonicity plays a key role in this development.

Our purpose in this paper is to discuss a new technique for constructing stationary solutions to (1) that does not depend on the recursion being of the type (2) with monotonicity conditions. To achieve this we have to weaken the solution concept, along lines that are familiar from the theory of stochastic differential equations; see, for instance, Stroock and Varadhan (1979). That is, rather than constructing a pathwise solution to (1), we construct a probability measure on another sample space and families of random variables on this space whose law gives a stationary solution to (1).

Our approach is based on the following construction: consider the product space $\Omega \times E$, endowed with the product σ-field $\mathscr{F} \otimes \mathscr{E}$ and the new measurable shift $\Theta : \Omega \times E \to \Omega \times E$ defined by

$$\Theta(\omega, x) = (\theta\omega, \varphi_0(\omega)[x]). \tag{3}$$

Note the following composition rule.

$$\Theta^n(\omega, x) = (\theta^n\omega, \varphi_0(\theta^{n-1}\omega)\varphi_0(\theta^{n-2}\omega) \dots \varphi_0(\omega)[x])$$

$$= (\theta^n\omega, \varphi_{n-1}(\omega)\varphi_{n-2}(\omega) \dots \varphi_0(\omega)[x]).$$

The problem of existence of a stationary solution is then translated into the problem of existence of a probability measure Q on $(\Omega \times E, \mathscr{F} \otimes \mathscr{E})$ that is invariant under Θ, and whose Ω marginal is P. Indeed, suppose it was possible to construct such a probability measure Q. Consider the new random variables

$$X_0(\omega, x) = x, \qquad \Phi_0(\omega, x) = \varphi_0(\omega),$$

defined on $\Omega \times E$ and let

$$X_n(\omega, x) = X_0(\Theta^n(\omega, x)), \qquad \Phi_n(\omega, x) = \Phi_0(\Theta^n(\omega, x))$$

for $n \geqslant 0$. X_0 takes values in $(E, \mathscr{E})$ and Φ_0 takes values in the space of measurable mappings from $(E, \mathscr{E})$ into itself. A simple calculation shows

$$X_n(\omega, x) = \varphi_{n-1}(\omega)\varphi_{n-2}(\omega) \dots \varphi_0(\omega)[x], \tag{4}$$

$$\Phi_n(\omega, x) = \varphi_n(\omega).$$

Thus, $\{X_n, n \geqslant 0\}$ solves the recursion

$$X_{n+1} = \Phi_n[X_n].$$

Since Q is Θ-invariant, $\{X_n, \Phi_n\}_n$ is stationary under Q. Since Q has Ω marginal P, $\{\Phi_n\}_n$ has the same distribution as $\{\varphi_n\}_n$. We refer to such a measure Q as a *weak stationary solution* for the recursion (1).

We note that a point of view similar to the one presented here has been taken by Crauel (1987) and Arnold and Crauel (1991): recursions of the form (1) are called random dynamical systems. Analogous systems are also defined in continuous time. In this work, E has the structure of a smooth manifold and $\varphi_0(\omega)$ is assumed to be a diffeomorphism on E. One then studies the random dynamical system in connection with the linearized diffeomorphism on the tangent bundle and its associated Lyapunov exponents. However, in discrete event systems and stochastic networks applications, such a smooth structure is unrealistic. The map $\varphi_0(\omega)$ cannot be assumed to be smooth or invertible. Stationary queueing systems have also been considered by Brandt et al. (1990): a concept of "stationary weak solution" is introduced and several examples, where the solution is constructed on an "enlarged probability space" (because one does not exist on the original probability space) are treated in detail. What is made clear in our formulation is that it suffices to consider the product $\Omega \times E$ as this enlarged probability space. Also, one should note that a technique for the construction of stationary solutions, based on the concept of "renovating events", has been developed by Borovkov (see, for instance, Borovkov (1988) for applications to communication networks and Foss (1986)).

In Section 2 we describe a technique to construct a probability measure Q on $\Omega \times E$ of the desired type, when a certain tightness condition is satisfied. We also give a simple way of checking this criterion without reference to the construction of the product space $\Omega \times E$. One should note that such a tightness criterion for stationarity has been a point of discussion in the applied probability community for quite some time now. Herein we just present a concrete proposal on how tightness implies stationarity. In Section 3 we illustrate our approach by reinterpreting some well-known existence results in simple queueing systems. We also show how our technique can be used to prove existence of solutions in non-monotonic recursions of type (2), which cannot be handled by previously available techniques. Of particular interest is the fact that a stationary recursion on a compact state space always admits a stationary solution in our sense. In Section 4 we discuss the question of uniqueness of stationary solutions. Here we give a theorem on uniqueness along lines familiar from the ergodic theory of positive Markov operators on spaces of continuous functions – see, for instance, Krengel (1985, Ch. 5). In Section 5 we demonstrate how the uniqueness theorem can be used to establish uniqueness of solutions in some of the examples of Section 3. Some concluding remarks are made in Section 6.

