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STATIONARY SOLUTIONS OF STOCHASTIC RECURSIONS DESCRIBING DISCRETE EVENT SYSTEMS

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Abstract

We consider recursions of the form $x_{n+1} = \varphi_n[x_n]$, where $\{\varphi_n, n \geq 0\}$ is a stationary ergodic sequence of maps from a Polish space $(E, \mathcal{E})$ into itself, and $\{x_n, n \geq 0\}$ are random variables taking values in $(E, \mathcal{E})$. The question of when stationary solutions exist for such recursions, whether they are unique, and whether there is convergence to a stationary solution starting from arbitrary initial conditions is of considerable interest in discrete event system applications. Currently available techniques can only answer such questions under strong simplifying assumptions on the statistics of $\{\varphi_n\}_n$ (such as Markov assumptions), or on the nature of these maps (such as monotonicity).

In this paper we introduce a new technique for studying stochastic recursions without such simplifying assumptions. To do so, we weaken the solution concept: rather than constructing a pathwise solution we construct a probability measure on another sample space and families of random variables on this space whose law gives a stationary solution to the recursion. The problem of existence of a stationary solution is then translated into the problem of establishing tightness of a sequence of probability distributions, and uniqueness questions can be addressed using techniques familiar from the ergodic theory of positive Markov operators on spaces of continuous functions.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We assume that $(\Omega, \mathcal{F})$ is a Polish space (i.e. Ω is a complete separable metric space and $\mathcal{F}$ is a sub σ-algebra of the Borel σ-algebra of Ω) – see, for instance, Parthasarathy [15] for background on Polish spaces. Let θ be a measurable map from $(\Omega, \mathcal{F})$ into itself such that P is θ-invariant: $P(\theta^{-1}(A)) = P(A)$ for all $A \in \mathcal{F}$. We assume that P is θ-ergodic: if $A \in \mathcal{F}$ is such that $\theta^{-1}(A) = A$ up to sets of P-probability 0, then A has P-probability 0 or 1. Our reference for basic ergodic theoretic concepts will be Krengel [12].

Let $(E, \mathcal{E})$ be a Polish space and φ_0 a random variable defined on $(\Omega, \mathcal{F}, P)$ that takes values in the space of measurable maps from $(E, \mathcal{E})$ into itself. Let $\varphi_n(\omega) = \varphi_0(\theta^n\omega)$. Then, under P, $\{\varphi_n, n \geq 0\}$ is a stationary ergodic sequence of random maps from $(E, \mathcal{E})$ into itself.

We are interested in solutions of recursions of the form

$$x_{n+1} = \varphi_n[x_n] \, . \tag{1}$$

Here $\{x_n, n \geq 0\}$ are random variables taking values in $(E, \mathcal{E})$, and to avoid confusion we have used square brackets to write $\varphi_n[x_n]$ for the evaluation of the map $\varphi_n(\omega)$ at $x_n(\omega) \in E$. Given an initial condition, say x_0 at time 0, there is of course no problem in constructing a pathwise solution to (1) for $n \geq 0$. In view of the stationarity and ergodicity of $\{\varphi_n\}_n$, one expects that, if some stability condition holds, the solutions constructed in this manner converge in some suitable sense to a stationary solution. By a stationary solution of the above recursion we mean a stationary process $\{x_n, n \geq 0\}$ that satisfies the recursion.

The question of when such stationary solutions exist, whether they are unique, and whether there is convergence to a stationary solution starting from arbitrary initial conditions, is of considerable interest in discrete event system applications. This is because discrete event systems subject to random influences can often be described by recursions of the form (1). At the present time it is often the case in recursions of practical interest that stability conditions can be derived – and answers to the existence, uniqueness,

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and convergence questions can be given – only if one is willing to make strong simplifying assumptions about the statistics of $\{\varphi_n\}_n$. This situation is far from ideal, since such simplifying assumptions are typically not met in practice. Thus there is an important role for the study of recursions of the form (1) in the general stationary ergodic context.

The canonical example of such a recursion is

$$x_{n+1} = \varphi(x_n, \xi_n) \tag{2}$$

where $\{\xi_n, n \geq 0\}$ is a stationary and ergodic sequence of random variables and φ is a deterministic function. If one assumes that φ is continuous in both variables and increasing in the first one, an idea due to Loynes [13] can be used in certain cases to identify stability conditions and to explicitly construct stationary solutions. This technique has been developed to deal with a number of applications – see Baccelli and Brémaud [2, 3], Bambos and Walrand [5], Konstantopoulos and Baccelli [11, 4], and Walrand [18], for several examples. In many of these examples uniqueness conditions and convergence theorems can also be derived. However, the assumed monotonicity plays a key role in this development.

