The Input–Output Map of a Monotone Discrete-Time Quasireversible Node

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***Abstract*—A class of discrete-time quasireversible nodes called *monotone*, which includes discrete-time analogs of the ./M/∞ and ./M/1 nodes is considered. For stationary ergodic nonnegative integer valued arrival processes, the existence and uniqueness of stationary regimes are proven when a natural rate condition is met. Coupling is used to prove the contractiveness of the input–output map relative to a natural distance on the space of stationary arrival processes that is analogous to Ornstein's $\bar{d}$ distance. A consequence is that the only stationary ergodic fixed points of the input–output map are the processes of independent and identically distributed Poisson random variables meeting the rate condition.**

***Index Terms*—Communication networks, Palm theory, quasireversible networks.**

I. Introduction

THE CONCEPT of quasireversibility of a queueing node in continuous time was introduced by Kelly, [8], and a probabilistic understanding of this concept was provided by Walrand [13]. Networks constructed from quasireversible nodes form a natural class of models for the performance analysis of communication networks; they offer modeling flexibility and admit *product form* stationary distributions, which makes the computation of stationary performance quantities easy. One of the characteristics of a continuous time quasireversible node is that, in stationarity, when the arrival process is a Poisson process so is the departure process. One can thus think of the Poisson process as a fixed point of the input output map of such a node, which we view as a map on the space of stationary processes (of course, this has to be appropriately formulated). A question of some interest then is whether the input output map has any other fixed points, apart from mixtures of Poisson processes, which are also trivially fixed.

For ./G/∞ nodes acting on stationary ergodic input processes, Vere-Jones proved a result of this type (for a precise formulation, see [12]). The problem for ./M/1 nodes has been circulating in the community for some time now. For example, it is mentioned as an old open problem in a recent paper of Glynn and Whitt [5] to prove that the stationary departure process from a long tandem of identical ./G/1 nodes fed by a renewal process becomes asymptotically Poisson as the length of the tandem tends to infinity. For ./M/1 nodes a natural approach to this problem would be to prove that Poisson processes are the only stationary ergodic fixed points. A recent contribution to the study of fixed points of the input output map of first come first served queues in a very general setup is due to Bambos and Walrand [2].

The purpose of this paper is to discuss the analogous question in discrete time. To do this, one first needs to settle on a definition of a quasireversible node in discrete time. A natural way of defining the concept of quasireversibility for queueing nodes in discrete time was developed by Walrand, [14]. Walrand first restricts attention to a simple class of discrete-time nodes which he calls S-queues. An S-queue admits batch arrivals and has batch service. Its service process is described via a family of independent and identically distributed random variables which are also independent of the arrival process—one might thus think of this as a virtual departure process. The virtual departure process prescribes the number of customers to release from the queue at each time, conditional on the queue size. Walrand defines such an S-queue to be quasireversible if, when it is fed by a sequence of independent Poisson distributed random variables and a natural rate condition is met, the queue admits a stationary situation in which the departure process is also a sequence of independent Poisson distributed random variables, and the past of the departure process is independent of the present state of the queue. This definition of quasireversibility in discrete time is thus a natural attempt to mimic the usual definition in continuous time. In [14], Walrand first completely characterizes those distributions for the virtual departure process under which an S-queue is quasireversible. He then shows that networks of quasireversible S-queues with Bernoulli routing whose arrival process is a sequence of independent Poisson distributed random variables admit a product form stationary distribution, which validates this as a useful concept.

In Section II, we first recapitulate the definition of an S-queues, following Walrand [14]. We then state Walrand's characterization of which S-queues are quasireversible. As previously stated, this characterization is purely in terms of the distribution of the virtual departure process and does not involve any assumptions on the arrival process. For the rest of the paper, we are interested in the input output map of quasireversible S-queues which are fed by general stationary ergodic arrival sequences, i.e., we make once and for all the assumption that the virtual departure processes

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of our S-queues satisfy the properties identified by Walrand as being necessary and sufficient for quasireversibility and then analyze such queues for general stationary and ergodic arrival processes. The main result in Section II is Lemma 1, which proves a simple property of quasireversible S-queues allowing us to set up a version of the virtual departure process in such a way as to satisfy some nice pathwise properties. In Section III, we introduce the concept of a pre-stationary regime for our queue following Baccelli and Bremaud [1], and use the Loynes construction, [9], to construct the minimal pre-stationary regime. This construction is done in Theorem 1 and is completely standard for readers familiar with Loynes' constructions.

From Section IV onwards, we restrict attention further to a smaller class of quasireversible S-queues, which we call monotone quasireversible nodes. These nodes are characterized by a certain monotonicity condition imposed on the distribution of the virtual departure process, so that once again their definition does not invlove any assumptions on the arrival process. They are an interesting class of nodes, because they include, for example, the natural discrete-time analogs of the ./M/∞ and ./M/1 nodes, see Walrand [14]. In Lemma 2, we prove a property of the virtual departure process of a monotone quasireversible node that allows us to assume a nice pathwise monotonicity property for the virtual departure process. In Theorem 2, we consider monotone quasireversible nodes fed by a general stationary and ergodic arrival process, and are able to give a simple rate condition on the arrival process under which the minimal prestationary regime is a stationary regime. Further, in Theorem 4, we prove the uniqueness of the stationary regime when it exists. In Theorem 6, we show that the departure process in this unique stationary regime is a stationary ergodic process with the same rate as the arrival process and that its distribution does not depend on the choice of the representation for the arrival process. The net conclusion of the development in this section is that we can view a monotone node as a map on stationary ergodic processes which meet the rate condition for stability.

In Section V, we introduce a distance on stationary arrival processes that is similar to the $\bar{d}$ distance studied by Ornstein [10]. This distance is defined via couplings of processes—it is for this reason that we are careful throughout to keep track of the sample spaces on which our processes are defined and to prove the necessary invariance properties of distributions to the choice of sample space. In Section VI, we demonstrate that the input-output map of a monotone quasireversible node is contractive relative to the distance introduced in Section V. This is in some sense the main result of this paper, and is proved in Theorem 8. Finally, in Theorem 9 of Section VII, we are able to state and prove the uniqueness of stationary ergodic fixed point for the input output map of monotone quasireversible nodes as a simple conclusion of this contractiveness property. Some concluding remarks are made in Section VIII, particularly relating the question of iterating the input output map, which corresponds to considering long tandems of identical monotone quasireversible nodes.

II. Discrete-Time Quasireversible Nodes

Let us first formally describe the class of discrete-time queues called S-queues by Walrand [14]. An S-queues has batch arrivals and batch services. Given an arbitrary arrival sequence $\{a_n, n \geq 0\}$ of $\boldsymbol{N} = \{0, 1, 2, \cdots\}$ valued random variables, the queue length process of an S-queue is given by

$$x_{n+1} = x_n + a_n - d_{n+1},$$

where

$$P(d_{n+1} = j \mid x_m, d_m, 0 \leq m \leq n; \quad a_k, k \geq 0, x_n + a_n = i) = S(i,j),$$

for $0 \leq j \leq i$ and $n \geq 0$. Here, x_0 is arbitrary and $d_0 = 0$. Notice from the definition that the operation of an S-queue can be alternately visualized as follows: There is a sequence of independent and identically distributed random variables $(d_n(i), i \geq 1)$, $-\infty < n < \infty$, with $P(d_n(i) = j) = S(i,j)$, $0 \leq j \leq i$, which is independent of the arrival process. This sequence can be thought of as a virtual departure process. At time $n + 1$, if the the queue size just prior to release of the departures is i, we release $d_{n+1}(i)$ customers. An S-queue is thus characterized by its arrival process and its virtual departure process, which are independent.

