Proceedings of the 28th Conference
on Decision and Control
Tampa, Florida • December 1989

THE STABILITY REGION OF THE FINITE-USER SLOTTED ALOHA PROTOCOL

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ABSTRACT

We consider a version of the discrete time slotted ALOHA protocol operating with finitely many buffered terminals. The stability region is defined to be the set of vectors of arrival rates $\lambda = (\lambda_1, \ldots, \lambda_M)$ for which there exists a vector of transmission probabilities such that the system is stable. We assume arrivals are independent from slot to slot, and assume the following model for the arrival distribution in a slot: The total number of arrivals in any slot is geometrically distributed, with the probability that such an arrival is at node i being $\lambda_i(\sum_k \lambda_k)^{-1}$, independent of the others. With this arrival model, we prove that the closure of the stability region of the protocol is the same as the the Shannon capacity region of the collision channel without feedback, as determined by Massey and Mathys. The basic probabilistic observation is that the stationary distribution and certain conditional distributions derived from it have positive correlations for bounded increasing functions. Similiar techniques may be of use in studying other interacting systems of queues. At present it is not clear to us if our result depends on the choice of arrival distribution.

INTRODUCTION

Slotted ALOHA, [1], is a technique for multiple access communication that is by now well known. An extensive discussion of several versions of ALOHA and other multiple access schemes is available in the recent text of Bertsekas and Gallager, [4]. We also refer the reader to the special issue, [0], of the IEEE Transactions on Information Theory, and in particular to [8] and [25]. The study of such multi-access schemes is by now almost two decades old, but the area continues to attract considerable research activity, witness e.g. [3], [9], [14], [15], [18], [20], [21], [22], [24], which have all appeared in print in the last two years.

Most of the successful analyses of ALOHA and related collision resolution protocols have relied on the infinite-user assumption. Here each node regards itself as a collection of virtual nodes, one for each arriving packet. The effect of this assumption is that one can ignore buffering effects at the individual nodes. It is well known that the original ALOHA protocol is theoretically incapable of sustaining any arrival rate under the infinite node assumption. A particularly simple proof follows from the result of [13]. On the other hand, ingenious distributed control schemes are known which can achieve positive throughput in the infinite user model, [11], [24].

Despite the existence of such control strategies, the original slotted ALOHA protocol continues to be of interest, mainly because of its extreme simplicity. A more realistic analysis of this protocol would explicitly take into account the effects of buffering at a finite set of nodes. The stability region of the system, i.e. the set of vectors of arrival rates that such a system can sustain, is of particular interest. Starting with the original paper of Abramson, [1], there has been a considerable amount of work addressing the problem of determining this stability region. We refer the reader to [14] for a discussion, and for references to this work. This problem has turned out to be more difficult than it might appear at first sight. For the protocol with two users, the stability region was determined almost ten years ago, [26]. However, when there are more than two users, only inner and outer bounds for the capacity region are known, [14], [20], [21].

In this paper we study the stability region of a version of the finite user slotted ALOHA protocol. For this version, we prove that the closure of the stability region is precisely the capacity region of the collision channel without feedback, as determined by Massey and Mathys, [17]. The meaning of the result is that a vector of arrival rates can be sustained if and only if it is also possible to sustain the arrival rate at each node individually even when all the other buffers are always full.

A result of this sort has been widely expected for finite user slotted ALOHA, [21]. One might reasonably expect that the stability region of the protocol should be the same for different arrival statistics, although care is needed because of the interaction, see the concluding remarks. At present it is not clear to us if we can translate our result to the usual arrival models, which assume that the number of arrivals at the individual nodes in a slot are independent.

Our proof uses some simple and powerful ideas which are well known in the study of interacting particle systems, but do not appear to have been used previously in the study of communication networks. It would be interesting if it turns out that these ideas have wider applicability in models of engineering interest.

SETUP

We consider the discrete time slotted ALOHA protocol operating with M buffered terminals over the collision channel. Thus, transmission attempts are made at discrete times

Research supported by NSF Grant No. NCR 8710840, an NSF PYI Award, IBM, and BellCore

- at each time n each terminal i attempts transmission with probability p_i, if there is a packet in buffer i at time n. If two or more packets attempt transmission at the same time, all attempting terminals are unsuccessful. Transmission attempts by a terminal are made according to coin flips which are independent from time to time, independent from terminal to terminal, and independent of the arrival processes. We may assume that each node makes a coin flip at each time whether or not it has a packet, allowing us to define a Bernoulli process of virtual attempts associated to each node. By convention we will assume that arrivals at any time occur just prior to transmission attempts.