Our purpose in this paper is to discuss a new technique for constructing stationary solutions to (1) that does not depend on the recursion being of the type (2) with monotonicity conditions. To achieve this we have to weaken the solution concept, along lines that are familiar from the theory of stochastic differential equations – see, for instance, Stroock and Varadhan [17]. That is, rather than constructing a pathwise solution to (1), we construct a probability measure on another sample space and families of random variables on this space whose law gives a stationary solution to (1).

Our approach is based on the following construction: consider the product space $\Omega \times E$, endowed with the product σ-field $\mathcal{F} \otimes \mathcal{E}$ and the new measurable shift $\Theta(\omega, x) : \Omega \times E \to \Omega \times E$ defined by

$$\Theta(\omega, x) = (\theta\omega,\ \varphi_0(\omega)[x]). \tag{3}$$

Note the following composition rule.

$$\begin{aligned}\Theta^n(\omega, x) &= \\ &= (\theta^n\omega,\ \varphi_0(\theta^{n-1}\omega)\varphi_0(\theta^{n-2}\omega)\dots\varphi_0(\omega)[x]) \\ &= (\theta^n\omega,\ \varphi_{n-1}(\omega)\varphi_{n-2}(\omega)\dots\varphi_0(\omega)[x])\ , \\ \Theta^{n+m}(\omega, x) &= \Theta^n(\Theta^m(\omega, x))\ .\end{aligned}$$

The problem of existence of a stationary solution is then translated into the problem of existence of a probability measure Q on $(\Omega \times E,\ \mathcal{F} \otimes \mathcal{E})$ that is invariant under Θ, and whose Ω marginal is P. Indeed, suppose it was possible to construct such a probability measure Q. Consider the new random variables

$$X_0(\omega, x) = x, \quad \Phi_0(\omega, x) = \varphi_0(\omega)\ ,$$

defined on $\Omega \times E$ and let

$$X_n(\omega, x) = X_0(\Theta^n(\omega, x)), \quad \Phi_n(\omega, x) = \Phi_0(\Theta^n(\omega, x))\ ,$$

for $n \geq 0$. X_0 takes values in $(E, \mathcal{E})$ and Φ_0 takes values in the space of measurable mappings from $(E, \mathcal{E})$ into itself. A simple calculation shows

$$\begin{aligned}X_n(\omega, x) &= \varphi_{n-1}(\omega)\varphi_{n-2}(\omega)\dots\varphi_0(\omega)[x]) \quad (4) \\ \Phi_n(\omega, x) &= \varphi_n(\omega)\ .\end{aligned}$$

Thus $\{X_n,\ n \geq 0\}$ solves the recursion

$$X_{n+1} = \Phi_n[X_n]\ .$$

Since Q is Θ-stationary, $\{X_n, \Phi_n\}_n$ is stationary under Q. Since Q has Ω marginal P, $\{\Phi_n\}_n$ has the same distribution as $\{\varphi_n\}_n$. We refer to such a measure Q as a *weak stationary solution* for the recursion (1).

We note that a point of view similar to the one presented here has been taken by Crauel [9], who calls a recursion of the form (1) a random dynamical system. The focus here is quite different from ours: it is assumed that φ_0 is a homeomorphism, and the concern is with the Markovian nature of the solutions. In fact, the idea to construct Θ as in equation (3) is an old one in ergodic theory, see for instance Krengel [12, pg. 261], or Petersen [16, pg. 11], where it is called a *skew-product*. Again, the concerns in ergodic theory are quite different: an underlying invariant measure on the space on which the skew-product acts is taken as given, and most of the interesting work deals with the case where E admits a group action, and φ_0 is a random element of the group.

Stationary queueing systems have also been considered by Brandt et al [8]: a concept of "stationary weak solution" is introduced and several examples, where the solution is constructed on an "enlarged probability space" (because one does not exist on the original probability space) are treated in detail. What is made clear in our formulation is that it suffices to consider the product $\Omega \times E$ as this enlarged probability space. Also, one should note that a technique for the construction of stationary solutions, based on the concept of "renovating events", has been developed by Borovkov (see, for instance, [7] for applications to communication networks) and Foss [10].