Mimicking the continuous time concept due to Kelly [8], an S-queue is called quasireversible if, when $\{a_n, n \geq 0\}$ is a Poisson arrival sequence such that the state admits an equilibrium distribution, then the sequence of actual departures $\{d_n, n \geq 0\}$ is Poisson in equilibrium and for all n $\{d_l, l \leq n\}$ and x_n are independent. Walrand [14] gave a necessary and sufficient condition on the distribution of the virtual departure process for an S-queue to be quasireversible. He proved that an S-queue is quasireversible, if and only if $S(i,j)$ has the following form:

$$S(0,0) = c(0) = 1, \tag{2.1a}$$

$$S(i,0) = c(i), \qquad i > 0, \tag{2.1b}$$

$$S(i,j) = \frac{c(i)}{j!}\alpha(i)\alpha(i-1)\cdots\alpha(i-j+1), \qquad 0 < j \leq i, \tag{2.1c}$$

where $\alpha(0) = 1$, $\alpha(j) > 0$ for $j > 0$ and $c(i)$ is such that

$$\sum_{j=0}^{i} S(i,j) = 1.$$

Further, the queue admits an equilibrium distribution π for a Poisson arrival sequence of mean λ, if and only if the normalizing constant c exists such that

$$\pi(i) = c\frac{\lambda^i}{\alpha(0)\cdots\alpha(i)}, i \geq 0$$

is a probability distribution.

Our main aim in this section is to prove a simple property of quasireversible S-queues which allows us to set up the virtual

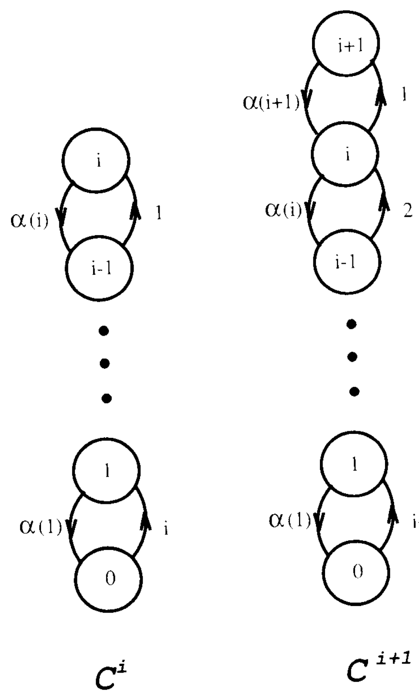

Fig. 1.

departure process so that it satisfies certain nice pathwise properties. This is done in Lemma 1. Note that this lemma simply discusses a property of numbers $S(i,j), 0 \le j \le i, i \ge 0$ which obey (2.1). It might be of independent interest in other contexts.

Lemma 1: Given $\alpha(0) = 1$ and $\alpha(j) > 0$ for $j > 0$, let $S(i,j)$ be as in (2.1). Then one can construct random variables $d(i), i \ge 0$ such that, with $r(i) = i - d(i)$, we have

$$P(d(i) = j) = S(i,j). \tag{2.2}$$

and

$$r(i+1) \ge r(i), \qquad i \ge 0. \tag{2.3}$$

Proof: (See Fig. 1). Let $\mathcal{C}^i$ denote the birth and death chain on $\{0,\cdots,i\}$ with birth rate $i-j$ and death rate $\alpha(j)$ in state j. This chain has stationary distribution $S(i, i-j)$. We couple $\mathcal{C}^i$ to $\mathcal{C}^{i+1}$ starting at 0. Namely, on $\{0,1,\cdots i\} \times \{0,1,\cdots,i+1\}$ we construct a Markov process $((X^i(t), X^{i+1}(t)), t \ge 0)$ with $(X^i(0), X^{i+1}(0)) = (0,0)$ and so that $(X^i(t), t \ge 0)$ (resp. $(X^{i+1}(t), t \ge 0)$) has the distribution of $\mathcal{C}^i$ (resp. $\mathcal{C}^{i+1}$). It is easy to see from the rate diagrams in Fig. 1 that we can construct such a coupling with $X^{i+1}(t) \ge X^i(t)$ for all $t \ge 0$. This is because the birth rates from states $0 \le j \le i$ in the former process are at least as large as those in the latter and the death rates are identical. The existence of such a coupling implies that for any $k \ge 0$

$$\sum_{j=k}^{i+1} S(i+1, i+1-j) \ge \sum_{j=k}^{i} S(i, i-j).$$

Note that with $r(i) = i - d(i)$, this reads

$$P(r(i+1) \ge k) \ge P(r(i) \ge k). \qquad k \ge 0.$$

Clearly, we can now construct $(d(i), i \ge 0)$ with the required distribution so that (2.3) holds pointwise. For example, on the unit interval $[0,1]$ with Lebesgue measure on the Borel σ-algebra, we can define $r(i,\omega) = k$ iff $\sum_{j=0}^{k-1} S(i, i-j) < \omega \le \sum_{j=0}^{k} S(i, i-j)$. □

In view of Lemma 1, we will choose a natural canonical representation of the virtual departure process of our quasireversible node. Let $\Omega_d = [0,1]^{\boldsymbol{Z}}$ with product measure P_d defined on the σ-algebra $\mathcal{F}_d$ generated by cylinder sets, where each factor has Lebesgue measure on the Borel σ-algebra. Given $\omega = (\omega_n, n \in \boldsymbol{Z})$, let $r_n(i,\omega) = k$ iff $\sum_{j=0}^{k-1} S(i, i-j) < \omega_n \le \sum_{j=0}^{k} S(i, i-j)$, and let $d_n(i) = i - r_n(i)$. Then, $((d_n(i), i \ge 0), n \in \boldsymbol{Z})$ are independent and have identical distribution given by (2.2). Further, we can now define $r_n(\infty) = \lim_{i\to\infty} r_n(i)$. Simple algebra shows we must necessarily have $r_n(\infty) = \infty$ a.s.

Note that the transformation $\theta_d : \Omega_d \to \Omega_d$ given by the left shift

$$\theta_d(\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) = (\cdots, \omega_0, \omega_1, \omega_2, \cdots)$$

is an invertible measure preserving transformation on $(\Omega_d, \mathcal{F}_d, P_d)$. Further, this transformation is ergodic. See, for example, Billingsley [3, ch. 1 for the terminology, p. 18 for a proof of ergodicity]. We also note that $d_n(i, \theta_d(\omega)) = d_{n+1}(i,\omega)$ for all $i \ge 0$ and $n \in \boldsymbol{Z}$.

Let $\boldsymbol{N}^{\boldsymbol{Z}}$ denote the set of two-sided infinite sequences of nonnegative integers with the σ-algebra $\mathcal{B}$ generated by the cylinder sets. We think of $\boldsymbol{N}^{\boldsymbol{Z}}$ endowed with the left shift θ_s, and call a measure on $\boldsymbol{N}^{\boldsymbol{Z}}$ stationary, resp. ergodic, if it is so with respect to the left shift. Let $\mathcal{M}_S(\lambda)$ and $\mathcal{M}_S^e(\lambda)$ denote, respectively, the space of stationary measures and the space of stationary ergodic measures on $\boldsymbol{N}^{\boldsymbol{Z}}$ with rate λ.

A stationary arrival process of rate λ into our node is specified in distribution by an element $\mu_a \in \mathcal{M}_S(\lambda)$ ($\mu_a \in \mathcal{M}_S^e(\lambda)$ if it is ergodic). We think of a stationary arrival process as prescribed by an $\boldsymbol{N}$ valued random variable a_0 given on a sample space $(\Omega_a, \mathcal{F}_a, P_a)$ supporting an invertible P_a preserving transformation θ_a. Then $a_n = a_0 \circ \theta_a^n$. Later we may need to use different representations for the same arrival distribution. Of course, each arrival process has a standard representation $(\boldsymbol{N}^{\boldsymbol{Z}}, \mathcal{B}, \mu_a, \theta_s)$ on which it is given by the marginal x_0 at time 0.

