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THE INPUT-OUTPUT MAP OF A MONOTONE DISCRETE TIME QUASIREVERSIBLE NODE

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ABSTRACT

We consider a class of discrete time quasireversible nodes called *monotone* which includes discrete time analogs of the ./M/∞ and ./M/1 nodes. For stationary ergodic nonnegative integer valued arrival processes we prove the existence and uniqueness of stationary regimes when a natural rate condition is met. We use coupling to prove the contractiveness of the input output map relative to a natural distance on the space of stationary arrival processes that is analogous to Ornstein's $\bar{d}$ distance. A consequence is that the only stationary ergodic fixed points of the input output map are the processes of independent and identically distributed Poisson random variables meeting the rate condition. The problem is of interest in connection with the construction of product form network models.

1. INTRODUCTION

The concept of quasireversibility of a queueing node in continuous time was introduced by Kelly, [9], and a probabilistic understanding of this concept was provided by Walrand, [13]. Networks constructed from quasireversible nodes form a natural class of models for the performance analysis of communication networks : they offer modelling flexibility and admit *product form* stationary distributions, which makes the computation of stationary performance quantities easy. One of the characteristics of a continuous time quasireversible node is that, in stationarity, when the arrival process is a Poisson process so is the departure process. One can thus think of the Poisson process as a fixed point of the input output map of such a node, which we view as a map on the space of stationary processes (of course, this has to be appropriately formulated). A question of some interest then is whether the input output map has any other fixed points, apart from mixtures of Poisson processes, which are also trivially fixed.

For ./G/∞ nodes acting on stationary ergodic input processes, Vere-Jones, [12], proved a result of this type (for a precise formulation, see [12]). The problem for ./M/1 nodes has been circulating in the community for some time now. For example, it is mentioned as an old open problem in a recent paper of Glynn and Whitt, [6], to prove that the stationary departure process from a long tandem of identical ./G/1 nodes fed by a renewal process becomes asymptotically Poisson as the length of the tandem tends to infinity. For ./M/1 nodes a natural approach to this problem would be to prove that Poisson processes are the only stationary ergodic fixed points. This has recently been established in [2], using techniques motivated by those in this paper. Another recent contribution to the study of fixed points of the input output map of first come first served queues in a very general setup is due to Bambos and Walrand, [4].

The purpose of this paper is to discuss a partial resolution of the analogous question in discrete time. Indeed the concept of quasireversibility and the fact that networks of quasireversible nodes with Bernoulli routing admit product form stationary distributions carry over to discrete time, as shown by Walrand, [14]. The role of the Poisson process here is played by the i.i.d. process whose each marginal is a Poisson random variable.

2. DISCRETE TIME QUASIREVERSIBLE NODES

We consider discrete time quasireversible queues in the sense of Walrand, [14]. A complete classification of such nodes was given in [14]; we remark that such a queue necessarily has batch arrivals and batch services. The quasireversible queues considered by Walrand are first of all S-queues, i.e., given an arbitrary arrival sequence $\{a_n, n \geq 0\}$ of $\mathrm{N} = \{0,1,2,\ldots\}$ valued random variables, the queue length process is given by

$$x_{n+1} = x_n + a_n - d_{n+1} ,$$

where for some $S(i,j)$, $0 \leq j \leq i$, we have, for $n \geq 0$

$$P(d_{n+1} = j \mid x_m, d_m, 0 \leq m \leq n; a_k, k \geq 0, x_n + a_n = i) = S(i,j) .$$

Here x_0 is arbitrary and $d_0 = 0$. Mimicking the continuous time concept due to Kelly, [9], such a queue is called quasireversible if, when $\{a_n, n \geq 0\}$ is a Poisson arrival sequence such that the state admits an equilibrium distribution, then $\{d_n, n \geq 0\}$ is Poisson in equilibrium and for all n $\{d_l, l \leq n\}$ and x_n are independent. Walrand, [14], proved that an S-queue is quasireversible if and only if $S(i,j)$ has the following form :

$$S(0,0) = c(0) = 1 , \qquad (2.1a)$$

$$S(i,0) = c(i) , \ i > 0 , \qquad (2.1b)$$

$$S(i,j) = \frac{c(i)}{j!}\alpha(i)\alpha(i-1)\ldots\alpha(i-j+1) , \quad 0 < j \leq i , \ (2.1c)$$

where $\alpha(0) = 1$, $\alpha(j) > 0$ for $j > 0$ and $c(i)$ is such that

$$\sum_{j=0}^{i} S(i,j) = 1 .$$

Further, the queue admits an equilibrium distribution π for a Poisson arrival sequence of mean λ if and only if the normalizing constant c exists such that

$$\pi(i) = c\frac{\lambda^i}{\alpha(0)\dots\alpha(i)}\ , \quad i \geq 0$$

is a probability distribution.