2. Existence of stationary solutions

Suppose $\{x_n, n \geq 0\}$ is a stationary solution to (1). In other words, this is a stationary sequence on the original probability space $(\Omega, \mathscr{F}, P)$ and can thus be termed a strong solution to (1). Consider the map $e: \Omega \to \Omega \times E$ given by

$$e(\omega) = (\omega, x_0(\omega)).$$

Let $Q = P \circ e^{-1}$. We note that Q has Ω marginal P, i.e., $Q(A \times E) = P(A)$ for all $A \in \mathscr{F}$. We also note that $e(\theta\omega) = \Theta(e(\omega))$. Indeed, from the definition of Θ in (3), $\Theta(e(\omega)) = (\theta\omega, \varphi_0(\omega)[x_0(\omega)])$; but $\varphi_0(\omega)[x_0(\omega)] = x_1(\omega)$, because of the recursion (1); in addition, from the stationarity assumption for $\{x_n, n \geq 0\}$, we have $x_1(\omega) = x_0(\theta\omega)$. So, $\Theta(e(\omega)) = (\theta\omega, x_0(\theta\omega)) = e(\theta\omega)$. We thus demonstrated that

$$Q \circ \Theta^{-1} = P \circ e^{-1} \circ \Theta^{-1} = P \circ \theta^{-1} \circ e^{-1} = P \circ e^{-1} = Q.$$

Hence, Q is invariant under Θ. In particular, the (constant) sequence $\{Q \circ \Theta^{-n}, n \geq 0\}$ is *tight* (see below for the definition of tightness, and Billingsley (1968), for more details).

On the other hand, let us start with an arbitrary probability Q_0 on $\Omega \times E$ whose Ω marginal is P. Let Q_n denote the probability distribution $Q_0 \circ \Theta^{-n}$ on $\Omega \times E$. A simple calculation shows $Q_n(A \times E) = P(\theta^{-n}A)$. By stationarity of P, $P(\theta^{-n}A) = P(A)$, so Q_n also has Ω marginal P for all $n \geqslant 0$.

Suppose now that the sequence $\{Q_n, n \geqslant 0\}$ is *tight*, i.e. that for any $\varepsilon > 0$ there is a compact set $K \subseteq \Omega \times E$ such that $Q_n(K) > 1 - \varepsilon$ for all $n \geqslant 0$. This is enough to demonstrate the existence of a Θ-invariant probability distribution Q on $\Omega \times E$ having Ω marginal P. To see this, let

$$\bar{Q}_n = \frac{1}{n}(Q_0 + \cdots + Q_{n-1}), \quad n \geqslant 1.$$

Then $\{\bar{Q}_n, n \geqslant 1\}$ is also tight, and each $\bar{Q}_n$ has Ω marginal P. Let Q be any subsequential limit of this sequence, say $Q = \lim_{k\to\infty} \bar{Q}_{n_k}$. (The existence of such a limit is guaranteed by Prohorov's theorem – see Billingsley (1968)). Clearly, Q has Ω marginal P. Also, for any measurable subset $C \subseteq \Omega \times E$,

$$\begin{aligned} Q(\Theta^{-1}(C)) &= \lim_{k\to\infty} \bar{Q}_{n_k}(\Theta^{-1}(C)) \\ &= \lim_{k\to\infty} \bar{Q}_{n_k}(C) - \lim_{k\to\infty} \frac{1}{n_k} Q_0(C) + \lim_{k\to\infty} \frac{1}{n_k} Q_{n_k}(C) \\ &= Q(C), \end{aligned}$$

so Q is Θ-invariant.

We have thus proved the following theorem.