Note it is possible to represent $\mu_a \in \mathcal{M}_S^e(\lambda)$ by a *non*-ergodic $(\Omega_a, \mathcal{F}_a, P_a, \theta_a, a_0)$, but it is not possible to represent a nonergodic $\mu_a \in \mathcal{M}_S(\lambda)$ by an ergodic $(\Omega_a, \mathcal{F}_a, P_a, \theta_a, a_0)$. This is an easy consequence of the definition of ergodicity. Thus, we will distinguish between the ergodicity of an arrival process and of its representation. Since the virtual departure process of our node is assumed independent of the arrival process, the node with its arrivals is completely described by a_0 and $(d_1(i), i \ge 0)$ on $(\Omega_a \times \Omega_d, \mathcal{F}_a \times \mathcal{F}_d, P_a \times P_d, \theta_a \times \theta_d)$. We abbreviate this to $(\Omega, \mathcal{F}, P, \theta)$. Note that if θ_a is P_a-ergodic then θ is P-ergodic.

We are somewhat careful about keeping track of the sample space on which our arrival process is defined because we will later need to couple different arrival processes (see Section V or the introductory remarks in Section I). We would like to make explicit the fact that our results about the input output map of a queue, which is a map on *distributions*, do not depend on the choice of the sample space.

III. Minimal Pre-Stationary Regime

We consider now an S-queue with a general stationary and ergodic arrival process. The first question is if it is possible to set up a stationary version of the queue size process, and if so, under what conditions. Yet another question is whether there is unique stationary version consistent with the recursion equations of the system. The usual approach in the Loynes' theory, as developed by Baccelli and Bremaud [1], is to first construct a minimal queue size process consistent with the recursion equations of the queueing system at hand without worrying about whether it is finite, and then to tackle the question of conditions under which it is finite, and condtions under which it is unique. We will pursue this approach. To answer the latter questions we will need to restrict attention to a more restricted class of quasireversible S-queues, to be introduced in the next section. Here, we just describe how to construct a minimal queue size process for a given stationary ergodic arrival process to a quasireversible S-queue. Our node is said to admit a *pre-stationary regime* if there is a nonnegative random variable x_0 (possibly infinite with positive probability) such that

$$x_0 \circ \theta = r_1(x_0 + a_0).$$

The reason for the terminology is the following : If the node admits a pre-stationary regime, let $x_n = x_0 \circ \theta^n$. Then, $(x_n, n \in \boldsymbol{Z})$ is a stationary process satisfying

$$x_{n+1} = r_{n+1}(x_n + a_n), \qquad n \in \boldsymbol{Z}.$$

Thus, it is a stationary queue size process seen by arrivals.

Our first result is the following theorem.

Theorem 1: Every discrete-time quasireversible node admits a pre-stationary regime for any stationary arrival process.

Remark: Note that Theorem 1 as stated is trivially true, since taking $x_0 = \infty$ defines a pre-stationary regime by virtue of the fact that $r_n(\infty) = \infty$ for all $n \in \boldsymbol{Z}$. However, the pre-stationary regime constructed in the proof of Theorem 1 is the *minimal* pre-stationary regime, in that if there is any other pre-stationary regime it must dominate this one pointwise.

Proof: Recall that we are working on $(\Omega, \mathcal{F}, P, \theta)$ defined above so that, while the arrival representation is arbitrary, the virtual departures have a canonical representation and the overall sample space is a product. We use the Loynes construction. See Loynes [9], Baccelli and Bremaud [1, pt. II, ch. 1] and Walrand, [15, ch. 7 for discussions of this construction]. For each $m \geq 0$, we construct a process $(x_n^m, n \geq -m)$ by

$$x_{-m}^m = 0, \tag{3.1a}$$

$$x_{n+1}^m = r_{n+1}(x_n^m + a_n) = x_n^m + a_n - d_{n+1}(x_n^m + a_n). \tag{3.1b}$$

One thinks of x_n^m as the queue size that would be seen by arrivals at time n if the node were started empty at time $-m$. We claim that $x_n^{m+1} \geq x_n^m$ for all $n \geq -m$. Indeed, $x_{-m}^{m+1} \geq 0 = x_{-m}^m$. Suppose $x_{n-1}^{m+1} \geq x_{n-1}^m$. Then,

$$\begin{aligned} x_n^{m+1} &= r_n(x_{n-1}^{m+1} + a_{n-1}) \\ &\geq r_n(x_{n-1}^m + a_{n-1}) \end{aligned}$$

by (2.3), giving the desired. Thus we may define

$$x_n^\infty = \lim_{m \to \infty} x_n^m \tag{3.2}$$

for all $n \in \boldsymbol{Z}$. Since $x_{n+1}^m = x_n^{m+1} \circ \theta$, (3.1b) becomes

$$x_n^{m+1} \circ \theta = r_{n+1}(x_n^m + a_n). \tag{3.3}$$

From (3.2), it follows that

$$x_n^\infty \circ \theta = x_{n+1}^\infty = r_{n+1}(x_n^\infty + a_n).$$

x_0^∞ is the desired pre-stationary regime. □

A pre-stationary regime is said to be a *stationary regime* if the random variable x_0 is a.s. finite. Clearly one needs some kind of rate condition on the arrival process for the existence of a stationary regime. To discuss this we now restrict attention to monotone quasireversible nodes. See Section VIII for some remarks on the need for this restriction.

IV. Monotone Quasireversible Nodes

We will call a discrete-time quasireversible S-queue *monotone* if the sequence $\alpha(i), i \geq 1$ is nondecreasing. Several natural examples of discrete-time nodes are monotone, including the analogs of the ./M/∞ and ./M/1 nodes, see [14]. For a monotone quasireversible node we have the following.

Lemma 2: Let $\alpha(0) = 1$, $0 < \alpha(1) \leq \alpha(2) \leq \cdots$, and let $S(i,j)$ be as in (2.1). Then one can construct random variables $d(i), i \geq 0$ such that, with $r(i) = i - d(i)$, we have (2.2) and (2.3) and also

$$d(i+1) \geq d(i), \qquad i \geq 0. \tag{4.1}$$

Proof: Let $\mathcal{D}^i$ denote the birth and death chain on $\{0, \cdots, i\}$ with birth rate $\alpha(i-j)$ and death rate j in state j. This chain has stationary distribution $S(i,j)$. We couple $\mathcal{D}^i$ to $\mathcal{D}^{i+1}$ starting at 0, i.e., we construct a $\{0, 1, \cdots i\} \times \{0, 1, \cdots, i+1\}$ valued Markov process $((Y^i(t), Y^{i+1}(t)), t \geq 0)$ with $(Y^i(0), Y^{i+1}(0)) = (0,0)$ and so that $(Y^i(t), t \geq 0)$ (resp. $(Y^{i+1}(t), t \geq 0)$) has the distribution of $\mathcal{D}^i$ (resp. $\mathcal{D}^{i+1}$). We can construct such a coupling with $Y^{i+1}(t) \geq Y^i(t)$ for all $t \geq 0$ because the birth rates from states $0 \leq j \leq i$ in the former process are at least as large as those in the latter and the death rates are identical (see Fig. 2). The existence of such a coupling implies that for any $k \geq 0$

$$\sum_{j=k}^{i+1} S(i+1, j) \geq \sum_{j=k}^{i} S(i,j).$$

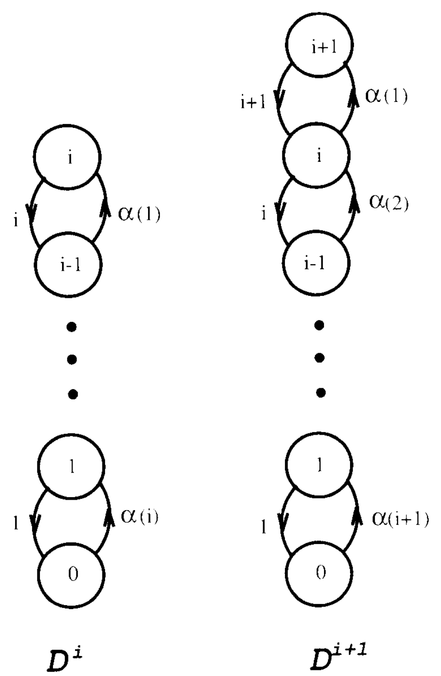

Fig. 2.