We first note the following :

Lemma 1 : Given $\alpha(0) = 1$ and $\alpha(j) > 0$ for $j > 0$, let $S(i,j)$ be as in (2.1). Then one can construct random variables $d(i), i \geq 0$ such that, with $r(i) = i - d(i)$, we have

$$P(d(i) = j) = S(i,j)\ , \tag{2.2}$$

and

$$r(i+1) \geq r(i)\ , \quad i \geq 0\ . \tag{2.3}$$

Proof : Let $\mathcal{C}^i$ be the birth and death chain on $\{0,\dots,i\}$ with birth rate $i-j$ and death rate $\alpha(j)$ in state j. This chain has stationary distribution $S(i,i-j)$. We couple $\mathcal{C}^i$ to $\mathcal{C}^{i+1}$ starting at 0. Namely, we construct a Markov process $((X^i(t), X^{i+1}(t)), t \geq 0)$ with $(X^i(0), X^{i+1}(0)) = (0,0)$ and so that $(X^i(t), t \geq 0)$ (resp. $(X^{i+1}(t), t \geq 0)$) has the distribution of $\mathcal{C}^i$ (resp. $\mathcal{C}^{i+1}$). It is easy to see that we can construct such a coupling with $X^{i+1}(t) \geq X^i(t)$ for all $t \geq 0$. The claim follows. •

In view of Lemma 1, we will choose a canonical representation of the virtual departure process of our quasireversible node. Let $\Omega_d = [0,1]^{\mathbf{Z}}$ with product measure P_d defined on the σ-algebra $\mathcal{F}_d$ generated by cylinder sets, where each factor has Lebesgue measure on the Borel σ-algebra. Given $\omega = (\omega_n, n \in \mathbf{Z})$, let $r_n(i,\omega) = k$ iff $\sum_{j=0}^{k-1} S(i,i-j) < \omega_n \leq \sum_{j=0}^{k} S(i,i-j)$, and let $d_n(i) = i - r_n(i)$. Then $((d_n(i), i \geq 0), n \in \mathbf{Z})$ are independent and have identical distribution given by (2.2). Further, we can define $r_n(\infty) = \lim_{i\to\infty} r_n(i)$. Simple algebra shows we must necessarily have $r_n(\infty) = \infty$ a.s..

Note that the transformation $\theta_d : \Omega_d \to \Omega_d$ given by the left shift

$$\theta_d(\dots,\omega_{-1},\omega_0,\omega_1,\dots) = (\dots,\omega_0,\omega_1,\omega_2,\dots)$$

is invertible measure preserving and ergodic. Note also that $d_n(i,\theta_d(\omega)) = d_{n+1}(i,\omega)$ for all $i \geq 0$ and $n \in \mathbf{Z}$.

Let $\mathbf{N}^{\mathbf{Z}}$ denote the set of two sided infinite sequences of nonnegative integers with the σ-algebra $\mathcal{B}$ generated by the cylinder sets. We think of $\mathbf{N}^{\mathbf{Z}}$ endowed with the left shift θ_s, and call a measure on $\mathbf{N}^{\mathbf{Z}}$ stationary, resp. ergodic, if it is so with respect to the left shift. Let $\mathcal{M}_S(\lambda)$ and $\mathcal{M}_S^e(\lambda)$ denote respectively the space of stationary measures and the space of stationary ergodic measures on $\mathbf{N}^{\mathbf{Z}}$ with rate λ.

A stationary arrival process of rate λ into our node is specified in distribution by an element $\mu_a \in \mathcal{M}_S(\lambda)$ ($\mu_a \in \mathcal{M}_S^e(\lambda)$ if it is ergodic). We think of a stationary arrival process as prescribed by an $\mathbf{N}$ valued random variable a_0 given on a sample space $(\Omega_a, \mathcal{F}_a, P_a)$ supporting an invertible P_a preserving transformation θ_a. Then $a_n = a_0 \circ \theta_a^n$. Later we may need to use different representations for the same arrival distribution. Of course each arrival process has a standard representation $(\mathbf{N}^{\mathbf{Z}}, \mathcal{B}, \mu_a, \theta_s)$ on which it is given by the marginal x_0 at time 0.