For the arrival process we will assume the following model - the rate of arrivals into node i is to be λ_i, $1 \leq i \leq M$. We assume that the total number of packets arriving in a time slot is geometrically distributed, and that each packet is likely to be a packet at node i with probability $\lambda_i / \sum_k \lambda_k$, independently of the others. This assumption has the effect of allowing us to think of the discrete time Markov chain describing the protocol as an embedded chain associated to a certain continuous time Markov process.

The usual arrival model takes the number of arrivals at the individual nodes in each time slot to be independent Poisson random variables of the appropriate means. At the present writing, our result appears to depend on our choice of arrival model, which is different from the usual one.

Let $A_i := (A_i(n))_{n \geq 0}$, and $V_i := (V_i(n))_{n \geq 0}$, $1 \leq i \leq M$, denote the arrival processes and the virtual attempt processes respectively. Let $Q_i := (Q_i(n))_{n \geq 0}$ denote the state of buffer i immediately after the arrival of the packet at time n, if any, and immediately before the transmission attempt, if any. The evolution of Q_i can be determined by defining

$$\tilde{V}_i(n) := V_i(n)\ 1(Q_i(n) > 0)\ ,$$

$$C(n) := 1(|\{i : \tilde{V}_i(n) = 1\}| > 1)\ ,$$

where $1(\cdot)$ denotes the function which is 1 if the event inside the brackets occurs, and is 0 otherwise, and $|\{\cdot\}|$ denotes the cardinality of the set $\{\cdot\}$. Then we have

$$Q_i(n_+) = Q_i(n) - \tilde{V}_i(n)\ 1(C(n) = 0)\ ,$$

$$Q_i(n+1) = Q_i(n_+) + A_i(n+1)\ .$$

Further, if $D_i := (D_i(n))_{n \geq 0}$ denotes the process of successful transmissions from terminal i, we have

$$D_i(n) = \tilde{V}_i(n)\ 1(C(n) = 0)\ .$$

$\mathbf{Q} = (Q_1, ..., Q_M)$ evolves according to a discrete-time Markov process with state space $\mathbf{Z}_+^M$. The system is said to be ***stable*** if this Markov chain admits a stationary probability distribution. A vector of arrival rates $\lambda = (\lambda_1, ..., \lambda_M)$ is said to be in the ***stability region*** of the system if there exists a vector of transmission attempt probabilities $\mathbf{p} = (p_1, ..., p_M)$, $0 \leq p_i \leq 1$, such that the resulting system is stable.

Consider the subset of $\mathbf{R}_+^M$ given by

$$C = \left\{ \text{vect} \left(p_i \prod_{j \neq i} (1 - p_j) \right) : 0 \leq p_i \leq 1,\ 1 \leq i \leq M \right\}\ .$$

It is elementary to see that C is a bounded region of $\mathbf{R}_+^M$, which is "co-ordinate convex" in the sense that the closed cube determined by the origin and a point of C is contained in C. For the origin of this terminology, see [2] and [7]. Such a set is sometimes called a "corner", see [6]. It is also known that the complement of C is convex, [19].

In [17], Massey and Mathys studied an information theoretic model for multiple access communication which shares some of the fundamental features of the slotted ALOHA protocol. They were able to determine the Shannon capacity region of this channel, which turns out to be C.

Our main result is the following :

Theorem 2.1 : The closure of the stability region of the slotted ALOHA protocol with the arrival model above is C.