We now see that the construction in Lemma 1 of $(d(i), i \geq 0)$ on the unit interval $[0, 1]$ with Lebesgue measure on the Borel σ-algebra has all the desired properties (2.2), (2.3), and (4.1). □

Thus, for a monotone quasireversible node when the virtual departure process has the canonical representation we have the added property that $d_n(i+1) \geq d_n(i)$ for all $i \geq 0$ and $n \in \boldsymbol{Z}$.

For monotone quasireversible nodes, we are able to answer the important question of necessary and sufficient conditions on the stationary ergodic arrival process for the existence of a stationary regime and also the question of uniqueness. The existence condition is just a simple rate condition—in fact it can be intuitively understood as follows: As long as the arrival process has a rate which is small enough that an arrival process of independent Poisson random variables of the same rate would result in a stable system, the monotone quasireversible S-queue with this arrival process will admit a stationary regime. To this end, we introduce

$$\lambda_0 = \sup\{\lambda : \sum_{i=0}^{\infty} \frac{\lambda^i}{\alpha(0)\cdots\alpha(i)} < \infty\}. \tag{4.2}$$

It is easily seen that $\lambda_0 > 0$. We may now state the following theorem.

Theorem 2: Consider a monotone quasireversible node fed by an arrival process $\mu_a \in \mathcal{M}_S^e(\lambda)$ where $\lambda < \lambda_0$. Let $(\Omega_a, \mathcal{F}_a, P_a, \theta_a, a_0)$ be an ergodic representation of μ_a. Then the node admits a stationary regime for this representation.

Proof: We will show that the minimal pre-stationary regime x_0^∞ previously constructed is a.s. finite. First. note that the event $\{x_0^\infty < \infty\}$ is θ-invariant. Since θ is P-ergodic this event has probability 0 or 1. Thus, it suffices to show that x_0^∞ cannot be ∞ with probability 1.

Since $\lambda < \lambda_0$, there is $\epsilon > 0$ such that with an i.i.d. Poisson arrival sequence of rate $\lambda + 2\epsilon$ the number of customers just prior to arrivals evolves as a Markov chain with stationary distribution

$$\pi(i) = c\frac{(\lambda + 2\epsilon)^i}{\alpha(0)\cdots\alpha(i)}, \qquad i \geq 0.$$

See Walrand [14], for a proof. In particular, we have

$$\sum_{i=0}^{\infty} \pi(i) E d_1(i) = \lambda + 2\epsilon. \tag{4.3}$$

Now, Lemma 2 informs us that $Ed_1(i)$ is nondecreasing in i. It follows from (4.3) that there is $K < \infty$ such that

$$Ed_1(i) > \lambda + \epsilon. \qquad \text{for all } i \geq K. \tag{4.4}$$

We now return to the construction of the minimal pre-stationary regime in the proof of Theorem 1. Equation (3.3) reads

$$x_n^{m+1} \circ \theta = x_n^m + a_n - d_{n+1}(x_n^m + a_n).$$

The θ invariance of P and the fact that $x_n^{m+1} \geq x_n^m$, imply that

$$Ex_n^{m+1} \circ \theta = Ex_n^{m+1} \geq Ex_n^m$$

Hence, (3.3) yields

$$\lambda \geq Ed_{n+1}(x_n^m + a_n). \tag{4.5}$$

Now suppose $x_n^\infty = \infty$ a.s. Then, for any $\delta > 0$ we can find $M_\delta < \infty$ so that $P(x_n^m > K) > 1 - \delta$ for all $m \geq M_\delta$. From (4.4) and (4.5), this gives

$$\lambda \geq (\lambda + \epsilon)(1 - \delta).$$

but this is a contradiction for δ sufficiently small. □

For the sake of completeness, we state the following result which shows that the characterization of stability in terms of the rate condition is tight in the sense that there are stationary ergodic arrival processes of rate exceeding λ_0 for which there is no stationary regime.

Theorem 3: A monotone quasireversible node fed by a sequence of i.i.d. Poisson random variables of rate exceeding λ_0 cannot admit a stationary regime in any representation.

Proof: We first note that an equivalent characterization of λ_0 is as

$$\lambda_0 = \lim_{i\to\infty} \alpha(i).$$

Indeed, the limit on the right exists by monotonicity, and if $\lambda < \lim_{i\to\infty} \alpha(i)$ the tail of the summation on the right-hand side of (4.2) is dominated above by a geometrically decreasing series, so $\lambda < \lambda_0$. Conversely, if $\lambda > \lim_{i\to\infty} \alpha(i)$ the tail of the summation on the right-hand side of (4.2) dominates by a geometrically increasing series, so $\lambda > \lambda_0$.

We may assume $\lambda_0 = \lim_{i\to\infty} \alpha(i) < \infty$, else the claim of the theorem is vacuous. Recall that

$$P(d_1(i) = j) = S(i,j) = \frac{c(i)}{j!}\alpha(i)\alpha(i-1)\cdots\alpha(i-j+1), \quad 0 < j \le i,$$

where

$$c(i) = (1 + \alpha(i) + \frac{\alpha(i)\alpha(i-1)}{2} + \cdots + \frac{\alpha(i)\cdots\alpha(1)}{i!})^{-1}.$$

We see that $c(i) \to \exp(-\lambda_0)$ as $i \to \infty$, so that

$$\lim_{i\to\infty} S(i,j) = \frac{(\lambda_0)^j}{j!}\exp(-\lambda_0). \tag{4.6}$$

By Lemma 2, $d_n(i)$ is nondecreasing in i in the canonical representation, so we can define

$$d_n(\infty) = \lim_{i\to\infty} d_n(i).$$

From (4.6), we see that $(d_n(\infty)n \in \boldsymbol{Z})$ is an i.i.d. sequence of Poisson random variables of rate λ_0.

Suppose we feed the node with a sequence of i.i.d. Poisson random variables of rate $\lambda > \lambda_0$ in any representation. We return to the construction of the minimal prestationary regime in the proof of Theorem 1. Let us define random variable $(z_n^m, n \ge -m)$ for each $m \ge 0$ by

$$z_{-m}^m = 0,$$

$$z_{n+1}^m = z_n^m + a_n - d_{n+1}(\infty).$$

We claim that $z_n^m \le x_n^m$ for all $n \ge -m$. Indeed $z_{-m}^m = x_{-m}^m = 0$. Suppose $z_n^m \le x_n^m$. Then,

$$\begin{aligned} z_{n+1}^m &= z_n^m + a_n - d_{n+1}(\infty) \\ &\le x_n^m + a_n - d_{n+1}(\infty) \\ &\le x_n^m + a_n - d_{n+1}(x_n^m + a_n) \\ &= x_{n+1}^m, \end{aligned}$$

see (3.1b). Now, it is obvious that

$$\lim_{m\to\infty} z_n^m = \infty,$$

for all $n \in \boldsymbol{Z}$. Indeed, z_n^m is the sum of $n-m+1$ independent random variables of positive mean and finite fourth moment, so the conclusion follows by a simple application of the Markov inequality and the Borel Cantelli Lemma, see e.g., Billingsley [4]. Hence,

$$x_n^\infty = \lim_{m\to\infty} x_n^m = \infty,$$

for all $n \in \boldsymbol{N}$. Since the minimal pre-stationary regime is infinity a.s. there cannot be any stationary regime. □

Note that if an ergodic arrival process satisfying the rate condition of Theorem 2 is given in a nonergodic representation, we may first construct a stationary regime on $(\Omega^*, \mathcal{F}^*, P^*, \theta^*)$ which is the product of the canonical representation of the arrival process with the canonical representation of the virtual departure process. We then take the composition of this stationary regime with the map

$$\Phi : (\Omega, \mathcal{F}, P, \theta) \to (\Omega^*, \mathcal{F}^*, P^*, \theta^*)$$

given by $\Phi((\omega_n)_n) = (a_n(\omega), d_n(\omega))_n$ to get a stationary regime on $(\Omega, \mathcal{F}, P, \theta)$. For a nonergodic arrival process, the rate condition is not enough to guarantee the existence of a stationary regime — for example a mixture of a sequence of i.i.d. Poisson random variables with rate exceeding λ_0 and one with rate below λ_0 can meet rate condition but cannot be supported by our monotone node.