Note it is possible to represent $\mu_a \in \mathcal{M}_S^e(\lambda)$ by a ***non*-ergodic** $(\Omega_a, \mathcal{F}_a, P_a, \theta_a, a_0)$, but it is not possible to represent a nonergodic $\mu_a \in \mathcal{M}_S(\lambda)$ by an ergodic $(\Omega_a, \mathcal{F}_a, P_a, \theta_a, a_0)$. This is an easy consequence of the definition of ergodicity. Thus we will distinguish between the ergodicity of an arrival process and of its representation. Since the virtual departure process of our node is assumed independent of the arrival process, the node with its arrivals is completely described by a_0 and $(d_1(i), i \geq 0)$ on $(\Omega_a \times \Omega_d, \mathcal{F}_a \times \mathcal{F}_d, P_a \times P_d, \theta_a \times \theta_d)$. We abbreviate this to $(\Omega, \mathcal{F}, P, \theta)$. Note that if θ_a is P_a-ergodic then θ is P-ergodic.

3. MINIMAL PRE-STATIONARY REGIME

The node is said to admit a *pre-stationary regime* if there is a non-negative random variable x_0 (possibly infinite with positive probability) such that

$$x_0 \circ \theta = r_1(x_0 + a_0)\ .$$

The reason for the terminology is the following : If the node admits a pre-stationary regime, let $x_n = x_0 \circ \theta^n$. Then $(x_n, n \in \mathbf{Z})$ is a stationary process satisfying

$$x_{n+1} = r_{n+1}(x_n + a_n)\ , \quad n \in \mathbf{Z}\ .$$

Thus it is a stationary queue size process seen by arrivals.

Our first result is the following :

Theorem 1 : Every discrete time quasireversible node admits a pre-stationary regime for any stationary arrival process.

Proof : Recall that we are working on $(\Omega, \mathcal{F}, P, \theta)$ defined above so that, while the arrival representation is arbitrary, the virtual departures have a canonical representation and the overall sample space is a product. We use the Loynes construction. See Loynes [10], Baccelli and Bremaud [3], Part II, Chapter 1 and Walrand, [15], Chapter 7 for discussions of this construction. For each $m \geq 0$ we construct a process $(x_n^m, n \geq -m)$ by

$$x_{-m}^m = 0\ , \tag{3.1a}$$

$$x_{n+1}^m = r_{n+1}(x_n^m + a_n) = x_n^m + a_n - d_{n+1}(x_n^m + a_n)\ . \tag{3.1b}$$

One thinks of x_n^m as the queue size that would be seen by arrivals at time n if the node were started empty at time $-m$. We claim that $x_n^{m+1} \geq x_n^m$ for all $n \geq -m$. Indeed, $x_{-m}^{m+1} \geq 0 = x_{-m}^m$. Suppose $x_{n-1}^{m+1} \geq x_{n-1}^m$. Then

$$\begin{aligned} x_n^{m+1} &= r_n(x_{n-1}^{m+1} + a_{n-1}) \\ &\geq r_n(x_{n-1}^m + a_{n-1}) \end{aligned}$$

by (2.3), giving the desired. Thus we may define

$$x_n^\infty = \lim_{m\to\infty} x_n^m \tag{3.2}$$

for all $n \in \mathbf{Z}$. Since $x_{n+1}^m = x_n^{m+1} \circ \theta$, (3.1b) becomes

$$x_n^{m+1} \circ \theta = r_{n+1}(x_n^m + a_n)\ . \tag{3.3}$$

From (3.2), it follows that

$$x_n^\infty \circ \theta = x_{n+1}^\infty = r_{n+1}(x_n^\infty + a_n) .$$

x_0^∞ is the desired pre-stationary regime. •

Note that Theorem 1 as stated is trivially true, since taking $x_0 = \infty$ defines a pre-stationary regime by virtue of the fact that $r_n(\infty) = \infty$ for all $n \in \mathbf{Z}$. However the pre-stationary regime constructed in the proof of Theorem 1 is the *minimal* pre-stationary regime, in that if there is any other pre-stationary regime it must dominate this one pointwise.

A pre-stationary regime is said to be a *stationary regime* if the random variable x_0 is a.s. finite. Clearly one needs some kind of rate condition on the arrival process for the existence of a stationary regime. To discuss this we now restrict attention to monotone quasireversible nodes. See Section 8 for some remarks on the need for this restriction.

4. MONOTONE QUASIREVERSIBLE NODES

We will call a discrete time quasireversible node *monotone* if the sequence $\alpha(i), i \geq 1$ is nondecreasing. Several natural examples of discrete time nodes are monotone, including the analogs of the ./M/∞ and ./M/1 nodes, see [14]. For a monotone quasireversible node we have the following :

Lemma 2 : Let $\alpha(0) = 1$, $0 < \alpha(1) \leq \alpha(2) \leq \ldots$, and let $S(i,j)$ be as in (2.1). Then one can construct random variables $d(i), i \geq 0$ such that, with $r(i) = i - d(i)$, we have (2.2) and (2.3) and also

$$d(i+1) \geq d(i) , \quad i \geq 0 . \tag{4.1}$$

Proof : Let $\mathcal{D}^i$ be the birth and death chain on $\{0,\ldots,i\}$ with birth rate $\alpha(i-j)$ and death rate j in state j. This chain has stationary distribution $S(i,j)$. We couple $\mathcal{D}^i$ to $\mathcal{D}^{i+1}$ starting at 0. We can clearly construct such a coupling with $Y^{i+1}(t) \geq Y^i(t)$ for all $t \geq 0$. The claim follows. •

Thus, for a monotone quasireversible node when the virtual departure process has the canonical representation we have the added property that $d_n(i+1) \geq d_n(i)$ for all $i \geq 0$ and $n \in \mathbf{Z}$. The next result is about stationary regimes for monotone nodes.