Theorem 1. *Let Q_0 be a probability distribution on $\Omega \times E$ whose Ω marginal is P. Let Q_n denote the probability distribution $Q_0 \circ \Theta^{-n}$ on $\Omega \times E$. Suppose that the sequence $\{Q_n, n \geqslant 0\}$ is tight. Then there is a stationary sequence $\{X_n, \Phi_n\}_n$ defined on $\Omega \times E$ with $\{X_n\}_n$ taking values in $(E, \mathscr{E})$ and $\{\Phi_n\}_n$ taking values in the space of measurable maps from $(E, \mathscr{E})$ into itself, such that $\{\Phi_n\}_n$ has the same distribution as $\{\varphi_n\}_n$ and $\{X_n\}_n$ obeys $X_{n+1} = \Phi_n[X_n]$, $n \geqslant 0$.*

Conversely, if the stochastic recursion $x_{n+1} = \varphi_n[x_n]$, $n \geqslant 0$, admits a stationary solution, there is a probability distribution Q on $\Omega \times E$ whose Ω marginal is P, and such that the sequence $\{Q \circ \Theta^{-n}, n \geqslant 0\}$ is constant (and hence tight).

At first sight it may appear that proving tightness of $\{Q_n, n \geqslant 0\}$ would be difficult. However, the following simple result shows that the question of the existence of a stationary regime in our sense can often be settled without reference to the product construction. The value of this result will become clear in the examples discussed later in the paper. Recall that a sequence of random elements is said to be tight iff the sequence of their distributions is tight – see Billingsley (1968). We now have the following:

Lemma 1. *Suppose that, for some $x \in E$, the sequence $\{\varphi_{n-1} \cdots \varphi_0[x], n \geqslant 1\}$ of random elements defined on $(\Omega, \mathscr{F}, P)$ is tight. Let Q_0^x denote the probability distribution $P \otimes \delta_x$ on $\Omega \times E$. Then the sequence $\{Q_n^x, n \geqslant 0\}$ is tight.*

Proof. We first note that for any sequence $\{Q_n, n \geqslant 0\}$ having Ω marginal P, tightness is equivalent to tightness of the E marginals. Thus, it suffices to show that the E marginals of $\{Q_n^x, n \geqslant 0\}$ are tight. But from (4), this is precisely the assumption of the lemma. □

3. Applications of the existence theorem

In this section we consider several examples of stochastic recursions arising in discrete event system applications to illustrate the scope of our idea.

3.1. The G/G/1 queue

A classical example of a stochastic recursion is the Lindley equation describing the workload seen by arriving customers into a G/G/1 queue. On a sample space $(\Omega, \mathscr{F}, P)$ admitting a shift θ under which P is ergodic, we are given nonnegative random variables (σ_0, τ_0) satisfying $E[\sigma_0] = \mu^{-1} < \infty$ and $E[\tau_0] = \lambda^{-1} < \infty$. Let $(\sigma_n, \tau_n) = (\sigma_0 \circ \theta^n, \tau_0 \circ \theta^n)$. Thus, $\{\sigma_n, \tau_n\}_n$ is a stationary ergodic sequence. The interpretation of σ_n is the work brought in by the nth customer to a server and τ_n denotes the interarrival time between the arrival of the nth customer and the $n+1$st customer. The server works at rate 1 if there is work in the system. Let W_n denote the workload in the system seen by the nth customer. Then, starting from some initial condition, the workload evolves according to the equation

$$W_{n+1} = (W_n + \sigma_n - \tau_n)^+. \tag{5}$$

This recursion is of the form (2). It is also monotone in the state variable, and continuous in the state variable and the value of the driving process. Thus, the existence of stationary solutions can be studied pathwise using the idea of Loynes (1962). For details, see Baccelli and Brémaud (1993), or Walrand (1988). The pathwise construction works under the assumption that the shift has a measurable inverse, and leads to the conclusion that there is a unique stationary solution to the recursion if $\lambda < \mu$. The pathwise construction of the stationary solution involves constructing variables $\{W_{m,n}, \ -\infty < m \leqslant n < +\infty\}$, with the initial condition $W_{m,m} = 0$. The monotonicity in the state variable gives that $W_{m,0}$ increases when m decreases, and the limit is the desired stationary solution. If $\lambda < \mu$, this can be shown to be proper.

On the other hand, our approach yields the existence of a stationary solution for $\lambda < \mu$ along slightly different lines. To begin with, we assume that $(\Omega, \mathscr{F})$ is a Polish space. Let $\xi_0 = \sigma_0 - \tau_0$, and $\xi_n = \xi_0 \circ \theta^n = \sigma_n - \tau_n$. Let also $\varphi_0(\omega)[x] = (x + \xi_0(\omega))^+$ and $W_{0,n}(\omega) = \varphi_{n-1}(\omega) \cdots \varphi_0(\omega)[0]$. Referring to Lemma 1 and the notation in the preceding paragraph, the existence of a stationary solution for $\lambda < \mu$ – which in this example means the existence of a Θ-invariant probability distribution on $\Omega \times \mathbb{R}_+$ having Ω-marginal P – follows if we can show that the sequence $\{W_{0,n}, n \geqslant 0\}$ is *tight*.