The next natural question is whether a monotone node can admit more than one stationary regime for a representation of an ergodic arrival process. This is answered by the next result.

Theorem 4: Given a monotone quasireversible node, let $\lambda < \lambda_0$ and consider an arrival process $\mu_a \in \mathcal{M}_S^e(\lambda)$. Let $(\Omega_a, \mathcal{F}_a, P_a, \theta_a, a_0)$ be an ergodic representation of μ_a. Then, the stationary regime guaranteed by Theorem 2 is unique.

Proof: Let x_0^∞ be the minimal pre-stationary regime constructed in the proof of Theorem 1, which we know by Theorem 2 is a stationary regime, and let $\tilde{x}_0$ be any other stationary regime. Since x_0^∞ is minimal, we have $\tilde{x}_0 \ge x_0^\infty$ a.s. We write

$$\tilde{x}_0 \circ \theta = r_1(\tilde{x}_0 + a_0) \tag{4.7}$$

and

$$x_0^\infty \circ \theta = r_1(x_0^\infty + a_0). \tag{4.8}$$

From (4.7), (4.8), and (4.1), we get

$$\tilde{x}_0 \circ \theta - x_0^\infty \circ \theta \le \tilde{x}_0 - x_0^\infty.$$

It follows that for any $0 \le K < \infty$, the event $\{\tilde{x}_0 - x_0^\infty \le K\}$ is θ-invariant and so must have probability 0 or 1. Since $\tilde{x}_0$ and x_0^∞ are both a.s. finite, there must be some $0 \le K < \infty$ such that $\tilde{x}_0 = x_0^\infty + K$. The theorem would be proved if we could show that $K = 0$.

Note that by the construction of x_0^∞ we have $x_0^\infty + a_0$ is independent of $(d_1(i), i \ge 0)$. Since x_0^∞ is a.s. finite, there is some $M < \infty$ such that $P(x_0^\infty + a_0 = M) > 0$. We get

$$\begin{aligned} P(\tilde{x}_0 \circ \theta = 0) &\ge P(\tilde{x}_0 + a_0 = M + K, d_1(M+K) \\ &\qquad = M + K) \\ &= P(x_0^\infty + a_0 = M, d_1(M+K) = M + K) \\ &= P(x_0^\infty + a_0 = M)P(d_1(M+K) \\ &\qquad = M + K) \\ &> 0. \end{aligned}$$

It follows that $P(\tilde{x}_0 = x_0^\infty) > 0$, so K must equal 0, completing the proof. □

If an arrival process $\mu_a \in \mathcal{M}_S^e(\lambda)$ with $\lambda < \lambda_0$ is given in a nonergodic representation, the events $\{\tilde{x}_0 = x_0^\infty + K\}$, $0 \le K < \infty$, are θ-invariant as before. Let $0 \le K < \infty$ be such that $P(\tilde{x}_0 = x_0^\infty + K) > 0$. Let $(\Omega^*, \mathcal{F}^*, P^*, \theta^*)$ denote the product of the canonical representation of the arrival process with the canonical representation of the virtual departure process. Let

$$\Phi : (\Omega, \mathcal{F}, P, \theta) \to (\Omega^*, \mathcal{F}^*, P^*, \theta^*)$$

be given by $\Phi((\omega_n)_n) = (a_n(\omega), d_n(\omega))_n$. Since $\Phi(\theta(\omega)) = \theta^*(\Phi(\omega))$, we see that $\Phi(\{\tilde{x}_0 = x_0^\infty + K\})$ is θ^*-invariant, and because $P(\{\tilde{x}_0 = x_0^\infty + K\}) > 0$, it must have probability 1. Consider the process $\tilde{z}_n = z_0^\infty + K$ on $(\Omega^*, \mathcal{F}^*, P^*, \theta^*)$, where z_0^∞ denotes the minimal pre-stationary regime on $(\Omega^*, \mathcal{F}^*, P^*, \theta^*)$. Then, $\tilde{x}_0 = \tilde{z}_0 \circ \Phi$ on $\{\tilde{x}_0 = x_0^\infty + K\})$, so (4.8) implies that $\tilde{z}_0$ is a stationary regime on $(\Omega^*, \mathcal{F}^*, P^*, \theta^*)$. But this is impossible unless $K = 0$.

Suppose we are given an ergodic representation of an arrival process $\mu_a \in \mathcal{M}_S^e(\lambda)$ with $\lambda < \lambda_0$. We wish to examine the stationary departure process from our monotone node in the unique stationary regime. We would like to know that the stationary departure process has rate λ and that it is uniquely specified in distribution independent of the representation. This is the content of the remaining two results in this section. Together they permit us to think of the node as defining a map T on $\mathcal{M}_S^e(\lambda)$. Properties of this map are studied in Section VI.

Theorem 5: Let the situation be as in the statement of Theorem 4. Let x_0^∞ denote the unique stationary regime, and let $(d_n(x_n^\infty + a_n), n \in \boldsymbol{Z})$ be the departure process in the stationary regime. Then this process is stationary and ergodic with mean λ, so it defines an element of $\mathcal{M}_S^e(\lambda)$.

Proof: We write

$$x_n^{m+1} \circ \theta = x_{n+1}^m = x_n^m + a_n - d_{n+1}(x_n^m + a_n). \quad (4.9)$$

Since $x_n^{m+1} \geq x_n^m$, taking expectations gives

$$\lambda = Ea_n \geq Ed_{n+1}(x_n^m + a_n),$$

for all $m \geq 0$. Since $d_{n+1}(x_n^m + a_n)$ increases monotonically to $d_{n+1}(x_n^\infty + a_n)$ as $m \to \infty$, we learn that it is integrable and $Ed_{n+1}(x_n^\infty + a_n) \leq \lambda$. But subtracting (4.9) for m from (4.9) for $m + 1$ gives

$$\begin{aligned} x_{n+1}^{m+1} - x_{n+1}^m &= r_{n+1}(x_n^{m+1} + a_n) - r_{n+1}(x_n^m + a_n) \\ &\leq x_n^{m+1} - x_n^m \end{aligned}.$$

where the second equality follows from (4.1). Iterating this inequality for decreasing n gives

$$x_{n+1}^{m+1} - x_{n+1}^m \leq a_{-m-1}.$$

for all $n \geq -m-1$. It follows that for any fixed n the random variables $(x_n^{m+1} - x_n^m, m \geq 0)$ are uniformly integrable. Further, since x_n^m increases monotonically to the a.s. finite random variable x_n^∞ as $m \to \infty$, we know that $x_n^{m+1} - x_n^m \to 0$ a.s. as $m \to \infty$. Together with uniform integrability this implies that $E(x_n^{m+1} - x_n^m) \to 0$ as $m \to \infty$. Returning to (4.9), taking expectations and the limit as $m \to \infty$, we conclude that $Ed_{n+1}(x_n^\infty + a_n) = \lambda$. The stationarity of the process $(d_n(x_n^\infty + a_n), n \in \boldsymbol{Z})$ is obvious by taking the limit as $m \to \infty$ in the equation

