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ON FAST SIMULATION OF THE TIME TO SATURATION OF SLOTTED ALOHA

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Abstract

Cottrell et al. (1983) have indicated how ideas from the large deviations theory lead to fast simulation schemes that estimate the mean time taken by the slotted ALOHA protocol to saturate starting empty. Such fast simulation schemes are particularly useful when the attempt probability is small. The remaining time to saturation when the protocol has been operating for a time is more accurately described by the quasi-stationary exit time from the stable regime. The purpose of this article is to prove that the ratio of the quasi-stationary exit time to the exit time starting empty approaches 1 as the attempt probability becomes small.

IMPORTANCE SAMPLING; LARGE DEVIATIONS; MULTIPLE-ACCESS; QUASI-STATIONARY DISTRIBUTION

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1. Introduction

Slotted ALOHA is a well-known multi-access communication protocol which originated in the early 1970s (Abramson [2]). For a historical discussion and for details about several versions of ALOHA, we refer the reader to the recent text of Bertsekas and Gallager [3], and to the special issue of the *IEEE Transactions of Information Theory* [1]. The protocol is usually analyzed by means of the following *infinite user model*.

Packets to be transmitted over the channel are assumed to arrive according to a Poisson process of rate λ. Time is divided into slots — the duration of a slot is equal to the transmission time of a packet, and this is assumed to be constant 1. A newly arrived packet immediately attempts to use the slot after the one during which it arrives. Thus the total number of fresh packets attempting transmission in a slot is a random variable with Poisson (λ) distribution, independent from slot to slot. Only one packet can be successfully transmitted in a slot — if more than one packet attempts to use a slot all attempting packets are blocked and need to be retransmitted in subsequent slots. Packets that have been blocked are called backlogged packets. Each packet that is backlogged at

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the beginning of a slot independently attempts to use the slot with probability p, independently from slot to slot. The attempt mechanism is independent of the arrival process.

With these assumptions, the slotted ALOHA protocol may be described by a discrete-time Markov chain $(X_n, n \geqq 0)$, with state space $\mathbb{Z}_+$, the set of non-negative integers, and with transition matrix $T = [t_{ij}]$ given by

$$t_{i,i-1} = \exp(-\lambda)\alpha(i) \quad \text{if } i \geqq 1, \qquad t_{i,i+1} = \lambda \exp(-\lambda)\beta(i) \quad \text{if } i \geqq 0,$$

$$t_{i,i+k} = \frac{\lambda^k}{k!}\exp(-\lambda) \quad \text{if } k \geqq 2 \text{ and } i \geqq 0$$

and

$$t_{ii} = \exp(-\lambda)(1-\alpha(i)) + \lambda \exp(-\lambda)(1-\beta(i)) \quad \text{if } i \geqq 0$$

where

$$\alpha(i) = ip(1-p)^{i-1} \quad \text{for } i \geqq 1 \quad \text{and} \quad \beta(i) = 1-(1-p)^i \quad \text{for } i \geqq 0.$$

Here X_n represents the number of backlogged packets at time n, i.e. immediately after the completion of the nth slot. For convenience, operation is assumed to start at time -1, and the 'initial' condition X_0 is prescribed after the end of the first slot $(-1, 0]$. Thus packets arriving in $(-1, 0]$ can compete for the slot $(0, 1]$ together with the X_0 backlogged packets at time 0.

The above equations are understood as follows. Suppose $X_n = i$. Then $X_{n+1} = i-1$ iff there are no arrivals in slot n and exactly one of the i backlogged packets at the end of slot n attempts transmission in slot $n+1$. $X_{n+1} = i+1$ iff exactly one fresh packet arrived in slot n and at least one of the i backlogged packets at the end of slot n attempts transmission in slot $n+1$ — since two or more packets would have attempted to transmit in slot $n+1$ in this scenario, no packet can get through. For $k \geqq 2$, $X_{n+1} = i+k$ iff exactly k packets arrived in slot n — it does not matter whether or not the backlogged packets attempt to transmit in slot $n+1$ since the k fresh packets all attempt to transmit, and so no packet gets through. In all the remaining scenarios, we have $X_{n+1} = i$.