From (5), we write

$$W_{0,n}(\omega) = \max(0, \xi_{n-1}(\omega), \xi_{n-1}(\omega) + \xi_{n-2}(\omega), \dots, \xi_{n-1}(\omega) + \xi_{n-2}(\omega) + \cdots + \xi_0(\omega)).$$

Since $\xi_n = \xi_0 \circ \theta^n$, and due to the P-ergodicity of θ, the sequence $\{\xi_n, n \geqslant 0\}$ is stationary and ergodic. Now, $\{\xi_n, n \geqslant 0\}$ can be extended to $\{\xi_n, n \in \mathbb{Z}\}$ on an appropriate sample space. Consider the stationary process $\{\tilde{\xi}_n, n \in \mathbb{Z}\}$ defined by $\tilde{\xi}_n = \xi_{-n}$. Then we have

$$\begin{aligned} W_{0,n} &\stackrel{\mathrm{d}}{=} \max(0, \tilde{\xi}_0, \dots, \tilde{\xi}_0 + \cdots + \tilde{\xi}_{n-1}) \\ &\leqslant_{\mathrm{st}} \max\left(0, \max_{k \geqslant 0} \sum_{l=0}^{k} \tilde{\xi}_l\right) \\ &= W^* \quad \text{say,} \end{aligned}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution, and $\leqslant_{\mathrm{st}}$ denotes stochastic ordering – for more on these concepts see, for example, Baccelli and Brémaud (1993). If $\lambda < \mu$, we have $E\tilde{\xi}_0 < 0$, from which it follows easily by Birkhoff's ergodic theorem that W^* is a proper random variable. Hence $\{W_{0,n}\}_n$ is tight.

At first sight it may not appear that this proof is significantly different from that of the Loynes construction. However, an essential point is that monotonicity of (5) has not been explicitly used anywhere in the argument. This point is clarified further in the example of Section 3.2 below, where our technique is used to handle a kind of non-monotone perturbation of Lindley's equation which the Loynes construction cannot handle.

3.2. *A non-monotone recursion*

In this section we consider a recursion which can be thought of as a non-monotone perturbation of the Lindley equation (5). The purpose is to illustrate the ease with which our technique yields the existence of stationary solutions for this recursion, albeit in a weak sense. Loynes' technique cannot be employed (at least, not in a straightforward way) to establish the existence of a stationary solution to this recursion using pathwise techniques.

Consider the function $g: \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leqslant x < 1, \\ 2(x-1) & \text{if } 1 \leqslant x < 2, \\ x & \text{if } 2 \leqslant x. \end{cases}$$

See Fig. 1 for a graph of g.

Let $(\Omega, \mathscr{F}, P)$ be a probability space supporting a real valued random variable ξ_0 with $E\xi_0 < 0$, and a measurable shift θ such that P is θ-invariant and ergodic. Let

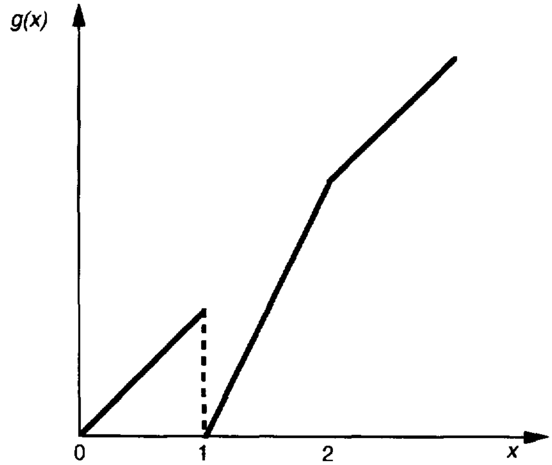

Fig. 1. Graph of the function g.

$\xi_n = \xi_0 \circ \theta^n$, $n \geqslant 0$. We consider the recursion

$$\tilde{W}_{n+1} = g(\tilde{W}_n + \xi_n) \tag{6}$$

and seek a stationary solution to this recursion. Recursion (6) can be thought of as a perturbation of (5). Note that the recursion (6) is not monotone in the state variable.