A sketch of the proof of Thm. 2.1 is as follows : It is elementary to prove that $C - \Delta(C)$ is contained in the stability region of the protocol, where $\Delta(C)$ denotes the boundary of C as a subset of $\mathbf{R}_+^M$ (its outer boundary). Thus, to prove Thm. 2.1, it is enough to show that the stability region is contained in C. Suppose the system is stable, admitting stationary distribution π. Since the system empties more easily if some of the buffers are empty, it is natural to expect that π has *positive correlations*, in the sense of [16]. As was mentioned above, we can think of the discrete time process $\mathbf{Q}$ as an embedded chain of a continuous time Markov process, $\mathbf{Y}$. In section 3, we study $\mathbf{Y}$. We observe that $\mathbf{Y}$ also has stationary distribution π. Further, we observe that it is *monotone*. We then prove a small generalization of the Harris correlation inequality, [5], [12], [16], by imitating the main step in the proof of Cox, [5]. This allows us to use the fact that every jump of $\mathbf{Y}$ is *up or down* in the partial order on $\mathbf{Z}_+^M$ to conclude that π has positive correlations for bounded monotone functions (π *is pcb*). This proves Thm. 2.1 for $M = 2$, but unfortunately does not appear to be enough to deal with general M. We are led to examine certain conditional distributions associated with π. In section 4, we use a technique parallel to that of section 3 to conclude that these conditional distributions are pcb. This concludes the proof of Thm. 2.1 for general M.

To close this section, we state the following result from [26] :

Lemma 2.2 : Let $\lambda = (\lambda_1, \ldots, \lambda_M)$, with $\lambda_i > 0$, $1 \leq i \leq M$. If there is $x > 0$ such that

$$x^{M-1} \geq \prod_{i=1}^{M} (x + \lambda_i)\ , \tag{2.1}$$

then $\lambda \in C$.

Proof : Suppose (2.1) holds. Let

$$q_i = \frac{\lambda_i}{x + \lambda_i}\ ,\ 1 \leq i \leq M\ .$$

Then $\lambda_i \leq q_i \prod_{j \neq i}(1 - q_j)$. Since C is co-ordinate convex, $\lambda \in C$.

♣

POSITIVE CORRELATIONS

We begin with some definitions. Let $(S, \leq)$ be a finite or countably infinite discrete partially ordered set. Except in the general statements below, S will be taken to be $\mathbf{Z}_+^M$ with the usual order given by

$$(x_1, ..., x_M) \leq (y_1, ..., y_M) \Leftrightarrow x_i \leq y_i \forall 1 \leq i \leq M \ .$$

A function $f : S \to \mathbf{R}$ is said to be *increasing* if $f(x) \leq f(y)$ for all $x, y \in S$ such that $x \leq y$. (Usually it is also required that f be continuous, but we will assume that S has the discrete topology, so that all functions are continuous). Let μ be a probability distribution on S. For a function $f : S \to \mathbf{R}$ we write $\mu(f)$ for $\sum_{x \in S} \mu(x) f(x)$. μ is said to have *positive correlations* if

$$f, g \ \text{ increasing } \Rightarrow \mu(fg) \geq \mu(f)\mu(g) \ .$$

If the above is true for all *bounded* increasing functions f and g, we will say that μ has positive correlations for bounded increasing functions. Below we abbreviate this to saying *μ is pcb*.

Let λ be a vector of arrival rates which can be stabilized by $\mathbf{p}$, $0 \leq p_i \leq 1$. Let π denote the stationary distribution of the resulting Markov chain $\mathbf{Q}$. We think of π as a probability distribution on the partially ordered set $(\mathbf{Z}_+^M, \leq)$. Our work in this section will be addressed at proving the following :

Lemma 3.1 : π is pcb.

To prove Lemma 3.1, we first observe that π is also the stationary distribution of the continuous time Markov process $\mathbf{Y} = (Y(t), t \geq 0)$ with state space $\mathbf{Z}_+^M$ and rate matrix Ω, given by

$$\Omega(x, x + e_i) = \lambda_i \ , \ 1 \leq i \leq M \ ,$$

$$\Omega(x, x - e_i) = p_i 1(x_i > 0) \prod_{j \neq i} (1 - p_j 1(x_j > 0)) \ , \ 1 \leq i \leq M \ ,$$

where e_i denotes the unit vector along the ith co-ordinate. To see this, observe that we may construct a version of $\mathbf{Y}$ as follows : We are given $M + 1$ independent Poisson processes $\mathcal{A}_i = (\mathcal{A}_i(t), t \geq 0)$, $1 \leq i \leq M$, of rates $\lambda_i, \ldots, \lambda_M$ respectively, and $\mathcal{D} = (\mathcal{D}(t), t \geq 0)$ of rate 1. In addition, we have M i.i.d. families of Bernoulli processes $V_i = (V_i(n), n \geq 0)$, $1 \leq i \leq M$, with $P(V_i(n) = 1) = p_i$. All these processes are independent. Define $\mathbf{Y}$ by