We observe that $\beta(i)$ is increasing in i, while $\alpha(i)$ increases to reach its maximum at $i_0 = \lfloor 1/p \rfloor$ after which it decreases. Let $d(i)$ denote the upward drift of the number of backlogged packets when the chain is in state i: $d(i) = \boldsymbol{E}[X_{n+1} - X_n \mid X_n = i]$. Then we can write

$$d(i) = \lambda - b(i) \tag{1.1}$$

where

$$b(i) = \exp(-\lambda)ip(1-p)^{i-1} + \lambda \exp(-\lambda)(1-p)^i. \tag{1.2}$$

We observe that $b(i)$ attains its maximum, and hence $d(i)$ attains its minimum at $i^* = \lceil (1-\lambda)(1-p)/p \rceil$. Further, $b(i)$ is increasing on $[0, i^*]$, and decreasing on

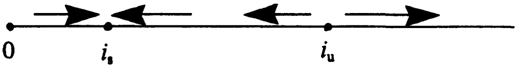
$[i^* + 1, \infty)$. So, if $\lambda < b(i^*)$, we see that there are uniquely defined i_s and i_u such that $0 < i_s < i^* < i_u$ and such that

$$b(i_s) \leqq \lambda < b(i_s + 1) \tag{1.3}$$

and

$$b(i_u) \geqq \lambda > b(i_u + 1) \tag{1.4}$$

i_s and i_u are called the stable and unstable critical points of the protocol respectively. The reason for this terminology should be apparent from a glance at Figure 1. It is to be intuitively expected that if the protocol is started to the left of i_u, it will spend a large proportion of time in the vicinity of i_s, until at some point a large excursion will take it beyond i_u, after which with high probability it will continue on to ∞.



Drift direction

Figure 1

We note that as $p \to 0$, $b(i^*) \to 1/e$, $i_s p \to c_s$ and $i_u p \to c_u$, where c_s and c_u are the two solutions of $c \exp(-c) = \lambda$. Throughout this paper, we assume that $\lambda < 1/e$.

Suppose the protocol has been operating reasonably for a long time and it is desired to know how long one can expect operation to continue before the protocol saturates. A natural measure of this time is the quantity $\boldsymbol{E}_\nu T$ where $T = \inf\{n > 0 : X_n \geqq i_u\}$, and ν is the quasi-stationary distribution of the process confined to $\{0, 1, \cdots, i_u\}$. This is the distribution given by

$$\nu(i) = \lim_{n \to \infty} P_\omega(X_n = i \mid X_m \in \{0, 1, \cdots, i_u\}, 0 \leqq m \leqq n - 1)$$

for $i \in \{0, 1, \cdots, i_u\}$. The limit is known to exist and to be independent of initial distributions ω which are supported on $\{0, 1, \cdots, i_u\}$, [5], and [7], pp. 90 ff. Further, as is evident from the definition, ν has the interpretation of the distribution in which the protocol will be found given that it has been operating satisfactorily for a long time.

$\boldsymbol{E}_\nu T$ is thus of considerable interest as a performance measure for the protocol; see Parekh et al. [9] and Schoute [12]. Since it is difficult to get a handle on $\boldsymbol{E}_\nu T$ analytically, it is natural to attempt to determine it by simulation. One of the difficulties in doing this simulation is that saturation is a rare event when $\lambda < 1/e$. Thus a straightforward regenerative simulation will see the system saturating very infrequently, and it will take many simulation runs to generate acceptable confidence bounds for the estimated time to saturation.

The quick simulation method of Cottrell et al. [4] may be used to speed up the estimation of $\boldsymbol{E}_0 T$, which is the mean time to reach i_u with the system starting empty. The idea here is to simulate a modified system in which saturation is more likely, and to generate estimates for E_0T by incorporating the likelihood ratio between the dynamics of the two systems. For more details, we refer to [4], [10], [13], and [14]. Since the real time of interest is E_vT rather than E_0T, it is natural to ask whether this method may be used also to effectively estimate the quasi-stationary exit time [9].

In this paper, we analyze the dynamics of the ALOHA protocol motivated by the above question. The main conclusion is rather intuitive. We prove that

$$\lim_{p\to 0} \frac{E_0T}{E_vT} = 1. \tag{1.5}$$

Thus, for small p (which is also required for the quick simulation method to be efficient), the quasi-stationary exit time does not differ very much from the time to exit starting from 0. The simulation-based estimates available in [4] may therefore be used with more confidence as representative of the actual quasi-stationary exit time.

In Section 2 we derive several preliminary lemmas, which will be used to prove the main result in Section 3. Several of the lemmas involve drift analysis, for which a good reference is [6]. The reader may wish to go directly to Section 3 after reading the statements of the lemmas in Section 2 before returning to their proofs. Some remarks on notation: let f and g be non-negative functions of p. We write $f = O(g)$ to mean $\limsup f(p)/g(p) \leqq K$ for some $K < \infty$ as $p \to 0$. We write $f = \Omega(g)$ to mean $\liminf f(p)/g(p) \geqq k$ for some $k > 0$ as $p \to 0$.