In Section 2 we describe a technique to construct a probability measure Q on $\Omega \times E$ of the desired type, when a certain tightness condition is satisfied. We also give a simple way of checking this criterion without reference to the construction of the product space $\Omega \times E$. In Section 3 we illustrate our approach by rederiving the well known existence result in the single server queue, which corresponds to finding a stationary solution for the Lindley equation. Our point of view is such that the monotonicity in the Lindley equation is not appealed to; for lack of space we have not explicated this further here, for details see [1]. We point out the interesting fact that a stationary recursion on a compact state space always admits a stationary solution in our sense. In Section 4, we discuss the question of uniqueness of stationary solutions. Here we give a theorem on uniqueness along lines familiar from the ergodic theory of positive Markov operators on spaces of continuous functions - see, for instance, Krengel [12, Chapter 5]. In Section 5 we demonstrate how the uniqueness theorem can be used to establish uniqueness of solutions in the problem of the single server queue. More examples are discussed in [1], but could not be included here. Some concluding remarks are made in Section 6.

2 Existence of stationary solutions

Suppose $\{x_n, n \geq 0\}$ is a stationary solution to (1). In other words, this is a stationary sequence on the original probability space $(\Omega,\ \mathcal{F},\ P)$ and can thus be termed a strong solution to (1). Consider the map $e : \Omega \rightarrow \Omega \times E$ given by

$$e(\omega) = (\omega, x_0(\omega)) .$$

Let $Q = P \circ e^{-1}$. We note that Q has Ω marginal P, i.e., $Q(A \times E) = P(A)$ for all $A \in \mathcal{F}$. Using the definition of Θ in (3), we have

$$Q \circ \Theta^{-1} = P \circ e^{-1} \circ \Theta^{-1} = P \circ \theta^{-1} \circ e^{-1} = P \circ e^{-1} = Q .$$

Thus Q is invariant under Θ. In particular, the (constant) sequence $\{Q \circ \Theta^{-n}, n \geq 0\}$ is *tight* (see below for the definition of tightness, and Billingsley [6], for more details).

On the other hand, let us start with an arbitrary probability distribution Q_0 on $\Omega \times E$ whose Ω marginal is P. Let Q_n denote the probability distribution $Q_0 \circ \Theta^{-n}$ on $\Omega \times E$. A simple calculation shows $Q_n(A \times E) = P(\theta^{-n} A)$. By stationarity of P, $P(\theta^{-n} A) = P(A)$, so Q_n also has Ω marginal P for all $n \geq 0$.

Suppose now that the sequence $\{Q_n, n \geq 0\}$ is *tight*, i.e. that for any $\varepsilon > 0$ there is a compact set $K \subseteq \Omega \times E$ such that $Q_n(K) > 1 - \varepsilon$ for all $n \geq 0$. This is enough to demonstrate the existence of a Θ invariant probability distribution Q on $\Omega \times E$ having Ω marginal P. To see this, let

$$\overline{Q}_n = \frac{1}{n}(Q_0 + \cdots + Q_{n-1}) , \quad n \geq 1 .$$

Then $\{\overline{Q}_n, n \geq 1\}$ is also tight, and each $\overline{Q}_n$ has Ω marginal P . Let Q be any subsequential limit of this sequence, say $Q = \lim_{n_k \rightarrow \infty} \overline{Q}_{n_k}$. Clearly, Q has Ω marginal P. Also, for any measurable subset $C \subseteq \Omega \times E$,

$$\begin{aligned} Q(\Theta^{-1}(C)) &= \lim_{n_k \rightarrow \infty} \overline{Q}_{n_k}(\Theta^{-1}(C)) \\ &= \lim_{n_k \rightarrow \infty} \overline{Q}_{n_k}(\Theta^{-1}(C)) \\ &= \lim_{n_k \rightarrow \infty} \overline{Q}_{n_k}(C) - \lim_{n_k \rightarrow \infty} \frac{1}{n_k} Q_0(C) + \lim_{n_k \rightarrow \infty} \frac{1}{n_k} Q_{n_k}(C) \\ &= Q(C) , \end{aligned}$$

so Q is Θ-invariant.