$$d\mathbf{Y}(t) = \sum_i d\mathcal{A}_i(t) - d\mathcal{D}_i(t) \ ,$$

where $\mathcal{D}_i = (\mathcal{D}_i(t), t \geq 0)$ are constructed by contention at the points of $\mathcal{D}$, i.e., we let d_n, $n \geq 0$, denote the points of $\mathcal{D}$, and let

$$\tilde{V}_i(n) := V_i(n) \ 1(Y_i(d_{n-}) > 0) \ ,$$

$$C(n) := 1(| \{i : \tilde{V}_i(n) = 1\} | > 1) \ ,$$

and

$$\mathcal{D}_i(t) = \sum_{d_n \leq t} \tilde{V}_i(n) 1(C(n) = 0) \ .$$

With this construction of $\mathbf{Y}$, and because Poisson arrivals see time averages, see e.g. [4] or [27], the time stationary distribution of $\mathbf{Y}$ is identical to the stationary distribution of the discrete time embedded Markov chain seen by the points of $\mathcal{D}$. But this Markov chain is just the one describing $\mathbf{Q}$.

♣

Returning to generalities, let $\mathbf{Y} = (Y_t, t \geq 0)$ be a continuous time Markov process on a countable discrete partially ordered state space $(S, \leq)$ with transition semigroup $(T_t, t \geq 0)$ and rate matrix Ω. Recall that Ω is called *uniform* if there is a uniform bound on the absolute values of its entries. Given $f : S \to \mathbf{R}$, we let $T_t f$ denote the function on S given by $T_t f(x) = \sum_{y \in S} p(t, x, y) f(y)$. $\mathbf{Y}$ is called *monotone* if for every $t \geq 0$, $T_t f$ is increasing whenever f is increasing. We have the following easy lemma :

Lemma 3.2 : Let $\mathbf{Y} = (Y_t, t \geq 0)$ be a continuous time Markov process, as above. Suppose that for every pair of initial conditions $x, y \in S$ such that $x \leq y$ there exists an $S \times S$ valued process $((Y^x(t), Y^y(t)), t \geq 0)$ such that $(Y^x(t), t \geq 0)$ is a version of $\mathbf{Y}$ with initial condition x, $(Y^y(t), t \geq 0)$ is a version of $\mathbf{Y}$ with initial condition y, and $Y^x(t) \leq Y^y(t)$ for all $t \geq 0$. Then $\mathbf{Y}$ is monotone.

♣

Let $\mathbf{Y} = (Y_t, t \geq 0)$ be a continuous time Markov process as above. Given a probability distribution μ on S, we write μT_t for the probability distribution on S given by $\mu T_t(y) = \sum_{x \in S} \mu(x) p(t, x, y)$. $\mathbf{Y}$ is said to have *positive correlations* if for every $t \geq 0$, μT_t has positive correlations whenever μ has positive correlations. We say that every jump of $\mathbf{Y}$ is *up or down* if $\mathbf{Y}$ always jumps between states which are comparable in the partial order. An important tool in the theory of interacting particle systems is the following, [5], [12], [16] :

Harris correlation inequality : A monotone process on a *finite* partially ordered set has positive correlations if and only if every jump of the process is up or down.

For our purposes we need a small generalization of the Harris correlation inequality, which we state as the following :

Lemma 3.3 : Let $\mathbf{Y} = (Y_t, t \geq 0)$ be a continuous time Markov process on a countable discrete partially ordered state space $(S, \leq)$ with transition semigroup $(T_t, t \geq 0)$ and rate matrix Ω. We say that $\mathbf{Y}$ has positive correlations for bounded increasing functions if for all $t \geq 0$, μT_t is pcb whenever μ is pcb. We abbreviate this to saying $\mathbf{Y}$ *is pcb*. Suppose that $\mathbf{Y}$ is monotone, and Ω is uniform. Then $\mathbf{Y}$ is pcb if and only if every transition of $\mathbf{Y}$ is up or down.

♣

We now apply the development above to our continuous time Markov process $\mathbf{Y}$ which was introduced following the statement of Lemma 3.1. First of all, $\mathbf{Y}$ is monotone. This may be seen by an application of Lemma 3.2. Given $x, y \in \mathbf{Z}_+^M$, wuch that $x \leq y$, we construct the coupling $((Y^x(t), Y^y(t)), t \geq 0)$ in the naive way on the sample space used in the construction which was outlined above for $\mathbf{Y}$. It is easy to see that this coupling preserves order for all time, which shows $\mathbf{Y}$ is monotone.