Theorem 2 : For a monotone quasireversible node, let

$$\lambda_0 = \sup\{\lambda \ : \ \sum_{i=0}^{\infty} \frac{\lambda^i}{\alpha(0)\ldots\alpha(i)}\} < \infty .$$

Note $\lambda_0 > 0$. Let $\lambda < \lambda_0$. Consider an arrival process $\mu_a \in \mathcal{M}_S^e(\lambda)$. Let $(\Omega_a, \mathcal{F}_a, P_a, \theta_a, a_0)$ be an ergodic representation of μ_a. Then the node admits a stationary regime for this representation.

Proof : We will show that the minimal pre-stationary regime x_0^∞ constructed above is a.s. finite. First note that the event $\{x_0^\infty < \infty\}$ is θ-invariant. Since θ is P-ergodic this event has probability 0 or 1. Thus it suffices to show that x_0^∞ cannot be ∞ with probability 1.

Since $\lambda < \lambda_0$, there is $\epsilon > 0$ such that with an i.i.d. Poisson arrival sequence of rate $\lambda + 2\epsilon$ the number of customers just prior to arrivals evolves as a Markov chain with stationary distribution

$$\pi(i) = c\frac{(\lambda + 2\epsilon)^i}{\alpha(0)\ldots\alpha(i)} , \quad i \geq 0 .$$

See Walrand, [14], for a proof. In particular, we have

$$\sum_{i=0}^{\infty} \pi(i) E d_1(i) = \lambda + 2\epsilon . \tag{4.2}$$

Now Lemma 2 informs us that $Ed_1(i)$ is nondecreasing in i. It follows from (4.2) that there is $K < \infty$ such that

$$Ed_1(i) > \lambda + \epsilon \ \text{ for all } \ i \geq K . \tag{4.3}$$

We now return to the construction of the minimal pre-stationary regime in the proof of Theorem 1. Equation (3.3) reads

$$x_n^{m+1} \circ \theta = x_n^m + a_n - d_{n+1}(x_n^m + a_n) .$$

The θ invariance of P and the fact that $x_n^{m+1} \geq x_n^m$, imply that

$$Ex_n^{m+1} \circ \theta = Ex_n^{m+1} \geq Ex_n^m$$

Hence (3.3) yields

$$\lambda \geq Ed_{n+1}(x_n^m + a_n) . \tag{4.4}$$

Now suppose $x_n^\infty = \infty$ a.s.. Then for any $\delta > 0$ we can find $M_\delta < \infty$ so that $P(x_n^m > K) > 1 - \delta$ for all $m \geq M_\delta$. From (4.3) and (4.4) this gives

$$\lambda \geq (\lambda + \epsilon)(1 - \delta) ,$$

but this is a contradiction for δ sufficiently small. •

Note that if an ergodic arrival process satisfying the rate condition of Theorem 2 is given in a nonergodic representation, we may first construct a stationary regime on $(\Omega^*, \mathcal{F}^*, P^*, \theta^*)$ which is the product of the canonical representation of the arrival process with the canonical representation of the virtual departure process. We then take the composition of this stationary regime with the map

$$\Phi \ : \ (\Omega, \mathcal{F}, P, \theta) \rightarrow (\Omega^*, \mathcal{F}^*, P^*, \theta^*)$$

given by $\Phi((\omega_n)_n) = (a_n(\omega), d_n(\omega))_n$ to get a stationary regime on $(\Omega, \mathcal{F}, P, \theta)$. For a nonergodic arrival process the rate condition is not enough to guarantee the existence of a stationary regime - for example a mixture of a sequence of i.i.d. Poisson random variables with rate exceeding λ_0 and one with rate below λ_0 can meet rate condition but cannot be supported by our monotone node.