We have thus proved the following theorem:

Theorem 1 *Let Q_0 be a probability distribution on $\Omega \times E$ whose Ω marginal is P. Let Q_n denote the probability distribution $Q_0 \circ \Theta^{-n}$ on $\Omega \times E$. Suppose that the sequence $\{Q_n, n \geq 0\}$ is* tight. *Then there is a stationary sequence $\{X_n, \Phi_n\}_n$ defined on $\Omega \times E$ with $\{X_n\}_n$ taking values in $(E, \mathcal{E})$ and $\{\Phi_n\}_n$ taking values in the space of measurable maps from $(E, \mathcal{E})$ into itself, such that $\{\Phi_n\}_n$ has the same distribution as $\{\varphi_n\}_n$ and $\{X_n\}_n$ obeys $X_{n+1} = \Phi_n[X_n]$, $n \geq 0$. Conversely, if the stochastic recursion $x_{n+1} = \varphi_n[x_n]$, $n \geq 0$, admits a stationary solution, there is a probability distribution Q on $\Omega \times E$ whose Ω marginal is P, and such that the sequence $\{Q \circ \Theta^{-n}, n \geq 0\}$ is tight.* □

At first sight it may appear that proving tightness of $\{Q_n, n \geq 0\}$ would be difficult. However the following simple result shows that the question of the existence of a stationary regime in our sense can often be settled without reference to the product construction. The value of this result becomes clear in the study of examples, see [1].

Lemma 1 *Suppose that, for some $x \in E$, the sequence $\{\varphi_{n-1}(\omega)\varphi_{n-2}(\omega)\ldots\varphi_0(\omega)[x], n \geq 1\}$, defined on $(\Omega,\ \mathcal{F},\ P)$ is tight. Let Q_0^x denote the probability distribution $P \otimes \delta_x$ on $\Omega \times E$. Then the sequence $\{Q_n^x, n \geq 0\}$ is tight.*

Proof. We first note that for any sequence $\{Q_n, n \geq 0\}$ having Ω marginal P, tightness is equivalent to tightness of the E marginals. Thus it suffices to show that the E marginals of $\{Q_n^x, n \geq 0\}$ are tight. But from (4), this is precisely the assumption of the lemma. □

3 Applications of the existence theorem

In this section we consider an example of a stochastic recursion arising in discrete event system applications to illustrate the scope of our idea.

3.1 The G/G/1 queue

A classical example of a stochastic recursion is the Lindley equation describing the workload seen by arriving customers into a G/G/1 queue. On a sample space $(\Omega, \mathcal{F}, P)$ admitting a shift θ under which P is ergodic, we are given nonnegative random variables (σ_0, τ_0) satisfying $E[\sigma_0] = \mu^{-1} < \infty$ and $E[\tau_0] = \lambda^{-1} < \infty$. Let $(\sigma_n, \tau_n) = (\sigma_0 \circ \theta^n, \tau_0 \circ \theta^n)$. Thus $\{\sigma_n, \tau_n\}_n$ is a stationary ergodic sequence. The interpretation of σ_n is the work brought in by the nth customer to a server and τ_n denotes the interarrival time between the arrival of the nth customer and the $n+1$st customer. The server works at rate 1 if there is work in the system. Let W_n denote the workload in the system seen by the nth customer. Then, starting from some initial condition, the workload evolves according to the equation

$$W_{n+1} = (W_n + \sigma_n - \tau_n)^+ \tag{5}$$

This recursion is of the form (2), is monotone in the state variable, and continuous in the state variable and the value of the driving process. Thus the existence of stationary solutions can be studied pathwise using the idea of Loynes [13]. For details, see Baccelli and Brémaud [2, 3], or Walrand [18]. The pathwise construction works under the assumption that the shift has a measurable inverse, and leads to the conclusion that there is a unique stationary solution to the recursion if $\lambda < \mu$. The pathwise construction of the stationary solution involves constructing variables $\{W_{m,n}, -\infty < m \leq n < +\infty\}$, with the initial condition $W_{m,m} = 0$. The monotonicity in the state variable gives that $W_{m,0}$ increases when m decreases, and the limit is the desired stationary solution. If $\lambda < \mu$, this can be shown to be proper.