Secondly, it is obvious that all jumps of $\mathbf{Y}$ are up or down. Clearly $\mathbf{Y}$ has uniform rate matrix. Hence Lemma 3.3 applies to show that $\mathbf{Y}$ is pcb. Start $\mathbf{Y}$ from the initial condition δ_0, which is pcb, and note that $\delta_0 T_t \to \pi$ as $t \to \infty$. Since $\mathbf{Y}$ is pcb, it follows that π is pcb, proving Lemma 3.1.

♣

Let us now introduce some more notation. For each $1 \leq i \leq M$ we define functions $f_i, g_i, h_i : \mathbf{Z}_+^M \to \mathbf{R}$ by

$$f_i(x) = 1(x_i > 0) \ ,$$

$$g_i(x) = 1(x_i = 0) + (1 - p_i)1(x_i > 0) \ .$$

$$h_i(x) = 1(x_i = 0) \ ,$$

Then we have

$$\pi(f_i) = \pi(\text{ buffer} i \text{ contains a packet}) \ ,$$

$$\pi(g_i) = \pi(\text{ terminal } i \text{ does not attempt transmission }) \ .$$

Further, we note that

$$\pi(g_i) = 1 - p_i \pi(f_i) \ .$$

We therefore have for each $1 \leq i \leq M$,

$$\lambda_i = p_i \pi(f_i \prod_{j \neq i} g_j) \tag{3.1}$$

With the notation above, we have :

<u>Proof of Thm. 2.1 for $M = 2$</u> : Let $\lambda = (\lambda_1, \lambda_2)$ be stabilized by $\mathbf{p} = (p_1, p_2)$. In Lemma 2.2, take $x = \pi(g_1 g_2)$. Clearly, $x > 0$. Observe that g_1 and g_2 are bounded decreasing functions on $(\mathbf{Z}_+^M, \leq)$. Since π is pcb, we have

$$x = \pi(g_1 g_2) \geq \pi(g_1)\pi(g_2) = (x + \lambda_1)(x + \lambda_2) \ , \tag{3.2}$$

where we have also used (3.1) to write (3.2). By Lemma 2.2, this means $\lambda \in C$.

♣

A STRONGER CORRELATION INEQUALITY

In view of (3.2), one might formally hope that for general M

$$(\pi(g_1 \cdots g_M))^{M-1} \geq \prod_{i=1}^{M} (\pi(g_1 \cdots g_M) + \lambda_i)$$

$$= \prod_{i=1}^{M} \pi(g_1 .. \hat{g}_i .. g_M) \ , \tag{4.1}$$

where the caret is used to indicate the missing element (thus $\pi(g_1 \hat{g}_2 g_3) = \pi(g_1 g_3)$). Clearly, this would prove Thm. 2.1. Unfortunately, one cannot conclude (4.1) purely on the basis of positive correlations.

Rather, one is led to examine the dynamics of the process more closely. Intuitively speaking, it is clear that the process is rather strongly positively correlated. A little thought will suggest that if we are told that a certain subfamily of the nodes are not attempting, then it should be even more likely that node 1 is not attempting than if we are given this information only for fewer nodes. Namely, we might hope that for any $k \leq M$

$$\pi(g_1 \mid g_2 \cdots g_k) \geq \pi(g_1 \mid g_2 \cdots g_{k-1}) \ . \tag{4.2}$$

This turns out to be a valid observation, and enough to complete the proof. The purpose of this section is to prove (4.2).

Let us now see how (4.2) leads to (4.1), and thus to Thm. 2.1. From (4.2) and permuted versions of it with $k = M$, we get, for each $1 \leq i \leq M - 1$,

$$\frac{\pi(g_1 \cdots g_M)}{\pi(g_1 .. \hat{g}_i .. g_M)} = \pi(g_i \mid g_1 .. \hat{g}_i .. g_M) \geq \pi(g_i \mid g_1 .. \hat{g}_i .. g_{M-1})$$

$$= \frac{\pi(g_1 \cdots g_{M-1})}{\pi(g_1 .. \hat{g}_i .. g_{M-1})} \ .$$

Hence

$$\prod_{i=1}^{M-1} \frac{\pi(g_1 \cdots g_M)}{\pi(g_1 .. \hat{g}_i .. g_M)} \geq \frac{\pi(g_1 \cdots g_{M-1})}{\pi(g_1 \cdots g_{M-2})} \prod_{i=1}^{M-2} \frac{\pi(g_1 \cdots g_{M-1})}{\pi(g_1 .. \hat{g}_i .. g_{M-1})} \ ,$$

which proves (4.1) by induction on M.

