A HYDRODYNAMIC LIMIT FOR A LATTICE CARICATURE OF DYNAMIC ROUTING IN CIRCUIT SWITCHED NETWORKS

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ABSTRACT

In simulations circuit switched networks with dynamic alternate routing exhibit hysteresis phenomena, which suggest that under dynamic routing there may be more than one stable regime of operation for the same offered traffic. This possibility also shows up in some analytical models of dynamic routing : one can write ODE limits for network occupancy probabilities which admit multiple equilibrium points for certain ranges of parameters. These ODE limits average out the spatial variation of the network. We attempt to preserve the spatial characteristics by considering a lattice model of dynamic routing. We derive a hydrodynamic equation for this lattice model. This is an integro-differential equation which describes how the spatial occupancy profile of the network evolves over time, and it admits multiple spatially homogeneous equilibrium solutions for certain ranges of the parameters. These solutions may be loosely thought of as the different operating regimes. Using this equation one can study questions like "for what parameter values is a hot spot of heavy loading in the system likely to take over the whole network by knock-on effects ?"

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* Research supported by NSF under NCR 8710840 and a PYI award, by IBM under a Faculty Development Award, and by BellCore Inc.

1. INTRODUCTION

Dynamic routing schemes in networks adaptively adjust traffic patterns in response to demand, so as to make better use of spare capacity, and to provide robustness to failures or overloads. Such schemes have been the topic of considerable recent interest, primarily because it has only recently been possible to implement them in practice, and because they offer improved performance over the traditional hierarchical routing schemes.

A difficulty associated with dynamic routing schemes is the potential for metastable states. Several simulation based studies of such routing schemes have revealed the existence of hysteresis phenomena, suggesting that the network may have several qualitatively different regimes of operation for the same offered traffic, spending long periods of time in one or the other regime and sometimes moving from one to the other in response to fluctuations in the demand. Intuitively a situation where most calls are using alternate routes is likely to persist for a while because arriving calls will then find the network close to saturation and will be unable to make their direct connections. On the other hand, for the same offered traffic, it might also be the case that if most of the calls in progress are using their direct route, arriving calls will be able to make their direct connection. Important performance characteristics of the network such as blocking probabilities typically differ considerably between regimes. All the same, the improvement in performance over hierarchical routing schemes is such that dynamic routing schemes are being implemented in real world networks, with control schemes such as trunk reservation for directly routed traffic. These, if suitably chosen, mitigate the effects of the potential multiplicity of operating regimes.

The possibility of metastable regimes of operation is also predicted in analytical models for dynamic routing such as the ones studied by Kelly, [7], Krupp, [8], Marbukh, [10], [11], and Gibbens, Hunt and Kelly, [6]. In [7] and [8], simple fixed point approximations for the blocking probability are written, and it is found they have multiple solutions for certain ranges of the parameters. The models in [6] and [10], [11], are more detailed. ODE limits are found for the fraction of network links that are in a given state as the network size becomes large. In Section 2 we briefly discuss the model of Gibbens, Hunt and Kelly, which primarily motivated this work.

In this paper we report on an attempt to understand the interaction between the operating regimes using particle system techniques, [9], [12]. To describe the dynamic exchange between different operating regimes, one needs simple equations that describe how the spatially distributed network state evolves over time. Motivated by this we consider a lattice model in Section 3, which is analogous to the model of Gibbens, Hunt and Kelly, [6]. We find a hydrodynamic equation for this lattice model. This is an integrodifferential equation describing the time evolution of the spatially distributed network occupancy profile. This equation also admits multiple spatially homogeneous time-invariant solutions for certain ranges of the parameters, which may be loosely thought of as the different operating regimes. The main results are stated as Theorem 1 and Theorem 2 in Section 3. The key idea of the proof is sketched in Section 4. Full details are available in [1].

2. ODE LIMITS

In this section we review the ODE limit of Gibbens, Hunt and Kelly, [6], in order to motivate the investigation in the following section.