The next natural question is whether a monotone node can admit more than one stationary regime for a representation of an ergodic arrival process. This is answered by :

Theorem 3 : Given a monotone quasireversible node, let $\lambda < \lambda_0$ and consider an arrival process $\mu_a \in \mathcal{M}_S^e(\lambda)$. For any representation of μ_a the stationary regime guaranteed by Theorem 2 is unique. •

Suppose we are given an ergodic representation of an arrival process $\mu_a \in \mathcal{M}_S^e(\lambda)$ with $\lambda < \lambda_0$. We wish to examine the stationary departure process from our monotone node in the unique stationary regime. We would like to know that the stationary departure process has rate λ and that it is uniquely specified in distribution independent of the representation. This is the content of the remaining two results in this section. Together they permit us to think of the node as defining a map T on $\mathcal{M}_S^e(\lambda)$. Properties of this map are studied in Section 6.

Theorem 4 : Let the situation be as in the statement of Theorem 3. Let x_0^∞ denote the unique stationary regime, and let $(d_n(x_n^\infty + a_n), n \in \mathbf{Z})$ be the departure process in the stationary regime. Then this process is stationary and ergodic with mean λ, so it defines an element of $\mathcal{M}_S^e(\lambda)$. •

In the above theorem, note that we have not argued the integrability of x_0^∞. Indeed, we do not know if this need always be true. By analogy with the Loynes construction for the workload process seen by arrivals in a FCFS queue with general stationary ergodic interarrival and service time process, see e.g. [3], there is no reason to expect this to be the case.

Theorem 5 : Let the situation be as in Theorem 4. Then the distribution of $(d_n(x_n^\infty + a_n), n \in \mathbf{Z})$ does not depend on the choice of the representation $(\Omega_a, \mathcal{F}_a, P_a, \theta_a, a_0)$ of the arrival process. •

The conclusion of the discussion in this section is that for each $\lambda < \lambda_0$ we can define a map $T \;:\; \mathcal{M}_S^e(\lambda) \to \mathcal{M}_S^e(\lambda)$ by letting $T(\mu_a) = \mu_d$, where μ_d is the distribution of the departure process in the unique stationary regime associated to any ergodic representation of the arrival process μ_a. We know that the process of independent Poisson random variables is a fixed point of T. Our goal is to prove that there is no other fixed point.

5. A METRIC ON ARRIVAL PROCESSES

In this section we introduce a metric on $\mathcal{M}_S(\lambda)$, which is analogous to the $\bar{d}$ distance introduced by Ornstein, [11]. Our metric is an example of a family of such generalizations introduced by Gray, Neuhoff and Shields, [8], who were motivated by application to information theory. Gray et. al. call their generalizations $\bar{\rho}$ distances, so we will use the notation $\bar{\rho}$ for our distance. The basic properties of the $\bar{\rho}$ distance are derived in [8] and also in Chapter 8 of the text by Gray, [7].

On $\mathbf{N}$ we consider the metric $\rho(u,v) = \mid u - v \mid$. Let $\mu, \nu \in \mathcal{M}_S(\lambda)$. Consider $\mathbf{N^Z} \times \mathbf{N^Z}$ with the σ-algebra $\mathcal{B} \times \mathcal{B}$ and the left shift $\theta_s \times \theta_s$. Let $\mathcal{M}_S(\lambda,\lambda)$ denote the space of stationary probability distributions on this space whose each marginal has rate λ. A *stationary coupling* $\mathcal{C}$ of μ and ν is specified by giving a measure $\alpha \in \mathcal{M}_S(\lambda,\lambda)$ with marginals μ and ν respectively.

Denote a generic element of $\mathbf{N^Z} \times \mathbf{N^Z}$ by $(u_n, v_n)_n$. The $\bar{\rho}$ distance of μ and ν is defined as

$$\bar{\rho}(\mu,\nu) = \inf_\alpha \alpha(\rho(u_0, v_0)) \;. \tag{5.1}$$

Thus, one considers the expected value of the difference between the number of arrivals in the first process and the second process at any time and takes the infimum over all stationary couplings.

First note the following :

Lemma 3 : $\mathcal{M}_S(\lambda)$ and $\mathcal{M}_S(\lambda,\lambda)$ are compact in the weak topology. •

The properties of $\bar{\rho}$ relevant to our discussion are summarized in the following result.

Theorem 6 : The $\bar{\rho}$ distance on $\mathcal{M}_S(\lambda)$ introduced above has the following properties

(i) $\bar{\rho}$ is a metric.

(ii) The infimum in the definition (5.1) is a minimum, i.e., there is stationary coupling that achieves the $\bar{\rho}$ distance.

(iii) If $\mu, \nu \in \mathcal{M}_S^e(\lambda)$, the infimum in (5.1) can be replaced by an infimum over stationary ergodic α, Further, this infimum is a minimum.

(iv) Let $(\mu_k, k \geq 1), \mu \in \mathcal{M}_S(\lambda)$. Suppose $\bar{\rho}(\mu_k, \mu) \to 0$ as $k \to \infty$. Then $\mu_k \to \mu$, where the convergence is in the weak topology of $\mathcal{M}_S(\lambda)$.