Next, we observe that (4.2) is equivalent to

$$\pi(g_1 g_k \mid g_2 \cdots g_{k-1}) \geq \pi(g_1 \mid g_2 \cdots g_{k-1}) \pi(g_k \mid g_2 \cdots g_{k-1}) \ . \tag{4.3}$$

Thus, in order to prove (4.2), it is enough to prove (4.3) for all $k \leq M - 2$. To show (4.3), it is enough to prove that for all $k \leq M - 2$ the conditional distribution $\pi(\cdot \mid g_1 \ldots g_k)$ is pcb. Fix k, and let this conditional distribution be denote $\tilde{\pi}$.

We now construct a continuous time Markov process $\tilde{\mathbf{Y}} = (\tilde{Y}(t), t \geq 0)$ having stationary distribution $\tilde{\pi}$. Let $\tilde{\Omega}$ be defined by

$$\tilde{\Omega}(x, y) = \frac{\Omega(x, y)}{\prod_{1 \leq i \leq k} [(1 - p_i) f_i(x) + h_i(x)]} = \frac{\Omega(x, y)}{\prod_{1 \leq i \leq k} g_i(x)} \ . \tag{4.4}$$

We let $\tilde{\mathbf{Y}}$ be the continuous time Markov proces with rate matrix $\tilde{\Omega}$. Since $\tilde{\pi}\tilde{\Omega} = 0$, we see that $\tilde{\mathbf{Y}}$ has stationary distribution $\tilde{\pi}$. The intermediate expression in (4.4) makes it clear that $\tilde{\mathbf{Y}}$ is just a time changed version of $\mathbf{Y}$. Indeed, let $B(x) = \{1 \leq l \leq k : x_l = 0\}$. Then in state x the rates of $\tilde{\mathbf{Y}}$ are $(\prod_{l \notin B(x)} (1 - p_l))^{-1}$ times as fast as those of $\mathbf{Y}$.

$\tilde{\mathbf{Y}}$ is monotone. We see this from Lemma 3.2 by constructing versions of $\tilde{\mathbf{Y}}$ on a sample space supporting $M + 1$ Poisson processes $\tilde{\mathcal{A}}_i = (\tilde{\mathcal{A}}_i(t), t \geq 0)$, $1 \leq i \leq M$, of rates $\lambda_i (\prod_{1 \leq l \leq k} (1 - p_l))^{-1}$ respectively, and $\tilde{\mathcal{D}} = (\tilde{\mathcal{D}}(t), t \geq 0)$ of rate $(\prod_{1 \leq l \leq k} (1 - p_l))^{-1}$, and M i.i.d. families of Bernoulli processes $V_i = (V_i(n), n \geq 0)$, $1 \leq i \leq M$, with $P(V_i(n) = 1) = p_i$. All these processes are independent. Given $x \leq y$, we construct the coupling $((\tilde{\mathbf{Y}}^x(t), \tilde{\mathbf{Y}}^y(t)), t \geq 0)$ as follows : Call the points of $\tilde{\mathcal{A}}_i$, $1 \leq i \leq M$, and $\tilde{\mathcal{D}}$ the *event times*. Consider the n th event time, say t. Let $((\tilde{\mathbf{Y}}^x(t_-), \tilde{\mathbf{Y}}^y(t_-)) = (z, w)$. Suppose t is a point of $\tilde{\mathcal{A}}_i$. We accept the arrival in the first (resp. second) co-ordinate only if $V_l(n) = 0$ for all $l \in B(z)$ (resp. all $l \in B(w)$). If t is a point of $\tilde{\mathcal{D}}$, we again accept it as an attempt time in the first (resp. second) co-ordinate iff $V_l(n) = 0$ for all $l \in B(z)$ (resp. all $l \in B(w)$). We use the $V_l(n)$, $l \notin B(z)$ (resp. the $V_l(n)$, $l \notin B(w)$) to give the attempt status of the nodes not in $B(z)$ (resp. not in $B(w)$) at this time. Note that the attempt status of the nodes in $B(z)$ (resp. $B(w)$) is irrelevant.