2. Preliminary lemmas

The first lemma shows that the time it takes for the number of backlogged packets to exceed a level is a decreasing function of the initial condition in the sense of stochastic order. For the definition and basic properties of stochastic ordering of random variables, see Ross [11].

Lemma 1. Let $i_a \leqq i_b \leqq i_r$. Let $\tau_r = \inf\{n > 0 : X_n \geqq i_r\}$. Let τ_{ar} have the distribution of τ_r started from i_a, and let τ_{br} have the distribution of τ_r started from i_b. Then $\tau_{ar} \geqq_s \tau_{br}$, where $\geqq_s$ denotes stochastic ordering of random variables.

Proof. We couple the processes from the two initial conditions, i.e. on the same sample space we construct versions $(A_n, n \geqq 0)$ and $(B_n, n \geqq 0)$ of $(X_n, n \geqq 0)$ such that $A_0 = i_a$, $B_0 = i_b$, and so that $A_n \leqq B_n$ for all $n \geqq 0$. Clearly this is enough to verify the claim. The proof is by induction on n. To start the induction, we are given $A_0 \leqq B_0$. Suppose $A_n \leqq B_n$. Identify A_n of the B_n backlogged packets in the B process at time n with the backlogged packets in the A process. Recall that the backlogged packets attempt to transmit in the nth slot independently of each other with probability p, and fresh packets arriving at time n immediately attempt to transmit in slot n. We consider several cases. If $A_n = B_n$ then $A_{n+1} = B_{n+1}$. If two or more packets arrive at time n, then $B_{n+1} - A_{n+1} = B_n - A_n \geqq 0$. If $B_n \geqq A_n + 1$, and exactly one packet arrives at time n, then $B_{n+1} \geqq B_n$ and $A_n + 1 \geqq A_{n+1}$, so $B_{n+1} \geqq A_{n+1}$. If $B_n \geqq A_n + 1$, and no packet arrives at time n, then $B_{n+1} \geqq B_n - 1$ and $A_n \geqq A_{n+1}$, so $B_{n+1} \geqq A_{n+1}$. Thus the inductive hypothesis propagates in all scenarios.

The next lemma considers the finite-state Markov chain one gets if the number of backlogged packets is not allowed to exceed i_u, i.e. any excess packets are simply discarded. It is shown that if this modified process is started with i_u backlogged packets, then the number of backlogged packets decreases relatively rapidly to about $i_s^\delta = \lfloor i_s + \delta/p \rfloor$.

Lemma 2. Fix $\delta > 0$. Consider the slotted ALOHA protocol on the interval $[i_s^\delta, i_u]$ and so that the state is returned to i_u whenever it exceeds i_u. Let $(X_n^f, n \geqq 0)$ denote the corresponding Markov chain. Let $\tau_\delta = \inf\{n > 0 : X_n^f = i_s^\delta\}$. Then $E_{i_u}\tau_\delta = O(1/p^2)$.

Proof. A calculation shows that we can find $0 < c < \infty$ such that $\boldsymbol{E}(X_{n+1}^f - X_n^f \mid X_n^f = i) < -cp$ for all $i_s^\delta \leqq i \leqq i_u$. Thus $M_n = X_n^f + ncp$ satisfies $\boldsymbol{E}(M_{n+1} - M_n \mid M_m, m \leqq n) < 0$, i.e. $(M_n, n \geqq 0)$ is a positive supermartingale. Clearly $\boldsymbol{E}_{i_u} M_0 = i_u$. By the optional stopping theorem for positive supermartingales (Neveu [8], Theorem II-2-13) we have $\boldsymbol{E}_{i_u} M_{\tau_\delta} = i_s^\delta + (\boldsymbol{E}\tau_\delta)cp \leqq i_u$ which yields

$$\boldsymbol{E}\tau_\delta \leqq \frac{i_u - i_s^\delta}{cp} = O\left(\frac{1}{p^2}\right).$$

We next show that the system takes a long time to work its way against the drift.