$$(d_n(x_n^m + a_n)) \circ \theta = d_{n+1}(x_{n+1}^{m-1} + a_{n+1}).$$

Ergodicity follows from the general fact that if $(\Omega_i, \mathcal{F}_i, P_i, \theta_i)$, $i = 1, 2$ are probability spaces where θ_i are invertible transformations preserving P_i, $i = 1, 2$, and $S : \Omega_1 \to \Omega_2$ is a measurable map such that $S(\theta_1(\omega)) = \theta_2(S(\omega))$ and $P_2(B) = P_1(S^{-1}(B))$ for all $B \in \mathcal{F}_2$, then the P_1-ergodicity of θ_1 implies the P_2-ergodicity of θ_2. To conclude the ergodicity of the distribution μ_d of $(d_n(x_n^\infty + a_n), n \in \boldsymbol{Z})$ take $(\Omega_1, \mathcal{F}_1, P_1, \theta_1) = (\Omega, \mathcal{F}, P, \theta)$ and $(\Omega_2, \mathcal{F}_2, P_2, \theta_2)$ to be $(\boldsymbol{N}^{\boldsymbol{Z}}, \mathcal{B}, \mu_d, \theta_s)$. □

In this theorem, note that we have not argued the integrability of x_0^∞. Indeed, we do not know if this need always be true. By analogy with the Loynes construction for the workload process seen by arrivals in a FCFS queue with general stationary ergodic interarrival and service time process, see e.g., [1], there is no reason to expect this to be the case.

Theorem 6: Let the situation be as in Theorem 5. Then the distribution of $(d_n(x_n^\infty + a_n), n \in \boldsymbol{Z})$ does not depend on the choice of the representation $(\Omega_a, \mathcal{F}_a, P_a, \theta_a, a_0)$ of the arrival process.

Proof: Let $(\Omega^*, \mathcal{F}^*, P^*, \theta^*)$ denote the product of the canonical representation of the arrival process with the canonical representation for the virtual departure process of the node. Clearly there is a measurable map S from $(\Omega, \mathcal{F}, P, \theta)$ to $(\Omega^*, \mathcal{F}^*, P^*, \theta^*)$ with the properties stated at the end of the proof of Theorem 5. The minimal pre-stationary regime on $(\Omega^*, \mathcal{F}^*, P^*, \theta^*)$ can be composed with S to give a stationary regime on $(\Omega, \mathcal{F}, P, \theta)$. By Theorem 4, this must coincide with the minimal pre-stationary regime on $(\Omega, \mathcal{F}, P, \theta)$. The conclusion is now obvious. □

The conclusion of the discussion in this section is that for each $\lambda < \lambda_0$ we can define a map $T : \mathcal{M}_S^e(\lambda) \to \mathcal{M}_S^e(\lambda)$ by letting $T(\mu_a) = \mu_d$, where μ_d is the distribution of the departure process in the unique stationary regime associated to any ergodic representation of the arrival process μ_a. We know that the process of independent Poisson random variables is a fixed point of T. Our goal is to prove that there is no other fixed point.

V. A Metric on Arrival Processes

In this section, we introduce a metric on $\mathcal{M}_S(\lambda)$, which is analogous to the $\bar{d}$ distance introduced by Ornstein, [10]. Our metric is an example of a family of such generalizations introduced by Gray, Neuhoff, and Shields [7], who were motivated by application to information theory. Gray *et al.* call their generalizations $\bar{\rho}$ distances, so we will use the notation $\bar{\rho}$ for our distance. The basic properties of the $\bar{\rho}$ distance are derived in [7] and also in Gray [6, ch. 8].

On $\boldsymbol{N}$ we consider the metric $\rho(u, v) = | u - v |$. Let $\mu, \nu \in \mathcal{M}_S(\lambda)$. Consider $\boldsymbol{N}^{\boldsymbol{Z}} \times \boldsymbol{N}^{\boldsymbol{Z}}$ with the σ-algebra $\mathcal{B} \times \mathcal{B}$ and the left shift $\theta_s \times \theta_s$. Let $\mathcal{M}_S(\lambda, \lambda)$ denote the space of stationary probability distributions on this space whose each marginal has rate λ. A *stationary coupling* $\mathcal{C}$ of μ and ν is specified by giving a measure $\alpha \in \mathcal{M}_S(\lambda, \lambda)$ with marginals μ and ν, respectively.

Denote a generic element of $\boldsymbol{N}^{\boldsymbol{Z}} \times \boldsymbol{N}^{\boldsymbol{Z}}$ by $(u_n, v_n)_n$. The $\bar{\rho}$ distance of μ and ν is defined as

$$\bar{\rho}(\mu, \nu) = \inf_{\alpha} \alpha(\rho(u_0, v_0)). \tag{5.1}$$

Thus, one considers the expected value of the difference between the number of arrivals in the first process and the second process at any time and takes the infimum over all stationary couplings.

First, note the following.

Lemma 3: $\mathcal{M}_S(\lambda)$ and $\mathcal{M}_S(\lambda, \lambda)$ are compact in the weak topology.

Proof: Consider a sequence of measures $(\mu_k)_k \in \mathcal{M}_S(\lambda)$. An easy application of the Markov inequality for each marginal shows that the finite dimensional distributions are tight, see Billingsley [4], or Parthasarathy [11], for a definition of tightness and other concepts connected with weak convergence. Thus each finite-dimensional distribution converges along a subsequence. A diagonal argument shows that there is a subsequence along which all the finite-dimensional distributions converge. This establishes the compactness of $\mathcal{M}_S(\lambda)$. A similar argument works for $\mathcal{M}_S(\lambda, \lambda)$. □

The properties of $\bar{\rho}$ relevant to our discussion are summarized in the following result.

Theorem 7: The $\bar{\rho}$ distance on $\mathcal{M}_S(\lambda)$ previously introduced has the following properties.

a) $\bar{\rho}$ is a metric.
b) The infimum in the definition (5.1) is a minimum, i.e., there is stationary coupling that achieves the $\bar{\rho}$ distance.
c) If $\mu, \nu \in \mathcal{M}_S^e(\lambda)$, the infimum in (5.1) can be replaced by an infimum over stationary ergodic α. Further, this infimum is a minimum.
d) Let $(\mu_k, k \geq 1), \mu \in \mathcal{M}_S(\lambda)$. Suppose $\bar{\rho}(\mu_k, \mu) \to 0$ as $k \to \infty$. Then $\mu_k \to \mu$, where the convergence is in the weak topology of $\mathcal{M}_S(\lambda)$.
e) Let $\mu_k \to \mu$ in the weak topology of $\mathcal{M}_S(\lambda)$. Then, for any $\nu \in \mathcal{M}_S(\lambda)$, we have

$$\liminf_{k \to \infty} \bar{\rho}(\mu_k, \nu) \geq \bar{\rho}(\mu, \nu).$$

Proof: Statements a) and b), and 3c) are proved, for example, in Gray [6, Theorem 8.3.1]. To prove d), let $\alpha_k \in \mathcal{M}_S(\lambda, \lambda)$ achieve the minimum in the definition of $\bar{\rho}(\mu_k, \mu)$. We first observe that by virtue of the compactness of $\mathcal{M}_S(\lambda)$ $(\mu_k)_k$ is a tight family of probability distributions. Hence it is enough to show that the finite dimensional distributions of μ_k converge to those of μ. This would follow if we could show for each $N \geq 1$ that $\mu_k(f) \to \mu(f)$ for all bounded functions f on $\boldsymbol{N}^{\boldsymbol{Z}}$ which only depend on the coordinates in $\{-N, \cdots, N\}$, because we could then apply this to the indicator functions of sequences restricted to take on specific values at the coordinates in $\{-N, \cdots, N\}$, thereby concluding the desired limit for finite-dimensional distributions. Given such an f, consider $g((u_n)_n, (v_n)_n) = \mid f((u_n)_n) - f((v_n)_n) \mid$ on $\boldsymbol{N}^{\boldsymbol{Z}} \times \boldsymbol{N}^{\boldsymbol{Z}}$. Clearly, we can find $M < \infty$ so that $g \leq M \sum_{n=-N}^{N} \mid u_n - v_n \mid$. Hence, we have

$$\mid \mu_k(f) - \mu(f) \mid \leq \alpha_k(g) \leq M \bar{\rho}(\mu_k, \mu) \to 0, \quad \text{as } k \to \infty,$$

from which d) follows.