With the construction above, $\tilde{\mathbf{Y}}^x$ and $\tilde{\mathbf{Y}}^y$ evolve as versions of $\tilde{\mathbf{Y}}$ started at x and y respectively. We also have $\tilde{\mathbf{Y}}^x(t) \leq \tilde{\mathbf{Y}}^y(t)$ for all $t \geq 0$. This can be checked by induction on the event times. Let the n th event time be t, let $((\tilde{\mathbf{Y}}^x(t_-), \tilde{\mathbf{Y}}^y(t_-)) = (z, w)$, and assume $z \leq w$, by inductive hypothesis. Note that $B(z) \supseteq B(w)$. Suppose t is a point of $\tilde{\mathcal{A}}_i$. By construction, if the arrival is accepted for the first marginal, it must be accepted for the second, so $\tilde{\mathbf{Y}}^x(t) \leq \tilde{\mathbf{Y}}^y(t)$. Next, suppose t is a point of $\tilde{\mathcal{D}}$. Now, it is possible that the virtual attempt is accepted by the second marginal, but not by the first. However, this can only happen if there is some node $l \in B(z) - B(w)$ with $V_l(n) = 1$. So, if there is a real departure from the second marginal at this attempt time, it must be a departure from node l, which is all right since then we still have $\tilde{\mathbf{Y}}^x(t) \leq \tilde{\mathbf{Y}}^y(t)$. If the virtual attempt is accepted as a real attempt time by both marginals, then the only nodes possibly contending at this time are those of $\{1, \ldots, M\} - B(z)$, and they have the same contention status in both marginals. Once again, $\tilde{\mathbf{Y}}^x(t) \leq \tilde{\mathbf{Y}}^y(t)$.

Clearly $\tilde{\mathbf{Y}}$ has uniform rate matrix, and all its jumps are up or down in the partial order on $\mathbf{Z}_+^M$. By Lemma 3.3, $\tilde{\pi}$ is pcb, proving (4.3) and hence completing the proof that the stability region is contained in C.

♣

CONCLUDING REMARKS

It remains to be seen to what extent these results depend on the arrival statistics. For example, one can imagine an extreme case of an arrival model with rates $(1/M, \ldots, 1/M)$, which is stabilized by $(1, \ldots, 1)$ when started empty; let $A(n) = (A_1(n), \ldots, A_M(n)) = e_i$ with probability $1/M$, $1 \leq i \leq M$. Observe that $(1/M, \ldots, 1/M) \notin C$. However this model is unstable from any other initial condition, and is likely to be pathological.

The usual model for the arrival statistics has the number of arrivals at the individual terminals in each time slot independent, with the arrival process at terminal i being i.i.d. having a general distribution with mean λ_i. Most common is to take the number of arrivals in a time slot to be Poisson random variables. Let a_n be the probability that the total number of arrivals to the system in a time slot is n in our model and b_n be that for the Poisson arrivals model. Then $a_n = \rho^n((\rho + 1)^{n+1})^{-1}$, whereas $b_n = (n!)^{-1}\rho^n e^{-\rho}$, where $\rho = \sum_i \lambda_i$. We see that $b_n < a_n$ for $n \geq 3$, suggesting that contention occurs less frequently in the Poisson arrivals model. But does this translate into a larger stability region ?

REFERENCES

[0] Special Issue on Random Access Communications, *IEEE Transactions on Information Theory*, Vol. IT-31, No. 2, Mar. 1985.

[1] N. Abramson, "The ALOHA System - another alternative for Computer Communications", *Proceedings of the AFIPS Conference*, Fall Joint Computer Conference, Vol. 37, pp. 281 -285, 1970.

[2] J. M. Aein and D. S. Kosovych, "Satellite Capacity Allocation", *Proceedings of the IEEE*, Vol. 65, 1977.

[3] D. Aldous, "Ultimate Instability of the Exponential Back-Off Protocol for Ack. Based Transmission Control of Random Access Communication Channels", *IEEE Transactions on Information Theory*, Vol. IT-33, No. 2, pp. 219 -223, 1987.