To address the question of existence of a stationary solution for (6), we assume that $(\Omega, \mathscr{F})$ is a Polish space. The existence of a stationary solution for (6) – in the sense of the existence of a probability distribution Q on $\Omega \times \mathbb{R}_+$ that is Θ-invariant and has Ω marginal P – will follow if we can show that the sequence $\{\tilde{W}_{0,n}, n \geqslant 0\}$ is *tight*, where $\tilde{W}_{0,n}$ denotes the solution of (6) started with $\tilde{W}_0 = \tilde{W}_{0,0} = 0$.

To this end, we compare the sequence $\{\tilde{W}_{0,n}, n \geqslant 0\}$ to the sequence $\{W_{0,n}, n \geqslant 0\}$ considered in Section 3.1. We claim that $\tilde{W}_{0,n} \leqslant W_{0,n}$ for all $n \geqslant 0$. This can be shown by induction on n. It is clearly true for $n = 0$. Further, we have

$$\begin{aligned} W_{0,n+1} &= (W_{0,n} + \xi_n)^+ \\ &\geqslant (\tilde{W}_{0,n} + \xi_n)^+ \quad \text{by inductive hypothesis} \\ &\geqslant g(\tilde{W}_{0,n} + \xi_n) \quad \text{from the definition of } g \\ &= \tilde{W}_{n+1,0}, \end{aligned}$$

which completes the inductive step. Thus, the already established tightness $\{W_{0,n}, n \geqslant 0\}$ implies the tightness of $\{\tilde{W}_{0,n}, n \geqslant 0\}$, which implies the existence of a stationary solution to (6).

Remark. It should be clear that a similar technique could be used to show the existence of a stationary solution for a recursion that is dominated in a suitable sense by another recursion for which the existence of a stationary solution is known. We will not attempt to formalize this statement.

3.3. The G/G/1/0 system

Consider a single server systems with no buffer space to hold waiting customers. This system is driven by a stationary ergodic sequence $\{\sigma_n, \tau_n\}_n$, where σ_n represents the work brought in by the nth customer and τ_n denotes the interarrival time between the arrival of the nth customer and the $n+1$st customer. We assume $E[\sigma_0] < \infty$ and $0 < E[\tau_0] < \infty$. Let W_n denote the workload in the system seen by the nth customer. This customer is admitted to the system iff $W_n = 0$, else he is rejected once and for all. The server works at rate 1 if there is work in the system.

The recursion describing the system above is

$$W_{n+1} = (W_n + \sigma_n 1(W_n = 0) - \tau_n)^+. \tag{7}$$

If this recursion is expressed in the form (2) it is seen to lack monotonicity in the state variable. Thus, Loynes' construction in its usual form cannot be used to analyze this system. This system has been analyzed in detail in Ch. 2 of Baccelli and Brémaud (1993), where examples are shown where (7) (i) does not admit a pathwise stationary solution, (ii) admits more than one stationary solution, and (iii) admits a unique stationary solution. It is pointed out that the existence of a stationary solution can be guaranteed in general if one is willing to augment the underlying sample space, and a general construction of stationary solution on an augmented sample space is described.

In this section we will show that the existence of a weak stationary solution is an immediate and simple consequence of our construction. (In particular, the augmented probability space can always be taken to be $\Omega \times E$). To this end, we assume the system is described on a sample space $(\Omega, \mathscr{F}, P)$ supporting σ_0 and τ_0 and a measurable shift θ such that P is θ-invariant and ergodic and such that $\{\sigma_n, \tau_n\}_n$ has the relevant statistics, where $(\sigma_n, \tau_n) = (\sigma_0, \tau_0) \circ \theta^n$, $n \geqslant 0$. We also assume that $(\Omega, \mathscr{F})$ is a Polish space. Let $W_{0,n}$ denote the solution of (7) starting with the initial condition $W_0 = W_{0,0} = 0$. It is enough to show the tightness of $\{W_{0,n}, n \geqslant 0\}$. This is a simple consequence of the following observation.