Gibbens, Hunt and Kelly, [6] consider a simplified model for dynamic alternate routing which bypasses the spatial features of the network. Consider a collection of N links, each of which consists of C circuits. At each link, calls arrive according to a Poisson process of rate ν. If its link is not saturated, the call occupies one circuit on the link. If its link is saturated, the call chooses two distinct links at random from the remaining $N-1$ links, and if neither is saturated, the call occupies one circuit from each of these two links. Otherwise the call is blocked and rejected from the system. Each occupied circuit is held for an independent exponential time of mean 1. (Note that when a call occupies two circuits after making a successful choice of alternate route, it is assumed that these circuits are released independently).

Let $\gamma_k^N(t)$, $0 \leq k \leq C$, denote the fraction of the N links that have k occupied circuits at time t. $(\gamma_0^N, \gamma_1^N, \ldots, \gamma_C^N)$ evolves as a Markov process on the C-dimensional simplex. In [6], an ODE limit is found for the evolution as $N \to \infty$. Namely, if the initial conditions $(\gamma_0^N(0), \gamma_1^N(0), \ldots, \gamma_C^N(0))$ converge weakly to a limit $(\gamma_0(0), \gamma_1(0), \ldots, \gamma_C(0))$, then the process converges to the deterministic process given by the equations

$$\begin{aligned} \dot{\gamma}_0 &= \gamma_1 - (\nu + 2\nu\gamma_C(1-\gamma_C))\gamma_0 \ , \\ \dot{\gamma}_k &= (k+1)\gamma_{k+1} + (\nu + 2\nu\gamma_C(1-\gamma_C))\gamma_{k-1} \\ &\quad - (k + \nu + 2\nu\gamma_C(1-\gamma_C))\gamma_k \ , 0 < k < C \ , \\ \dot{\gamma}_C &= -C\gamma_C + (\nu + 2\nu\gamma_C(1-\gamma_C))\gamma_{C-1} \ , \end{aligned} \tag{2.1}$$

with the appropriate initial conditions.

When one looks for equilibrium points of eqns. (2.1), one finds the following : The equilibrium points are given by the solutions of

$$\begin{aligned} \gamma_k^* &= \frac{(\nu + 2\nu\gamma_C^*(1-\gamma_C^*))^k C!}{(\nu + 2\nu\gamma_C^*(1-\gamma_C^*))^C k!} \gamma_C^* \ , \\ \gamma_C^* &= E(\nu + 2\nu\gamma_C^*(1-\gamma_C^*), C) \ . \end{aligned} \tag{2.2}$$

When C is large enough, one finds that there is a range of ν in which eqns. (2.2) admit three solutions, two of which are stable. To the left and right of this range there is a unique stable solution.

Notice that the spatial characteristics of the network are averaged out in writing the above differential equations.

3. LATTICE CARICATURE

We now introduce a lattice caricature for dynamic alternate routing which has the virtue of preserving spatial features of the system. We will write a hydrodynamic limit for this lattice model. This is an integrodifferential equation which describes how the spatially distributed network state evolves over time, see eqn. (3.1). The main results are Theorem 1 and Theorem 2 below.

Let $\mathbf{Z}^d/M$ denote the lattice in $\mathbf{R}^d$ consisiting of points all of whose co-ordinates are rational with denominator dividing M. The points of $\mathbf{Z}^d/M$ are called *sites*. Let W denote $\{0,1,\ldots,C\}$. We consider a Markov process $(\eta_t^M, t \geq 0)$ on $W^{\mathbf{Z}^d/M}$ which caricatures a circuit switched network with dynamic routing (the statements below are true for any d, but the situations $d = 1$, and $d = 2$ are likely to be of most interest). We use η to denote a generic element of $W^{\mathbf{Z}^d/M}$, and call $\eta(x)$ the *value* at site x. Let M^* denote $\binom{(2M+1)^d - 1}{2}$. The Markov process is described by the transitions

$$\eta(x) \longrightarrow \eta(x) - 1 \quad \text{at rate } \eta(x) \ ,$$

$$\eta(x) \longrightarrow \eta(x) + 1 \quad \text{at rate } \nu \text{ if } \eta(x) \neq C \ ,$$