Lemma 3. Fix $0 < \alpha < \beta$. Let $i_s^\alpha = \lfloor i_s + \alpha/p \rfloor$ and $i_s^\beta = \lfloor i_s + \beta/p \rfloor$. Consider the slotted ALOHA protocol on the interval $[i_s^\alpha, i_s^\beta]$, and assume that the state is returned to i_s^α whenever it attempts to fall below i_s^α, and returned to i_s^β whenever it attempts to exceed i_s^β. Let $(X_n^r, n \geqq 0)$ denote the resulting process. Let $\tau_\beta = \inf\{n > 0 : X_n^r = i_s^\beta\}$. Then $\boldsymbol{E}_{i_s^\alpha}\tau_\beta = \Omega(\exp(1/p))$.

Proof. Let π^r denote the stationary distribution of $(X_n^r, n \geqq 0)$. Let us accept that we can find $\rho < 1$ independent of p such that

$$\pi^r(i_s^\beta) \leqq \rho^{1/p} \pi^r(i_s^\alpha). \tag{2.1}$$

Start $(X_n^r, n \geqq 0)$ at i_s^α and split the path by means of times $T_1, S_1, T_2, S_2, \cdots$ defined as follows:

$$T_k = \inf\left\{m > 0 : n = \sum_{l=1}^{k-1} T_l + \sum_{l=1}^{k-1} S_l + m \text{ has } X_n^r = i_s^\beta\right\}$$

$$S_k = \inf\left\{m > 0 : n = \sum_{l=1}^{k} T_l + \sum_{l=1}^{k-1} S_l + m \text{ has } X_n = i_s^\alpha\right\}.$$

Note that $T_1 =^d \tau_\beta$. Also, $\boldsymbol{E}S_k = O(1/p^2)$ by Lemma 2 and a straightforward coupling argument. Now, observing that the intervals $[\Sigma_{l=1}^{k-1} T_l + \Sigma_{l=1}^{k-1} S_l, \Sigma_{l=1}^{k} T_l + \Sigma_{l=1}^{k} S_l)$ split the path of $(X_n^r, n \geqq 0)$ into i.i.d. cycles, and observing that the path visits i_s^β at least once in each such cycle we get

$$\left(\frac{1}{\rho}\right)^{1/p} \leqq \frac{\pi^r(i_s^\alpha)}{\pi^r(i_s^\beta)} \leqq \boldsymbol{E}T_k + \boldsymbol{E}S_k.$$

From this $E\tau_\beta = \Omega(\exp(1/p))$ follows.

It remains to show (2.1). First observe that we can find $c > 0$ such that, with $d^r(i) = E[X^r_{n+1} - X^r_n \mid X^r_n = i]$, we have $d^r(i) < -c$, $i^\alpha_s < i \leqq i^\beta_s$. Now introduce the notation $\mathscr{X}_i$ for $[i, i^\beta_s]$, $i^\alpha_s \leqq i \leqq i^\beta_s$. Let $(X^{r,i}_n, n \geqq 0)$ denote the Markov chain $(X^r_n, n \geqq 0)$ watched in $\mathscr{X}_i$. See Walrand [13], p. 68, for a definition of this concept. Let $\pi^{r,i}$ denote the stationary distribution of $(X^{r,i}_n, n \geqq 0)$. Then we have

$$\pi^{r,i}(j) = \frac{\pi^r(j)}{\pi^r(\mathscr{X}_i)}, \qquad i \leqq j \leqq i^\beta_s.$$

Let $d^{r,i}(j) = E[X^{r,i}_{n+1} - X^{r,i}_n \mid X^{r,i}_n = j]$, $i \leqq j \leqq i^\beta_s$. Then $d^{r,i}(j) = d^r(j) < -c$, $i < j \leqq i^\beta_s$, while $d^{r,i}(i) \leqq \lambda$. But the net drift from all states in stationarity must be 0. This gives

$$\sum_{j \in \mathscr{X}_i} \pi^{r,i}(j) d^{r,i}(j) = 0.$$

Hence

$$\lambda \pi^{r,i}(i) - c \sum_{i < j \leqq i^\beta_s} \pi^{r,i}(j) \geqq 0.$$

But this means that

$$\pi^r(i) \geqq \frac{c}{\lambda} \sum_{i < j \leqq i^\beta_s} \pi^{r,i}(j).$$

From this it is straightforward algebra to deduce (2.1).

The next lemma estimates the quasi-stationary distribution.

Lemma 4. Let ν denote the quasi-stationary distribution of the slotted ALOHA protocol on the interval $[0, i_u]$. Given $\delta > 0$, let i^δ_s denote $\lfloor i_s + \delta/p \rfloor$. Then $\lim_{p \to 0} \nu([0, i^\delta_s]) = 1$.