Lemma 2. *Let $\{(\sigma_n, \tau_n), n \geqslant 0\}$ be stationary and ergodic, with $E[\sigma_0] < \infty$ and $0 < E[\tau_0] = \lambda^{-1} < \infty$. Then $\max_{k \geqslant 0}(\sigma_k - (\tau_0 + \tau_1 + \cdots + \tau_k))$ is a.s. finite.*

Proof. Let $T_k = \tau_0 + \tau_1 + \cdots + \tau_k$. It suffices to show that

$$P\{\sigma_k > T_k \text{ i.o.}\} = 0,$$

where i.o. stands for "infinitely often". Observe that

$$\begin{aligned} P\{\sigma_k > T_k \text{ i.o.}\} &\leqslant P\{(\sigma_k \geqslant k\lambda^{-1}/2 \text{ i.o.}) \text{ or } (T_k \leqslant k\lambda^{-1}/2 \text{ i.o.})\} \\ &\leqslant P\{\sigma_k \geqslant k\lambda^{-1}/2 \text{ i.o.}\} + P\{T_k \leqslant k\lambda^{-1}/2 \text{ i.o.}\}. \end{aligned} \tag{8}$$

But $\sum_{k\geqslant 0} P\{\sigma_k \geqslant k\lambda^{-1}/2\} = \sum_{k\geqslant 0} P\{\sigma_0 \geqslant k\lambda^{-1}/2\} < \infty$, because of the finiteness of the expectation of σ_0. Hence, the Borel–Cantelli lemma shows that the first term of (8) is zero. Since $T_k/k \overset{\text{a.s.}}{\to} \lambda^{-1}$, it follows that the second term of (8) is also zero, and this completes the proof. □

Now, $\{(\sigma_n, \tau_n), n \geqslant 0\}$ can be extended to $\{(\sigma_n, \tau_n), n \in \mathbb{Z}\}$ on an appropriate sample space. We define the stationary process $\{(\tilde{\sigma}_n, \tilde{\tau}_n), n \in \mathbb{Z}\}$ by $(\tilde{\sigma}_n, \tilde{\tau}_n) = (\sigma_{-n}, \tau_{-n})$. We next observe the following bound for the workload seen by the nth arrival, $n \geqslant 0$, when arrival 0 sees the system empty

$$\begin{aligned} W_{0,n} &\leqslant \max_{1\leqslant k\leqslant n} (\sigma_{n-k} - (\tau_{n-1} + \cdots + \tau_{n-k}))^+ \\ &\leqslant_{\text{st}} \max_{k\geqslant 0} (\tilde{\sigma}_k - (\tilde{\tau}_0 + \tilde{\tau}_1 + \cdots + \tilde{\tau}_k))^+ \\ &= W^* \quad \text{say,} \end{aligned}$$

where the first step follows because the customer seen in the system by the nth arrival, if any, must be one of the previous customers. From Lemma 2 applied to $\{(\tilde{\sigma}_n, \tilde{\tau}_n), n \geqslant 0\}$, W^* is a.s. finite. Tightness of $\{W_{0,n}, n \geqslant 0\}$ follows, and from this the existence of a stationary solution to the recursion (7).

3.4. Compact state space

A simple but rather nice consequence of Theorem 1 is the following.

Theorem 2. *Let $(\Omega, \mathscr{F}, P)$ be a probability space, and θ a measurable map from $(\Omega, \mathscr{F})$ into itself such that P is θ-invariant and ergodic. We assume that $(\Omega, \mathscr{F})$ is a Polish space. Let $(E, \mathscr{E})$ be a compact Polish space and φ_0 a random variable defined on $(\Omega, \mathscr{F}, P)$ that takes values in the space of measurable maps from $(E, \mathscr{E})$ into itself. Let $\varphi_n(\omega) = \varphi_0(\theta^n\omega)$.*

Then the recursion (1) always admits a stationary solution, in the sense that there is a probability distribution Q on $\Omega \times E$ that is Θ-invariant and has Ω marginal P, where Θ is the shift on $\Omega \times E$ defined in (3).

Proof. This is an immediate consequence of Lemma 1. Indeed, any sequence of probability distributions on a compact Polish space is tight. □

4. Uniqueness

In this section we prove a uniqueness theorem that applies to recursions of type (1) under a broad range of conditions. This theorem is similar to theorems on unique ergodicity in topological dynamics – see, for instance, Theorem 9.2 of Mañé (1987, p. 58) – more generally, in the ergodic theory of Markov operators on spaces of continuous functions – see, for instance, Proposition 1.3 Krengel (1985, p. 178). Let $C_b(\Omega \times E)$ denote the space of bounded continuous functions on $\Omega \times E$.