(v) Let $\mu_k \to \mu$ in the weak topology of $\mathcal{M}_S(\lambda)$. Then for any $\nu \in \mathcal{M}_S(\lambda)$, we have

$$\liminf_{k \to \infty} \bar{\rho}(\mu_k, \nu) \geq \bar{\rho}(\mu, \nu) \;.$$

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6. CONTRACTIVENESS OF I-O MAP

The central observation of this paper is the following :

Theorem 7 : Let $\mu, \nu \in \mathcal{M}_S^e(\lambda)$. $\mu \neq \nu$. Then

$$\bar{\rho}(T(\mu), T(\nu)) < \bar{\rho}(\mu, \nu) \;.$$

Proof : Since $\mu \neq \nu$, by Theorem 6 (i), we know that $\bar{\rho}(\mu,\nu) > 0$. Let $(\mathbf{N^Z} \times \mathbf{N^Z}, \mathcal{B} \times \mathcal{B}, \alpha, \theta_s \times \theta_s, (u_0, v_0))$ be a stationary ergodic coupling achieving $\bar{\rho}(\mu,\nu)$. The existence of such a coupling is ensured by (ii) of Theorem 6. Taking the product with the canonical representation of the virtual departure process gives $(\Omega, \mathcal{F}, P, \theta)$ supporting $(a_0, \tilde{a}_0)) = (u_0, v_0)$ and $(d_1(i), i \geq 0)$. Observe that $P(a_0 = \tilde{a}_0) < 1$ because $\bar{\rho}(\mu,\nu) > 0$.

We jointly construct the minimal pre-stationary regimes for the two arrival processes under consideration following the Loynes scheme and a colouring idea. Let $a_0^Y = \min(a_0, \tilde{a}_0)$, $a_0^R = a_0 - a_0^Y$, and $a_0^B = \tilde{a}_0 - a_0^Y$. Note that

$$\bar{\rho}(\mu,\nu) = 2\lambda - E a_0^Y \;.$$

Let $(a_n, \tilde{a}_n, a_n^Y, a_n^R, a_N^B) = (a_0, \tilde{a}_0, a_0^Y, a_0^R, a_0^B) \circ \theta^n$. then $a_n = a_n^R + a_n^Y$ and $\tilde{a}_n = a_n^B + a_n^Y$. Think of a_n^Y arrivals coloured yellow, a_n^R arrivals coloured red and a_n^B arrivals coloured blue at time n. Note that $a_0^R a_0^B = 0$.

For each $m \geq 0$ we construct random variables

$$(z_n^m, x_n^{m,Y}, x_n^{m,R}, x_n^{m,B}, x_n^m, \tilde{x}_n^m, n \geq -m) .$$

Let

$$z_{-m}^m = x_{-m}^{m,Y} = x_{-m}^{m,R} = x_{-m}^{m,B} = x_{-m}^m = \tilde{x}_{-m}^m = 0 ,$$

$$z_{n+1}^m = r_{n+1}(z_n^m + a_n^Y)$$

$$x_{n+1}^{m,Y} = r_{n+1}(x_n^{m,Y} + a_n^Y + \min(x_n^{m,R}, a_n^B) + \min(x_n^{m,B}, a_n^R)) \quad (6.1)$$

$$x_{n+1}^{m,R} = r_{n+1}(x_n^m + a_n) - x_{n+1}^{m,Y}$$

$$x_{n+1}^{m,B} = r_{n+1}(\tilde{x}_n^m + \tilde{a}_n) - x_{n+1}^{m,Y}$$

$$x_{n+1}^m = r_{n+1}(x_n^m + a_n)$$

$$\tilde{x}_{n+1}^m = r_{n+1}(\tilde{x}_n^m + \tilde{a}_n) .$$

The interpretation of z_n^m is the number of customers in the node at time n if it is started empty at time $-m$ and fed with only yellow customers. The rest of the variables involve a recolouring procedure. At any time the node can have yellow and red customers or yellow and blue customers, but never simultaneously have red and blue customers. When arrivals come in the yellow arrivals are added on to the existing yellow customers, as many red arrivals as possible are merged with any existing blue customers, becoming yellow customers (a red arrival and a existing blue customer merge into a single customer) and as many blue arrivals as possible are merged with the existing red customers, becoming yellow customers. After the merging procedure it is once again true that the node has either yellow and red customers or yellow and blue customers but cannot have both red and blue customers. Further, if it has no blue customers the yellow customers represent the situation with the second arrival process and if it has no red customers the yellow customers represent the situation with the first arrival process. Now we examine the virtual departure variable corresponding to the total number of yellow customers and to the total number of customers respectively. The latter is no smaller than the former, and the difference between them is bounded by the number of nonyellow customers, by virtue of Lemma 2. We release as many yellow customers as required by the former, and release as many nonyellow customers as required by the difference between the latter and the former. Now we have determined the state of the node at the next time instant. Thus $x_n^{m,Y}$ is the number of yellow customers in the node, $x_n^{m,R}$ the number of red customers, $x_n^{m,B}$ the number of blue customers, x_n^m the total number of customers corresponding to the first arrival process and $\tilde{x}_n^m$ the total number of customers corresponding to the second arrival process, all at time n when the node is started empty at time $-m$.