$$(\eta(x), \eta(y), \eta(z)) \longrightarrow (\eta(x), \eta(y) + 1, \eta(z) + 1) \quad \text{at rate } \nu/M^*$$

$$\text{if } x, y, z \text{ are distinct sites with}$$

$$\eta(x) = C, \eta(y) < C, \eta(z) < C \text{ and } y, z \in x + [-1,1]^d \ .$$

There is no difficulty constructing such a Markov process even though the number of sites is infinite. See Liggett, [9], Chapter 1, Section 3, for details; Theorem 3.9 of that section applies directly.

We think of each site in the lattice as representing a link in our network, which consists of C circuits. We think of the value at a site as giving the number of occupied circuits in the corresponding link. Occupied circuits become free at rate 1. At each link there is a Poisson process of calls with rate ν. Each call occupies one circuit on its link if available; if the link is saturated the call randomly picks two other links which are in its $[-1,1]^d$ neighbourhood, and uses one circuit from each of these links if possible. Otherwise the call is blocked and rejected from the system. Note that because we have a compressed lattice, the interaction actually has range M on the scale of links.

For $x \in \mathbb{Z}^d/M$, let $u_M(t,x,k)$ denote $P(\eta_t^M(x) = k)$, $0 \leq k \leq C$. We extend the definition of $u_M(t,\cdot,k)$ to $\mathbb{R}^d$ by setting $u_M(t,x,k) = u_M(t,[x]_M,k)$ for $x \in \mathbb{R}^d$, where $[x]_M$ denotes the minimum element in $\mathbb{Z}^d/M$ which dominates x in the usual partial order on $\mathbb{R}^d$. Let $u(0,x,k)$, $0 \leq k \leq C$, be continuous functions with bounded derivative and with $\sum_{k=0}^{C} u(0,x,k) = 1$. Let $u(t,x,k)$, $0 \leq k \leq C$ denote the solution of the integrodifferential equations

$$\begin{aligned}
\frac{\partial u(t,x,0)}{\partial t} =& u(t,x,1) \\
&-\nu(1+2^{1-2d}\int\int_{q,r\in[-1,1]^d} u(t,x+q,C)(1-u(t,x+q+r,C))dqdr)u(t,x,0)\ , \\
\frac{\partial u(t,x,k)}{\partial t} =& (k+1)u(t,x,k+1) \\
&+\nu(1+2^{1-2d}\int\int_{q,r\in[-1,1]^d} u(t,x+q,C)(1-u(t,x+q+r,C))dqdr)u(t,x,k-1) \\
&-(k+\nu(1+2^{1-2d}\int\int_{q,r\in[-1,1]^d} u(t,x+q,C)(1-u(t,x+q+r,C))dqdr)u(t,x,k)\ , \\
&\text{for } 0<k<C\ , \\
\frac{\partial u(t,x,C)}{\partial t} =& -Cu(t,x,C) \\
&+\nu(1+2^{1-2d}\int\int_{q,r\in[-1,1]^d} u(t,x+q,C)(1-u(t,x+q+r,C))dqdr)u(t,x,C-1)\ .
\end{aligned} \tag{3.1}$$

Then we have the following :

Theorem 1 : Fix $T < \infty$. Suppose that we start $(\eta_t^M)_t$ with initial configuration the product measure having $P(\eta_0^M(x) = k) = u_M(0,x,k)$, $0 \leq k \leq C$. If $u_M(0,x,k) \to u(0,x,k)$ unformly on compact sets, $0 \leq k \leq C$, then $u_M(t,x,k) \to u(t,x,k)$ for all $t \in [0,T]$, $x \in \mathbb{R}^d$ and $0 \leq k \leq C$.