Proof. ν admits the following interpretation: run the protocol and whenever the state exceeds i_u, restart the system in the distribution ν. Then the stationary distribution of the resulting finite-state Markov chain is ν.

From this it is easy to see that if instead we restart the system at i_u whenever it exceeds i_u, then the resulting Markov chain $(X^f_n, n \geqq 0)$ on $[0, i_u]$ has stationary distribution π^+ which stochastically dominates ν in the linear order on $[0, i_u]$. In particular

$$\pi^+[0, i^\delta_s] \leqq \nu[0, i^\delta_s]. \tag{2.2}$$

Let $\alpha = \delta/2$ and $\beta = \delta$. Let i^α_s and i^β_s denote $\lfloor i_s + \alpha/p \rfloor$ and $\lfloor i_s + \beta/p \rfloor$ respectively. Now, start this chain from i_u and split the path into times $T_1, S_1, T_2, S_2, \cdots$ defined as follows:

$$T_k = \inf\left\{ m > 0 : n = \sum_{l=1}^{k-1} T_l + \sum_{l=i}^{k-1} S_l + m \text{ has } X_n \in [0, i^\alpha_s] \right\}$$

$$S_k = \inf\left\{ m > 0 : n = \sum_{l=1}^{k} T_l + \sum_{l=1}^{k-1} S_l + m \text{ has } X_n \in [i^\beta_s, i_u] \right\}.$$

Since the intervals

$$\left[\sum_{l=1}^{k} T_l + \sum_{l=1}^{k-1} S_l, \sum_{l=1}^{k+1} T_l + \sum_{l=1}^{k} S_l\right)$$

split the path of $(X_n^f, n \geqq 0)$ into i.i.d. cycles for $k \geqq 2$, and since the time spent by the path in $(i_s^\beta, i_u]$ on the kth cycle is no more than T_{k+1} while the time spent in $[0, i_s^\beta]$ is at least S_k, we have then

$$\frac{\pi^+((i_s^\beta, i_u])}{\pi^+([0, i_s^\beta])} \leqq \frac{ET_{k+1}}{ES_k} \qquad \text{for all } k \geqq 2. \tag{2.3}$$

Further, an easy coupling shows that

$$ET_k \leqq E\tau_\alpha = E_{i_u}[\inf\{n > 0 : X_n^f = i_s^\alpha\}]. \tag{2.4}$$

Note that $E\tau_\alpha = O(1/p^2)$ by Lemma 2. Also,

$$ES_k \geqq E\tau_\beta = E_{i_s^\alpha}[\inf\{n > 0 : X_n^r \geqq i_s^\beta\}] \tag{2.5}$$

in the chain $(X_n^r, n \geqq 0)$ where the state is returned to i_s^α whenever it attempts to fall below i_s^α and returned to i_s^β whenever it attempts to exceed i_s^β. Note that $E\tau_\beta = \Omega(\exp(1/p))$ by Lemma 3.

From (2.2), (2.3), (2.4), and (2.5) the claim follows.

We next show that the system reaches the stable equilibrium point relatively rapidly when it is started empty.

Lemma 5. Consider the slotted ALOHA protocol. Let $\tau_s = \inf\{n > 0 : X_n \geqq i_s\}$. Then

$$E_0\tau_s = O(1/p^2).$$

Proof. We compute the rightward drift $d(i)$ of the protocol at states $0 \leqq i < i_s$. From (1.1), (1.2) and (1.3), we see easily that $d(i)$ is decreasing and strictly positive on $[0, i_s - 1]$. From (1.2), we can also write

$$\frac{b(i+1)}{b(i)} = (1-p)\left[\frac{(i+1)p + \lambda(1-p)}{ip + \lambda(1-p)}\right].$$

From this, and because $b(i_s) \leqq \lambda$, it is a straightforward calculation to see that

$$d(i_s - 1) \geqq \frac{\lambda p}{(1-p)}\left[\frac{1}{i_s p + \lambda(1-p)} - 1\right].$$

which is strictly bounded below by a constant time p for all sufficiently small p. (Observe that $\lambda < 1/e$, and $i_s p$ is asymptotically c_s, where c_s solves $c_s \exp(-c_s) = \lambda$, so that $c_s < 1$.) The lemma is now easily verified by a drift analysis similar to the proofs of Lemmas 1 and 2.

Finally, even if there is a substantial number of backlogged packets, the system is likely to approach the stable equilibrium rather than saturate.