We claim that z_n^m, $x_n^{m,Y}$, x_n^m and $\tilde{x}_n^m$ are nondecreasing in m for each fixed n for which they are defined, i.e. $n \geq -m$. This is immediate from Theorem 1 for all these variables except $x_n^{m,Y}$. To see this, first note that $x_{-m}^{m+1,Y} \geq 0 = x_{-m}^{m,Y}$. Now suppose $x_{n-1}^{m+1,Y} \geq x_{n-1}^{m,Y}$ for some $n \geq -m$. Either $a_{n-1}^R = 0$ or $a_{n-1}^B = 0$; assume the former. Then from (6.1) we have

$$\begin{aligned} x_n^{m+1,Y} &= r_n(x_{n-1}^{m+1,Y} + a_{n-1}^Y + \min(x_{n-1}^{m+1,R}, a_{n-1}^B)) \\ &= r_n(a_{n-1}^Y + \min(x_{n-1}^{m+1}, x_{n-1}^{m+1,Y} + a_{n-1}^B)) \\ &\geq r_n(a_{n-1}^Y + \min(x_{n-1}^m, x_{n-1}^{m,Y} + a_{n-1}^B)) \\ &= x_n^{m,Y} \end{aligned}$$

where we have used the induction hypothesis in the third step.

Thus we can define z_n^∞, $x_n^{\infty,Y}$, x_n^∞ and $\tilde{x}_n^\infty$ by taking the pointwise limit as $m \to \infty$. We can also define $x_n^{\infty,R} = x_n^\infty - x_n^{\infty,Y}$ and $x_n^{\infty,B} = \tilde{x}_n^\infty - x_n^{\infty,Y}$. Clearly z_0^∞ is the unique stationary regime corresponding to the yellow arrivals, x_0^∞ the unique stationary regime corresponding to the sum of yellow and red arrivals, i.e. to the first arrival process, and $\tilde{x}_0^\infty$ the unique stationary regime corresponding to the sum of yellow and blue arrivals, i.e. to the second arrival process. The departure process corresponding to the first arrival process is $(d_n(x_{n-1}^\infty + a_{n-1}), n \in \mathbf{Z})$ and that corresponding to the second arrival process is $(d_n(\tilde{x}_{n-1}^\infty + \tilde{a}_{n-1}), n \in \mathbf{Z})$.

Let the number of yellow departures in the stationary regime at time n be denoted

$$d_n^Y = d_n(x_{n-1}^{\infty,Y} + a_{n-1}^Y + \min(x_{n-1}^{\infty,R}, a_{n-1}^B) + \min(x_{n-1}^{\infty,B}, a_{n-1}^R)) . \quad (6.2)$$

The process of red departures is denoted

$$d_n^R = d_n(x_{n-1}^\infty + a_{n-1}) - d_{n-1}^Y$$

and the process of blue departures is denoted

$$d_n^B = d_n(\tilde{x}_{n-1}^\infty + \tilde{a}_{n-1}) - d_{n-1}^Y .$$

Note that the sample space $(\Omega, \mathcal{F}, P, \theta)$ now supports a natural stationary coupling between distributions $T(\mu)$ and $T(\nu)$. Hence we get

$$\bar{\rho}(T(\mu), T(\nu)) \leq 2\lambda - Ed_n^Y .$$

To prove the claim, it suffices to establish that $Ed_n^Y > Ea_n^Y$. Further observing that $d_n(z_{n-1}^\infty + a_{n-1}^Y)$ is the departure process corresponding to the yellow arrivals, so that, by Theorem 4 $Ed_n(z_{n-1}^\infty + a_{n-1}^Y) = Ea_n^Y$, we see that it is enough to prove $Ed_n^Y > Ed_n(z_{n-1}^\infty + a_{n-1}^Y)$. Note that (6.2) already implies that $Ed_n^Y \geq Ed_n(z_{n-1}^\infty + a_{n-1}^Y)$.