Theorem 3. *Suppose that for every Θ-invariant probability distribution Q on $\Omega \times E$ having Ω marginal P, and all $f \in C_b(\Omega \times E)$*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(\Theta^j(\omega, x)) = \mathscr{A}(f) \tag{9}$$

exists Q-a.s. and is a constant for Q-a.a. (ω, x). Then there is a unique Θ-invariant probability distribution Q on $\Omega \times E$ having Ω marginal P. Conversely, if there is a unique Θ-invariant probability distribution Q on $\Omega \times E$ having Ω marginal P, then for all $f \in C_b(\Omega \times E)$ the limit in (9) exists Q-a.s. and is constant for Q-a.a. (ω, x).

Proof. Suppose there are two Θ-invariant probability distributions Q_1 and Q_2 on $\Omega \times E$ having Ω marginal P. Hence convergence (9) to a constant $\mathscr{A}_i(f)$ holds Q_i-a.s., $i = 1, 2$. Since, a priori, the measures Q_1, Q_2 may have disjoint supports, we introduce the mixture $Q = \frac{1}{2}Q_1 + \frac{1}{2}Q_2$. Then Q is also Θ-invariant, and has Ω marginal P. It follows from the assumption of the theorem that for any $f \in C_b(\Omega \times E)$, $\frac{1}{n}\sum_{j=0}^{n-1} f(\Theta^j(\omega, x))$ converges to a constant $\mathscr{A}(f)$ for Q-a.a. (ω, x). Hence, $\mathscr{A}_i(f) = \mathscr{A}(f)$, Q_i-a.s., $i = 1, 2$. Since f is bounded, the dominated convergence theorem implies that

$$\lim_{n \to \infty} \int_{\Omega \times E} \frac{1}{n} \sum_{j=0}^{n-1} f(\Theta^j(\omega, x))\,\mathrm{d}Q_i = \mathscr{A}(f), \quad i = 1, 2. \tag{10}$$

However Q_i is Θ-invariant, $i = 1, 2$, so that the left-hand side of (10) is just $\int_{\Omega \times E} f(\omega, x)\,\mathrm{d}Q_i$ for all $n \geqslant 1$. Since a probability distribution on $\Omega \times E$ is determined by its integrals against bounded continuous functions, it follows that $Q_1 = Q_2$. Thus, if there is a Θ-invariant probability distribution on $\Omega \times E$ having Ω marginal P, it is unique.

For the converse, let Q be a Θ-invariant probability distribution on $\Omega \times E$ having Ω marginal P. By Birkhoff's ergodic theorem – see, for instance, Krengel (1985, p. 9), the limit on the left-hand side of (9) exists Q-a.s., call it $\mathscr{A}(f, \omega, x)$. In fact, $\mathscr{A}(f, \omega, x)$ is equal to the Q-conditional expectations of f with respect to the σ-field of Θ-invariant measurable subsets of $\Omega \times E$. Suppose that for some $f \in C_b(\Omega \times E)$ this limit is not Q-a.s. constant. Then there exists some scalar a such that

$$0 < Q\{(\omega, x)\colon \mathscr{A}(f, \omega, x) > a\} < 1.$$

But, if C denotes $\{\mathscr{A}(f, \omega, x) > a\}$, it is easily seen that $\Theta^{-1}(C) = C$. Hence, Q is not ergodic. Thus, there exist probability distributions $Q_1 \neq Q_2$ on $\Omega \times E$ which are Θ-invariant and such that $Q = pQ_1 + (1 - p)Q_2$ for some $0 < p < 1$. This means we also have $Q \circ \pi^{-1} = pQ_1 \circ \pi^{-1} + (1 - p)Q_2 \circ \pi^{-1}$, where $\pi\colon \Omega \times E \to \Omega$ denotes the projection onto the first co-ordinate. Note that

$$Q \circ \pi^{-1} \circ \theta^{-1} = Q \circ \Theta^{-1} \circ \pi^{-1} = Q \circ \pi^{-1},$$

so that $Q \circ \pi^{-1}$ is Θ-invariant, and a similar argument implies $Q_i \circ \pi^{-1}$ is Θ-invariant, $i = 1, 2$. Since Q has Ω marginal P, in fact $Q \circ \pi^{-1}$ is just P. But P is ergodic, so $0 < p < 1$ and the Θ-invariance of $Q_i \circ \pi^{-1}, i = 1, 2$, implies that $Q_i \circ \pi^{-1} = P, i = 1, 2$. Thus, we have found two distinct Θ-invariant probability distributions on $\Omega \times E$ each having Ω marginal P, which would be impossible if there were a unique such distribution. □

An immediate consequence of the uniqueness theorem is the following.