To prove e), let $\alpha_k \in \mathcal{M}_S(\lambda, \lambda)$ achieve the minimum in the definition of $\bar{\rho}(\mu_k, \nu)$. By Lemma 3, $\alpha_k \to \alpha \in \mathcal{M}_S(\lambda, \lambda)$ along a subsequence. Clearly α has first marginal μ and second marginal ν. Now, consider the bounded continuous functions on $\boldsymbol{N}^{\boldsymbol{Z}} \times \boldsymbol{N}^{\boldsymbol{Z}}$ given by $g_K = \min(\mid u_0 - v_0 \mid, K)$, $K \geq 1$. Then $\alpha_k(g_K) \to \alpha(g_K)$ along the same subsequence, so $\liminf \alpha_k(\mid u_0 - v_0 \mid) \geq \alpha(g_K)$ along the subsequence. Taking $K \to \infty$ and observing that $\alpha(\mid u_0 - v_0 \mid) \geq \bar{\rho}(\mu, \nu)$ gives the desired. □

VI. Contractiveness of Input–Output Map

The central observation of this paper is the following theorem.

Theorem 8: Let $\mu, \nu \in \mathcal{M}_S^e(\lambda)$, $\mu \neq \nu$. Then

$$\bar{\rho}(T(\mu), T(\nu)) < \bar{\rho}(\mu, \nu).$$

Proof: Since $\mu \neq \nu$, by Theorem 7a), we know that $\bar{\rho}(\mu, \nu) > 0$. Let $(\boldsymbol{N}^{\boldsymbol{Z}} \times \boldsymbol{N}^{\boldsymbol{Z}}, \mathcal{B} \times \mathcal{B}, \alpha, \theta_s \times \theta_s, (u_0, v_0))$ be a stationary ergodic coupling achieving $\bar{\rho}(\mu, \nu)$. The existence of such a coupling is ensured by Theorem 7b). Taking the product with the canonical representation of the virtual departure process gives $(\Omega, \mathcal{F}, P, \theta)$ supporting $(a_0, \tilde{a}_0) = (u_0, v_0)$ and $(d_1(i), i \geq 0)$. Observe that $P(a_0 = \tilde{a}_0) < 1$ because $\bar{\rho}(\mu, \nu) > 0$.

We jointly construct the minimal pre-stationary regimes for the two arrival processes under consideration following the Loynes scheme and a coloring idea. Let $a_0^Y = \min(a_0, \tilde{a}_0)$, $a_0^R = a_0 - a_0^Y$, and $a_0^B = \tilde{a}_0 - a_0^Y$. Note that

$$\begin{aligned} \bar{\rho}(\mu, \nu) &= E[\mid a_0 - \tilde{a}_0 \mid] \\ &= E[a_0 + \tilde{a}_0 - 2\min(a_0, \tilde{a}_0)] = 2\lambda - 2Ea_0^Y. \end{aligned}$$

Let $(a_n, \tilde{a}_n, a_n^Y, a_n^R, a_N^B) = (a_0, \tilde{a}_0, a_0^Y, a_0^R, a_0^B) \circ \theta^n$. then $a_n = a_n^R + a_n^Y$ and $\tilde{a}_n = a_n^B + a_n^Y$. Think of a_n^Y arrivals colored yellow, a_n^R arrivals colored red and a_n^B arrivals colored blue at time n. Note that $a_0^R a_0^B = 0$.

For each $m \geq 0$, we construct random variables $(z_n^m, x_n^{m,Y}, x_n^{m,R}, x_n^{m,B}, x_n^m, \tilde{x}_n^m, n \geq -m)$. Let

$$z_{-m}^m = x_{-m}^{m,Y} = x_{-m}^{m,R} = x_{-m}^{m,B} = x_{-m}^m = \tilde{x}_{-m}^m = 0,$$

$$z_{n+1}^m = r_{n+1}(z_n^m + a_n^Y)$$

$$x_{n+1}^{m,Y} = r_{n+1}(x_n^{m,Y} + a_n^Y + \min(x_n^{m,R}, a_n^B) + \min(x_n^{m,B}, a_n^R)) \tag{6.1}$$

$$x_{n+1}^{m,R} = r_{n+1}(x_n^m + a_n) - x_{n+1}^{m,Y}$$

$$x_{n+1}^{m,B} = r_{n+1}(\tilde{x}_n^m + \tilde{a}_n) - x_{n+1}^{m,Y}$$

$$x_{n+1}^m = r_{n+1}(x_n^m + a_n)$$

$$\tilde{x}_{n+1}^m = r_{n+1}(\tilde{x}_n^m + \tilde{a}_n).$$

The interpretation of z_n^m is the number of customers in the node at time n if it is started empty at time $-m$ and fed with only yellow customers. The rest of the variables involve a recoloring procedure. At any time the node can have yellow and red customers or yellow and blue customers, but never simultaneously have red and blue customers. When arrivals come in the yellow arrivals are added on to the existing yellow customers, as many red arrivals as possible are merged with any existing blue customers, becoming yellow customers (a red arrival and a existing blue customer merge into a single customer) and as many blue arrivals as possible are merged with the existing red customers, becoming yellow customers. After the merging procedure, it is once again true that the node has either yellow and red customers or yellow and blue customers but cannot have both red and blue customers. Further, if it has no blue customers the yellow customers represent the situation with the second arrival process and if it has no red customers the yellow customers represent the situation with the first arrival process. Now, we examine the virtual departure variable corresponding to the total number of yellow customers and to the total number of customers respectively. The latter is no smaller than the former, and the difference between them is bounded by the number of nonyellow customers, by virtue of Lemma 2. We release as many yellow customers as required by the former, and release as many nonyellow customers as required by the difference between the latter and the former. Now, we have determined the state of the node at the next time instant. Thus, $x_n^{m,Y}$ is the number of yellow customers in the node, $x_n^{m,R}$ the number of red customers, $x_n^{m,B}$ the number of blue customers, x_n^m the total number of customers corresponding to the first arrival process and $\tilde{x}_n^m$ the total number of customers corresponding to the second arrival process, all at time n when the node is started empty at time $-m$.

We claim that z_n^m, $x_n^{m,Y}$, x_n^m and $\tilde{x}_n^m$ are nondecreasing in m for each fixed n for which they are defined, i.e., $n \geq -m$. This is immediate from Theorem 1 for all these variables except $x_n^{m,Y}$. To see this, first note that $x_{-m}^{m+1,Y} \geq 0 = x_{-m}^{m,Y}$. Now suppose $x_{n-1}^{m+1,Y} \geq x_{n-1}^{m,Y}$ for some $n \geq -m$. Either $a_{n-1}^R = 0$ or $a_{n-1}^B = 0$; assume the former. Then, from (6.1), we have

$$\begin{aligned} x_n^{m+1,Y} &= r_n(x_{n-1}^{m+1,Y} + a_{n-1}^Y + \min(x_{n-1}^{m+1,R}, a_{n-1}^B)) \\ &= r_n(a_{n-1}^Y + \min(x_{n-1}^{m+1}, x_{n-1}^{m+1,Y} + a_{n-1}^B)) \\ &\geq r_n(a_{n-1}^Y + \min(x_{n-1}^{m}, x_{n-1}^{m,Y} + a_{n-1}^B)) \\ &= x_n^{m,Y} \end{aligned}$$

where we have used the induction hypothesis in the third step.