Theorem 1 is a statement about pointwise convergence of probabilities. There is a corresponding functional limit theorem. This functional limit theorem allows us to describe the limit behaviour of an arbitrary choice of spatial integrals $\phi^{(1)}, \ldots, \phi^{(n)}$ at times $t_1, \ldots, t_n \in [0, T]$ as long as the $\phi^{(i)}$ decrease sufficiently rapidly. This allows, for example, to describe the evolution in time of spatial averages of the state over compact regions of the lattice (which caricatures compact regions of our network with dynamic routing). Let $\mathcal{S}(\mathrm{R}^d)$ denote the Schwarz space of rapidly decreasing functions on R^d and $\mathcal{S}^*(\mathrm{R}^d)$ the space of Schwarz distributions, which is its topological dual. We recall that $\mathcal{S}(\mathrm{R}^d)$ consists of precisely those infinitely differentiable functions $f \in C^\infty(\mathrm{R}^d)$ such that, with

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathrm{R}^d} \mid x^\alpha D^\beta f(x) \mid$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\beta = (\beta_1, \ldots, \beta_d)$, $x^\alpha = \prod_{j=1}^d x_j^{\alpha_j}$ and $D^\beta f = \frac{\partial^{\beta_1 + \cdots + \beta_d}}{\partial^{\beta_1} x_1 \ldots \partial^{\beta_d} x_d} f$, we have each $\|f\|_{\alpha,\beta} < \infty$ for all α, β. Further, $\mathcal{S}(\mathrm{R}^d)$ is a locally convex topological linear space with topology given by the family of seminorms $\|f\|_{\alpha,\beta}$. Then we have :

Theorem 2 : Given $\phi_k \in \mathcal{S}(\mathrm{R}^d)$, $0 \leq k \leq C$, let

$$X_t^M(\phi) = \frac{1}{M^d} \sum_{x \in \mathrm{Z}^d/M} \sum_{k=0}^{C} \phi_k(x) 1(\eta_t^M(x) = k) .$$

View X^M as an element of $D([0,T], (\mathcal{S}^*(\mathrm{R}^d))^{C+1})$. Then $X_\cdot^M \Rightarrow X_\cdot$, where $\Rightarrow$ denotes weak convergence in $D([0,T], (\mathcal{S}^*(\mathrm{R}^d))^{C+1})$, and

$$X_\cdot(\phi) = \sum_{k=0}^{C} \int_{x \in \mathrm{R}^d} \phi_k(x) u(\cdot, x, k) \, dx .$$

When we look for spatially homogeneous solutions of eqns. (3.1) which are time invariant, we are led to the same equations (2.2) found by Gibbens, Hunt and Kelly. Thus we see that for large enough C, there is a range of ν over which eqns. (3.1) admit three spatially homogeneous solutions. These may be loosely thought of as different phases associated to the network. The exchange between the phases can be studied by numerically integrating the eqns. from the appropriate initial conditions. Such work is currently in progress, [4]. Similar equations can also be written for more complicated dynamic routing schemes, including trunk reservation, using the techniques of the next section. The study of these equations may also be useful in making comparisons between schemes.

4. SKETCH OF THE PROOFS

The main idea is to consider a branching tree running backwards in time, which is in some sense a dual to the forward time process. We start from $x \in \mathbf{R}^d$ at time t, with a single "particle" alive at x. At time $t - s$ (reversed time s), there is a certain set $\mathcal{P}_s$ of "particles" which are alive. A particle alive at reversed time s stays alive on reversed time $[s, t]$. Each live particle p is at a point $x(p) \in \mathbf{R}^d$ and does not move. Further, at each reversed time s, we are given a map