On the other hand, our approach yields the existence of a stationary solution for $\lambda < \mu$ along slightly different lines. To begin with, we assume that $(\Omega, \mathcal{F})$ is a Polish space. Referring to Lemma 1 and the notation in the preceding paragraph, the existence of a stationary solution for $\lambda < \mu$ – which in this example means the existence of a Θ-invariant probability distribution on $\Omega \times \mathbf{R}_+$ having Ω-marginal P – follows if we can show that the sequence $\{W_{0,n}, n \geq 0\}$ is *tight*. Let $\xi_0 = \sigma_0 - \tau_0$, and $\xi_n = \xi_0 \circ \theta^n = \sigma_n - \tau_n$. From (5), we write

$$W_{0,n}(\omega) = \max(0,\ \xi_{n-1}(\omega), \xi_{n-1}(\omega) + \xi_{n-2}(\omega), \dots, \xi_{n-1}(\omega) + \xi_{n-2}(\omega) + \cdots + \xi_0(\omega)) .$$

Now, $\{\xi_n, n \geq 0\}$ can be extended to $\{\xi_n, n \in \mathbf{Z}\}$ on an appropriate sample space. Consider the stationary process $\{\tilde{\xi}_n, n \in \mathbf{Z}\}$ defined by $\tilde{\xi}_n = \xi_{-n}$. Then we have

$$\begin{aligned} W_{0,n} &\stackrel{d}{=} \max(0,\ \tilde{\xi}_0, \dots,\ \tilde{\xi}_0 + \cdots + \tilde{\xi}_{n-1}) \\ &\leq_{\text{st}} \max(0,\ \max_{k \geq 0} \sum_{l=0}^{k} \tilde{\xi}_l) \\ &= W^*, \text{ say }, \end{aligned}$$

where $\stackrel{d}{=}$ denotes equality in distribution, and $\leq_{\text{st}}$ denotes stochastic ordering – for more on these concepts see, for example, Baccelli and Brémaud [3]. If $\lambda < \mu$, we have $E\tilde{\xi}_0 < 0$, from which it follows easily by the strong law of large numbers that W^* is a proper random variable. Hence $\{W_{0,n}\}_n$ is tight.

At first sight it may not appear that this proof is significantly different from that of the Loynes construction. However, an essential point is that monotonicity of (5) has not been explicitly used anywhere in the argument. The reader can easily verify this by playing around with equations that are non-monotone perturbations of Lindley's equation, but match it for large enough workloads; an explicit example along these line is developed in [1], along with other examples which have been omitted for lack of space.

3.2 Compact state space

A simple but rather nice consequence of Theorem 1 is the following:

Theorem 2 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and θ a measurable map from $(\Omega, \mathcal{F})$ into itself such that P is θ-invariant and ergodic. We assume that $(\Omega, \mathcal{F})$ is a Polish space. Let $(E, \mathcal{E})$ be a* compact *Polish space and φ_0 a random variable defined on $(\Omega, \mathcal{F}, P)$ that takes values in the space of measurable maps from $(E, \mathcal{E})$ into itself. Let $\varphi_n(\omega) = \varphi_0(\theta^n \omega)$.*
Then the recursion (1) always *admits a stationary solution, in the sense that there is a probability distribution Q on $\Omega \times E$ that is Θ-invariant and has Ω marginal P, where Θ is the shift on $\Omega \times E$ defined in (3).*

Proof. This is an immediate consequence of Lemma 1. Indeed, any sequence of probability distributions on a compact Polish space is tight. □

4 Uniqueness

In this section we state a uniqueness theorem that applies to recursions of type (1) under a broad range of conditions. This theorem is similar to theorems on unique ergodicity in topological dynamics – see, for instance, Theorem 9.2 on page 58 of Mañé [14] – or more generally, in the ergodic theory of Markov operators on spaces of continuous functions – see, for instance, Proposition 1.3 on page 178 of Krengel [12]. Let $C_b(\Omega \times E)$ denote the space of bounded continuous functions on $\Omega \times E$.

Theorem 3 *Suppose that for every Θ-invariant probability distribution Q on $\Omega\times E$ having Ω marginal P, and all $f \in C_b(\Omega \times E)$*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(\Theta^j(\omega, x)) = \mathcal{A}(f) \qquad (6)$$

exists Q-a.s. and is a constant for Q-a.a. (ω, x). Then, if there is a Θ-invariant probability distribution Q on $\Omega \times E$ having Ω marginal P, it is unique. Conversely, if there is a unique Θ-invariant probability distribution Q on $\Omega\times E$ having Ω marginal P, then for all $f \in C_b(\Omega \times E)$ the limit in (6) exists Q-a.s.. and is constant for Q-a.a. (ω, x).

The reader is referred to [1] for a complete proof of Theorem 3.