Corollary 1. *If the conditions of Theorem* 3 *are satisfied, then the unique* weak stationary solution Q *is* Θ*-ergodic.*

5. An application of the uniqueness theorem

In this section we illustrate the use of Theorem 3. There are several well known and important examples of recursions where the uniqueness of stationary solutions does not hold, including the example of Section 3.3 in certain cases – see, for example, Baccelli and Brémaud (1993). Non-uniqueness should therefore be thought of as being relatively common, and certainly not pathological. In such situations, one is interested in classifying the different stationary regimes. Theorem 3 is clearly not of much use for such investigations. Nevertheless, it provides a useful tool to prove uniqueness in some examples, as this section hopes to demonstrate.

We consider the Lindley equation (5) of Section 3.1 for the G/G/1 queue. In Section 3.1 we have demonstrated the existence of a probability distribution Q on $\Omega \times \mathbb{R}_+$ that is Θ-invariant and has Ω marginal P. To show the uniqueness of this distribution, by Theorem 3 we need to show that for every such distribution and every $f \in C_b(\Omega \times E)$ the limit on the left hand side of (9) exists Q-a.s. and is constant for Q-a.a. (ω, x).

The Q-a.s. existence of the limit is a consequence of Birkhoff's ergodic theorem, and we may call it $\mathscr{A}(f, \omega, x)$. Note that for Q-a.a. x, $\mathscr{A}(f, \omega, x)$ is well-defined for Q-a.a. ω (or equivalently for P-a.a. ω). Let us now show that for all $x_1 \neq x_2$ such that $\mathscr{A}(f, \omega, x_i)$ is well-defined for Q-a.a. ω, $i = 1, 2$, we have

$$\mathscr{A}(f, \omega, x_1) = \mathscr{A}(f, \omega, x_2) \quad Q\text{-a.s.} \tag{11}$$

This is an immediate consequence of the following lemma.

Lemma 3. *For all* $x_1 \neq x_2$, *there exists a* P*-a.s. finite random time* $\kappa(\omega)$ *such that*

$$\Theta^j(\omega, x_1) = \Theta^j(\omega, x_2) \quad \textit{for all } j \geqslant \kappa(\omega). \tag{12}$$

Proof. Assume without loss of generality that $x_1 < x_2$, and define

$$\kappa_2(\omega) = \inf\{ j \geqslant 0 \colon \Theta^j(\omega, x_2) = 0 \}.$$

($\kappa_1(\omega)$ is defined similarly.) The P-a.s. finiteness of $\kappa_2(\omega)$ follows from the strong law of large numbers for the sequence $\{\xi_n, n \geqslant 0\}$ and the fact that $E\xi_0 < 0$. Indeed,

$$P\{\omega : \kappa_2(\omega) > m\} \leqslant P\left\{\omega : x + \sum_{j=0}^{n-1} \xi_j(\omega) > 0, \quad \text{for all } n \leqslant m\right\},$$

and the latter probability goes to zero as $m \to \infty$ since the partial sums $\sum_{j=0}^{n-1} \xi_j(\omega)$ diverge to $-\infty$, P-a.s. An examination of (5) reveals that $\kappa_1(\omega) \leqslant \kappa_2(\omega)$ (admittedly, we use monotonicity here). Letting $\kappa(\omega) = \kappa_2(\omega)$, (12) now follows immediately from the above observations. □

That Lemma 3 implies (11) is easily seen by examining the left-hand sides of (9) for x_1 and x_2.

We have thus demonstrated that $\mathscr{A}(f, \omega, x)$ is Q-a.s. a function only of ω, so we may write $\mathscr{A}(f, \omega, x) = \mathscr{A}(f, \omega)$. But $\mathscr{A}(f, \omega, x)$ is Θ-invariant, and being Q-a.s. equal to a function $\mathscr{A}(f, \omega)$ defined on Ω, this latter function is θ-invariant. But then, by ergodicity of P, it must be P-a.s. constant. Theorem 3 now implies the uniqueness of the weak stationary solution for the Lindley equation.

6. Concluding remarks

We have proposed an approach to the study of stationary solutions for stochastic recursions driven by stationary ergodic processes, that we believe is novel, and which can be used to handle a much wider range of situations than the currently available techniques. Questions about the existence of stationary regimes for such recursions are of considerable importance in discrete event systems applications. Our results are clearly only a start in this direction, in that questions regarding the classification of stationary regimes in situations where there is more than one such, and questions of convergence to stationary regimes from arbitrary initial conditions have not been addressed in this work. Nevertheless, several previously intractable non-monotonic recursions might conceivably be amenable to a partial treatment using this approach.

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