$$F_s : W \to 2^{W^{\mathcal{P}_s}}$$

If $C \in W^{\mathcal{P}_s}$ is such that $C \in F_s(l)$, this means that if the point $x(p)$ has value $C(p)$ for each $p \in \mathcal{P}_s$, then tracing the forward time process from time $t - s$ to t, the resulting value at x is l. Each particle p alive at reversed time s has associated with it several independent Poisson processes. There are C Poisson processes $D(p,k), 1 \leq k \leq C$ of rate 1 : at a time of $D(p,k)$, $F_s(l)$ is reevaluated for each $l \in W$ by thinking of this point as requiring a call to leave in forward time if the number of calls in progress is at least k. There is a Poisson process $A(p)$ of rate ν : at the times of $A(p)$ $F_s(l)$ is reevaluated for each $l \in W$ by thinking of such a point as an exogenous arrival in forward time. (This process does not participate in alternate routing). Finlly, there is a Poisson process $Q(p)$ of rate 2ν : at the times of $Q(p)$ p generates $z(1), z(2) \in [-1,1]^d$ independently and uniformly and places one new particle at $x(p) + z(1)$ and one new particle at $x(p) + z(1) + z(2)$. $F_s(l)$ is reevaluated for each $l \in W$ by thinking of

$x(p) + z(1)$ as being a link which is potentially saturated and is performing alternate routing to $x(p)$ and $x(p) + z(1) + z(2)$.

From the branching tree on $\mathbf{R}^d$, we construct a branching tree on $\mathbf{Z}^d/M$ by moving particles to the nearest lattice point that dominates their assigned point. Now, more than one particle may be alive at a site. We need to show that this happens with vanishingly small probability as $M \to \infty$. We also need to show that if we construct branching trees starting from two distinct sites $x, y \in \mathbf{Z}^d/M$, the probability that they intersect, (i.e. that some site is occupied by particles from each process) has vanishingly small probability as $M \to \infty$. This ensures that the joint probability terms that show up in the generator equations for the finite M process asymptotically become products and give the desired hydrodynamic equations for the evolution of the occcupancy profile.

ACKNOWLEDGEMENTS

I am greatly indebted to Rick Durrett for his considerable interest in this work, and for guiding me to the relevant particle systems literature. I would also like to thank Roland Dobrushin, Gaston Giroux, Lawrence Gray, Vadim Malyshev, Roberto Schonmann, and Glen Swindle for useful comments.

REFERENCES

[1] V. Anantharam, "A Hydrodynamic Limit for a Lattice Caricature of Dynamic Routing in Circuit Switched Networks", Preprint, 1989.

[2] J. M. Akinpelu, "The Overload Performance of Engineered Networks with Nonhierarchical and Hierarchical Routing", *AT & T Bell Laboratories Technical Journal*, Vol. 63, No. 7, pp. 1261 -1281, 1984.

[3] V. Anantharam, "Metastability and Phase Transitions Associated to Dynamic Routing in Networks", *Proceedings of the 28th IEEE CDC*, Tampa, Florida, 1989.

[4] M. Benchekroun, *PhD. Dissertation*, in preparation, Cornell University, 1990.

[5] R. J. Gibbens, *Some Aspects of Dynamic Routing in Circuit-Switched Telecommunication Networks*, Rayleigh Prize Essay, University of Cambridge, 1986.

[6] R. J. Gibbens, P. J. Hunt and F. Kelly, "Bistability in Communication Networks", *Preprint*, 1988.

[7] F. Kelly, "Blocking Probabilities in Large Circuit Switched Networks", *Advances in Applied Probability*, Vol. 18, pp. 473 -505, 1986.

[8] R. S. Krupp, "Stabilization of Alternate Routing Networks", *International Communications Conference*, 31.2.1 -31.2.5, Philadelphia, 1982.

[9] T. Liggett, *Interacting Particle Systems*, Grund. der Math. Wiss., Vol. 276, Springer Verlag, 1985.

[10] V. V. Marbukh, "Asymptotic Investigation of a Complete Communications Network with a large number of points and bypass routes", *Problems of Information Transmission*, pp. 212 -216, 1982.

[11] V. V. Marbukh, "Fully Connected Message-Switching Network with a Large number of Nodes, Bypass routes and a limited number of Waiting places at the nodes", *Problems of Information Transmission*, pp. 154 -161, 1985.

[12] G. Swindle, "A Hydrodynamic Limit for the Contact process with Large range", *Ph.D. Dissertation*, Center for Applied Mathematics, Cornell University, Ithaca, NY, 1988.