5 Applications of the uniqueness theorem

In this section we illustrate the use of Theorem 3. Before doing so, let us note that there are several well known and important examples of recursions where the uniqueness of stationary solutions does not hold, see, for example, Baccelli and Brémaud, [2, 3]. Non-uniqueness should therefore be thought of as being relatively common, and certainly not pathological. In such situations, one is interested in classifying the different stationary regimes. Theorem 3 is clearly not of much use for such investigations. Nevertheless, it provides a useful tool to prove uniqueness in some examples, as this section hopes to demonstrate.

5.1 The G/G/1 queue

We consider again the Lindley equation (5) of Section 3.1. In Section 3.1 we have demonstrated the existence of a probability distribution Q on $\Omega \times \mathbf{R}_+$ that is Θ-invariant and has Ω marginal P, under the stability condition $\lambda < \mu$. To show the uniqueness of this distribution, by Theorem 3 we need to show that for every such distribution and every $f \in C_b(\Omega \times E)$ the limit on the left hand side of (6) exists Q-a.s. and is constant for Q-a.a. (ω, x).

The Q-a.s. existence of the limit is a consequence of Birkhoff's ergodic theorem, and we may call it $\mathcal{A}(f,\omega,x)$. Note that for Q-a.a. x, $\mathcal{A}(f,\omega,x)$ is well-defined for Q-a.a. ω (or equivalently for P-a.a. ω). Let us now show that for all $x_1 \neq x_2$ such that $\mathcal{A}(f,\omega,x_i)$ is well-defined for Q-a.a. ω, $i = 1, 2$, we have

$$\mathcal{A}(f,\omega,x_1) = \mathcal{A}(f,\omega,x_2) \quad Q\text{-a.s.} \ . \qquad (7)$$

This is an immediate consequence of the following lemma.

Lemma 2 *For all $x_1 \neq x_2$, there exists a P-a.s. finite random time $\kappa(\omega)$ such that*

$$\Theta^j(\omega, x_1) = \Theta^j(\omega, x_2) \quad \textit{for all} \;\; j \geq \kappa(\omega) \ . \qquad (8)$$

Proof. Assume without loss of generality that $x_1 < x_2$, and define

$$\kappa_2(\omega) = \inf\{j \geq 0 \ : \ \Theta^j(\omega, x_2) = 0\} \ .$$

($\kappa_1(\omega)$ is defined similarly.) The P-a.s. for the sequence $\{\xi_n, n \geq 0\}$ and the fact that $E\xi_0 < 0$. Indeed,

$$P\{\omega : \kappa_2(\omega) > m\}$$
$$\leq P\{\omega : x + \sum_{j=0}^{n-1} \xi_j(\omega) > 0, \text{ for all } n \leq m\},$$

and the latter probability goes to zero as $m \to \infty$ since the partial sums $\sum_{j=0}^{n-1} \xi_j(\omega)$ converge to $-\infty$, P-a.s.. An examination of (5) reveals that $\kappa_1(\omega) \leq \kappa_2(\omega)$ (admittedly, we use monotonicity here). Letting $\kappa(\omega) = \kappa_2(\omega)$, (8) now follows immediately from the above observations. □

That Lemma 2 implies (7) is easily seen by examining the left hand sides of (6) for x_1 and x_2.

Note that the excluded null event in Lemma 2 can be taken to be the same for all pairs x_1, x_2. We have thus demonstrated that $\mathcal{A}(f,\omega,x)$ is Q-a.s. a function only of ω, so we may write $\mathcal{A}(f,\omega,x) = \mathcal{A}(f,\omega)$. But $\mathcal{A}(f,\omega,x)$ is Θ-invariant, and being Q-a.s. equal to a function $\mathcal{A}(f,\omega)$ defined on Ω, this latter function is θ-invariant. But then, by ergodicity of P, it must be P-a.s. constant. Theorem 3 now implies the uniqueness of the weak stationary solution for the Lindley equation.

6 Concluding remarks

We have proposed an approach to the study of stationary solutions for stochastic recursions driven by stationary ergodic processes that we believe is novel, and which can be used to handle a much wider range of situations than the currently available techniques. Questions about the existence of stationary regimes for such recursions are of considerable importance in discrete event systems applications. Our results are clearly only a start in this direction, in that questions regarding the classification of stationary regimes in situations where there is more than one such, and questions of convergence to stationary regimes from arbitrary initial conditions have not been addressed in this work. Nevertheless, several previously intractable non-monotonic recursions might conceivably be amenable to a partial treatment using this approach.

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