Thus, we can define z_n^∞, $x_n^{\infty,Y}$, x_n^∞ and $\tilde{x}_n^\infty$ by taking the pointwise limit as $m \to \infty$. We can also define $x_n^{\infty,R} = x_n^\infty - x_n^{\infty,Y}$ and $x_n^{\infty,B} = \tilde{x}_n^\infty - x_n^{\infty,Y}$. Clearly z_0^∞ is the unique stationary regime corresponding to the yellow arrivals, x_0^∞ the unique stationary regime corresponding to the sum of yellow and red arrivals, i.e., to the first arrival process, and $\tilde{x}_0^\infty$ the unique stationary regime corresponding to the sum of yellow and blue arrivals, i.e., to the second arrival process. The departure process corresponding to the first arrival process is $(d_n(x_{n-1}^\infty + a_{n-1}), n \in \mathbf{Z})$ and that corresponding to the second arrival process is $(d_n(\tilde{x}_{n-1}^\infty + \tilde{a}_{n-1}), n \in \mathbf{Z})$.

Let the number of yellow departures in the stationary regime at time n be denoted

$$\begin{aligned} d_n^Y = d_n(x_{n-1}^{\infty,Y} &+ a_{n-1}^Y \\ &+ \min(x_{n-1}^{\infty,R}, a_{n-1}^B) + \min(x_{n-1}^{\infty,B}, a_{n-1}^R)). \end{aligned} \tag{6.2}$$

The process of red departures is denoted

$$d_n^R = d_n(x_{n-1}^\infty + a_{n-1}) - d_{n-1}^Y$$

and the process of blue departures is denoted

$$d_n^B = d_n(\tilde{x}_{n-1}^\infty + \tilde{a}_{n-1}) - d_{n-1}^Y.$$

Note that the sample space $(\Omega, \mathcal{F}, P, \theta)$ now supports a natural stationary coupling between distributions $T(\mu)$ and $T(\nu)$. Hence, we get

$$\bar{\rho}(T(\mu), T(\nu)) \leq 2\lambda - 2Ed_n^Y.$$

To prove the claim, it suffices to establish that $Ed_n^Y > Ea_n^Y$. Further, observing that $d_n(z_{n-1}^\infty + a_{n-1}^Y)$ is the departure process corresponding to the yellow arrivals, so that, by Theorem 5 $Ed_n(z_{n-1}^\infty + a_{n-1}^Y) = Ea_n^Y$, we see that it is enough to prove $Ed_n^Y > Ed_n(z_{n-1}^\infty + a_{n-1}^Y)$. Note that (6.2) already implies that $Ed_n^Y \geq Ed_n(z_{n-1}^\infty + a_{n-1}^Y)$.

Since $P(a_0 = \tilde{a}_0) < 1$ and because μ and ν are both ergodic, it follows that there must exist some $1 \leq N < \infty$ such that

$$\begin{aligned} P(a_n^R > 0, a_m^R = a_m^B = 0, \quad &\text{for all } n+1 \\ &\leq k \leq n+N-1, a_{n+N}^B > 0) > 0. \end{aligned} \tag{6.3}$$

Indeed, $P(a_0 = \tilde{a}_0) < 1$ implies $P(a_n^R > 0) > 0$ and also $P(a_n^B > 0) > 0$. Consider $\cup_{m=-\infty}^k \{a_m^R > 0\}$, which is mapped into itself by the left shift. Since it is an event determined by an ergodic distribution, and has nonzero probability, it has probability 1. So, for some $l < k+1$ we have $P(a_l^R > 0, a_{k+1}^B > 0) > 0$. From this, it is a simple matter to conclude (6.3).

From (6.3) and the fact that the stationary regimes are a.s. finite, we conclude that there must be $0 < M_1 < \infty$, $0 < M_2 < \infty$, and $0 < M_3 < \infty$ such that

$$\begin{aligned} P(&\max(x_n^\infty + a_n, \tilde{x}_n^\infty + \tilde{a}_n) \\ &= M_1, a_n^R = M_2, a_m^R = a_m^B = 0, \\ &\text{for all } n+1 \leq k \leq n+N-1, a_{n+N}^B = M_3) > 0. \end{aligned}$$

Since the variables involved in the above event are all functions of the arrival processes and of the virtual departure process upto time n, they are independent of the variables $(d_k(i), i \geq 0), k \geq n+1)$. Hence, there is a positive probability that in addition to the event above we also have $\{d_{n+1}(M_1) = 0, d_{n+2}(M_1) = 0, \cdots, d_{n+N-1}(M_1) = 0\}$. But this implies that there is a positive probability that there is a merge at one of the times from n to $n+N$, i.e., the creation of a new yellow from a red and a blue. Hence, the rate at which yellows leave the node must strictly exceed the rate at which they enter, which was to be established. □

VII. Main Result

We are now in a position to state and prove the result claimed in the introduction.

Theorem 9: Consider a monotone discrete-time quasireversible node. Let $\lambda < \lambda_0$, and let $\mu \in \mathcal{M}_S^e(\lambda)$ be such that $T(\mu) = \mu$, i.e., μ is a stationary ergodic fixed point of the input output map. Then μ is the distribution of an independent sequence of Poisson random variables of rate λ.

Proof: Let μ^0 denote the distribution of an independent sequence of Poisson random variables. We know that $T(\mu^0) = \mu^0$, see [14]. Suppose $\mu \neq \mu^0$. Since $\bar{\rho}$ is a metric by Theorem 7 (i), we have $\bar{\rho}(\mu, \mu^0) > 0$. By Theorem 8, $\bar{\rho}(\mu, \mu^0) > \bar{\rho}(T(\mu), T(\mu^0)) = \bar{\rho}(\mu, \mu^0)$. This is an absurdity. Hence $\mu = \mu^0$. □

VIII. Concluding Remarks

Lack of monotonicity of the quasireversible node in the above would cause Lemma 2 to fail, so that the entire structure of the proof would collapse. Note that in the absence of monotonicity we do not even know if the existence of a stationary regime is ensured by a simple rate condition as in Theorem 2. Nevertheless, it is difficult to imagine how the result can fail to be true even in the absence of monotonicity.

Another question left unanswered at the moment is the interesting one of what happens when a stationary ergodic arrival process meeting the rate condition is put through a long tandem of identical monotone quasireversible nodes. One would expect that the stationary departure process converges weakly to the unique fixed point as the length of the tandem goes to infinity. In the previous notation, to prove this we would have to show that $T^k(\mu) \to \mu^0$ as $k \to \infty$, where T^k denotes T iterated k times, and μ^0 is as defined in Theorem 9. Let $K \subset \mathcal{M}_S(\lambda)$ denote the set of weak limit points of the sequence $(T^k(\mu), k \geq 1)$. K is closed, and hence compact by Lemma 3. Since $\mu \to \bar{\rho}(\mu, \mu^0)$ is a lower semicontinuous function on the compact set K, it attains it minimum at some $\tilde{\mu}$. Suppose $\tilde{\mu} \neq \mu^0$. Then, by Theorem 7a), $\bar{\rho}(\tilde{\mu}, \mu^0) > 0$. If we could show that $\tilde{\mu} \in \mathcal{M}_S^e(\lambda)$ then we arrive at a contradiction by Theorem 8, and it is a simple step to show that $(T^k(\mu))_k$ converges weakly to μ^0. The problem is that it is *a priori* possible for the sequence of ergodic processes $(T^k(\mu))_k$ to have a nonergodic subsequential limit, although this seems extremely unlikely in the problem at hand. We are currently attempting to resolve this issue.

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