Lemma 6. Consider the slotted ALOHA protocol started from $i_s^\delta = \lfloor i_s + \delta/p \rfloor$. Then

$$\lim_{p \to 0} P_{i_s^\delta}(X_n \text{ hits } i_s \text{ before it exceeds } i_u) \geqq 1 - \frac{\delta}{c_u - c_s}.$$

Proof. Consider the system on $[i_s, i_u]$ which is absorbed at i_u when it attempts to exceed i_u, and absorbed at i_s when it hits i_s. Denote the corresponding process by $(X_n^a, n \geqq 0)$. Note that $d^a(i) = \boldsymbol{E}[X_{n+1}^a - X_n^a \mid X_n^a = i] \leqq 0$ for all $i_s \leqq i \leqq i_u$, so $(X_n^a, n \geqq 0)$ is a positive supermartingale. Let $\eta = \inf\{n > 0 : X_n^a = i_s \text{ or } X_n^a = i_u\}$. By the optional stopping theorem for positive supermartingales (Neveu [8], Theorem II-2-13), we have

$$\begin{aligned} i_s^\delta \geqq \boldsymbol{E}_{i_s^\delta} X_\eta^a &= \boldsymbol{P}_{i_s^\delta}(X_\eta^a = i_s) i_s + \boldsymbol{P}_{i_s^\delta}(X_\eta^a = i_u) i_u \\ &= i_u - \boldsymbol{P}_{i_s^\delta}(X_\eta^a = i_s)(i_u - i_s). \end{aligned}$$

But

$$\boldsymbol{P}_{i_s^\delta}(X_\eta^a = i_s) = \boldsymbol{P}_{i_s^\delta}(X_n \text{ hits } i_s \text{ before it exceeds } i_u).$$

The lemma follows directly.

3. Main result

In this short section, we bring together the above lemmas to complete the proof of (1.5). First observe that $\boldsymbol{E}_0 T \geqq \boldsymbol{E}_\nu T$ by Lemma 1. Hence, to prove (1.5), it suffices to show that

$$\liminf_{p \to 0} \frac{\boldsymbol{E}_\nu T}{\boldsymbol{E}_0 T} \geqq 1.$$

It is easy to see that

$$\boldsymbol{E}_\nu T \geqq \nu([0, i_s^\delta]) \boldsymbol{E}_{i_s^\delta} T. \tag{3.1}$$

From Lemma 4, it follows that

$$\liminf_{p \to 0} \frac{\boldsymbol{E}_\nu T}{\boldsymbol{E}_{i_s^\delta} T} \geqq 1. \tag{3.2}$$

But, by the strong Markov property, we also have $\boldsymbol{E}_{i_s^\delta} T \geqq \boldsymbol{P}_{i_s^\delta}(\text{hit } i_s \text{ before hit } i_u) \boldsymbol{E}_{i_s} T$. Hence, by Lemma 6, we have

$$\liminf_{p \to 0} \frac{\boldsymbol{E}_{i_s^\delta} T}{\boldsymbol{E}_{i_s} T} \geqq 1 - \frac{\delta}{c_u - c_s}. \tag{3.3}$$

Now, it is also easy to show that $\boldsymbol{E}_0 T \leqq \boldsymbol{E}_0 \tau_s + \boldsymbol{E}_{i_s} T$. Further, $\boldsymbol{E}_{i_s} T \geqq \boldsymbol{E}_{i_s^\delta} T = \Omega(\exp(1/p))$ by Lemma 3, whereas $\boldsymbol{E}_0 \tau_s = O(1/p^2)$ by Lemma 5, Hence we have

$$\liminf_{p \to 0} \frac{\boldsymbol{E}_{i_s} T}{\boldsymbol{E}_0 T} \geqq 1. \tag{3.4}$$

From (3.2), (3.3), and (3.4), and letting $\delta \to 0$, we get (3.1) as desired, completing the proof of (1.5).

4. Concluding remarks

In this letter we have established that as the attempt probability tends to 0, the quasi-stationary time to saturation of slotted ALOHA is asymptotically the same as the time to saturation starting empty. This work was motivated by the observation that the former is a more meaningful performance measure for slotted ALOHA than the latter. For the time to saturation started empty, the quick simulation method of Cottrell et al. [4] gives an efficient estimation scheme, based on ideas from the large deviation theory. Further, this scheme is particularly efficient for small attempt probabilities. The contribution of this paper is to point out that these numbers can be taken as representative also of the quasi-stationary time to saturation, and hence used with more confidence as a representative performance measure.

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