Since $P(a_0 = \tilde{a}_0) < 1$ and because μ and ν are both ergodic, it follows that there must exist some $1 \leq N < \infty$ such that

$$P(a_n^R > 0, a_m^R = a_m^B = 0 \text{ for all } n+1 \leq k \leq n+N-1,$$
$$a_{n+N}^B > 0) > 0 . \quad (6.3)$$

Indeed, $P(a_0 = \tilde{a}_0) < 1$ implies $P(a_n^R > 0) > 0$ and also $P(a_n^B > 0) > 0$. Consider $\cup_{m=-\infty}^k \{a_m^R > 0\}$ which is mapped into itself by the left shift. Since it is an event determined by an ergodic distribution, and has nonzero probability, it has probability 1. So, for some $l < k+1$ we have $P(a_l^R > 0, a_{k+1}^B > 0) > 0$. From this it is a simple matter to conclude (6.3).

From (6.3) and the fact that the stationary regimes are a.s. finite, we conclude that there must be $0 < M_1 < \infty$, $0 < M_2 < \infty$, and $0 < M_3 < \infty$ such that

$$P(\max(x_n^\infty + \sigma_n,\ \tilde{x}_n^\infty + \tilde{a}_n) = M_1, a_n^R = M_2,$$
$$a_m^R = a_m^B = 0 \text{ for all } n+1 \leq k \leq n+N-1,$$
$$a_{n+N}^B = M_3) > 0 .$$

Since the variables involved in the above event are all functions of the arrival processes and of the virtual departure process upto time n, they are independent of the variables $(d_k(i), i \geq 0), k \geq n+1)$. Hence there is a positive probability that in addition to the event above we also have $\{d_{n+1}(M_1) = 0, d_{n+2}(M_1) = 0, \ldots, d_{n+N-1}(M_1) = 0\}$. But this implies that there is a positive probability that there is a merge at one of the times from n to $n+N$, i.e. the creation of a new yellow from a red and a blue. Hence the rate at which yellows leave the node must strictly exceed the rate at which they enter, which was to be established. •

7. MAIN RESULT

We are now in a position to state and prove the result claimed in the introduction.

Theorem 8 : Consider a monotone discrete time quasireversible node. Let $\lambda < \lambda_0$, and let $\mu \in \mathcal{M}_S^e(\lambda)$ be such that $T(\mu) = \mu$, i.e. μ is a stationary ergodic fixed point of the input output map. Then μ is the distribution of an independent sequence of Poisson random variables of rate λ.

Proof : Let μ^0 denote the distribution of an independent sequence of Poisson random variables. We know that $T(\mu^0) = \mu^0$, see [14]. Suppose $\mu \neq \mu^0$. Since $\bar{\rho}$ is a metric by Theorem 6 (i), we have $\bar{\rho}(\mu, \mu^0) > 0$. By Theorem 7, $\bar{\rho}(\mu, \mu^0) > \bar{\rho}(T(\mu), T(\mu^0)) = \bar{\rho}(\mu, \mu^0)$. This is an absurdity. Hence $\mu = \mu^0$. •

8. CONCLUDING REMARKS

Lack of monotonicity of the quasireversible node in the above would cause Lemma 2 to fail, so that the entire structure of the proof would collapse. Note that in the absence of monotonicity we do not even know if the existence of a stationary regime is ensured by a simple rate condition as in Theorem 2. Nevertheless, it is difficult to imagine how the result can fail to be true even in the absence of monotonicity.

Another question left unanswered at the moment is the interesting one of what happens when a stationary ergodic arrival process meeting the rate condition is put through a long tandem of identical monotone quasireversible nodes. One would expect that the stationary departure process converges weakly to the unique fixed point as the length of the tandem goes to infinity. In the above notation, to prove this we would have to show that $T^k(\mu) \to \mu^0$ as $k \to \infty$, where T^k denotes T iterated k times, and μ^0 is as defined in Theorem 8. Let $K \subset \mathcal{M}_S(\lambda)$ denote the set of weak limit points of the sequence $(T^k(\mu), k \geq 1)$. K is closed, and hence compact by Lemma 3. Since $\mu \to \bar{\rho}(\mu, \mu^0)$ is a lower semicontinuous function on the compact set K, it attains it minimum at some $\tilde{\mu}$. Suppose $\tilde{\mu} \neq \mu^0$. Then, by Theorem 6 (i), $\bar{\rho}(\tilde{\mu}, \mu^0) > 0$. If we could show that $\tilde{\mu} \in \mathcal{M}_S^e(\lambda)$ then we arrive at a contradiction by Theorem 7, and it is a simple step to show that $(T^k(\mu))_k$ converges weakly to μ^0. The problem is that it is a priori possible for the sequence of ergodic processes $(T^k(\mu))_k$ to have a nonergodic subsequential limit, although this seems extremely unlikely in the problem at hand. We are currently attempting to resolve this issue.

For full details of the proofs of the theorem stated without proof, see [1].

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