[4] D. Bertsekas and R. Gallager, *Data Networks*, Prentice-Hall Inc., 1987.

[5] J. T. Cox, "An Alternate Proof of a Correlation Inequality of Harris", *Annals of Probability*, Vol. 12, No. 1, pp. 272 -273, 1984.

[6] I. Csiszar, J. Korner, L. Lovasz, K. Marton and G. Simonyi, "Entropy Splitting for Antiblocking Pairs and Perfect Graphs", *Preprint*, June 1988.

[7] G. J. Foschini, B. Gopinath, and J. F. Hayes, "Optimum Allocation of Servers to Two Types of Competing Customers", *IEEE Transactions on Communications*, Vol. COM-29, No. 7, pp. 1051 -1055, 1981.

[8] R. Gallager, "A Perspective on Multiaccess Channels", *IEEE Transactions on Information Theory*, Vol. IT-31, No. 2, pp. 124 -142, Mar. 1985.

[9] S. Ghez, S. Verdu and S. C. Schwartz, "Stability Properties of Slotted ALOHA with Multipacket reception Capability", *IEEE Transactions on Automatic Control*, Vol. AC-33, No. 7, pp. 640 -649, 1988.

[10] B. Hajek, "Hitting Time and Occupation Time Bounds implied by Drift Analysis, with Applications", *Advances in Applied Probability*, 1982.

[11] B. Hajek and T. vanLoon, "Decentralized Dynamic Control of a Multiaccess Broadcast Channel", *IEEE Transactions on Automatic Control*, Vol. AC-27, No. 3, pp. 559 -569, 1982.

[12] T. E. Harris, "A Correlation Inequality for Markov Processes in Partially Ordered State Spaces", *Annals of Probability*, Vol. 5, No. 3, pp. 451 -454, 1977.

[13] M. A. Kaplan, "A Sufficient Condition for Nonergodicity of a Markov Chain", *IEEE Transactions on Information Theory*, Vol. IT-25, No. 4, pp. 470 -471, July 1979.

[14] M. Karatzoglu and A. Ephremides, "Ergodicity of M-d. Random Walks and Random Access Systems", *Proceed-*

ings of the 26th IEEE Conference on Decision and Control, Los Angeles, pp. 2270 -2273, 1987.

[15] F. Kelly and I. MacPhee, "The Number of Packets Transmitted by Collision-detect Random Access Schemes", *Annals of Probability*, Vol. 15, No. 4, pp. 1557 -1568, 1987.

[16] T. M. Liggett, *Interacting Particle Systems*, Grundlehren der Mathematischen Wissenschaften, Vol. 276, Springer Verlag, 1985.

[17] J. Massey and P. Mathys, "The Collision Channel Without Feedback", *IEEE Transactions on Information Theory*, Vol. IT-31, No. 2, pp. 192 -204, Mar. 1985.

[18] S. Parekh, F. Schoute and J. Walrand, "Instability and Geometric Transience of the ALOHA Protocol", *Proceedings of the 26th IEEE Conference on Decision and Control*, Los Ageles, CA., pp. 1073 -1077, 1987.

[19] K. A. Post, "Convexity of the Nonachievable Rate Region for the Collision Channel Without Feedback", *IEEE Transactions on Information Theory*, Vol. IT-31, No. 2, pp. 205 -206, Mar. 1985.

[20] R. Rao, "Stability Analysis of Interacting Queues in the ALOHA system", *Proceedings of the 26th IEEE Conference on Decision and Control*, Los Angeles, CA., pp. 2058 -2059, 1987.

[21] R. Rao and A. Ephremides, "On the Stability of Interacting Queues", to appear in *IEEE Transactions on Information Theory*, 1988.

[22] R. L. Rivest, "Network Control by Bayesian Broadcast", *IEEE Transactions on Information Theory*, Vol. IT-33, NO. 3, PP. 323 -328, 1987.

[23] T. N. Saadawi and A. Ephremides, "Analysis, Stability and Optimization of Slotted ALOHA with a Finite Number of Buffered Users", *IEEE Transactions on Automatic Control*, Vol. AC-26, No. 3, pp. 680 -689, June 1981.

[24] J. Tsitsiklis, "Analysis of a Multiaccess Control Scheme" *IEEE Transactions on Automatic Control*, Vol. AC-32, No. 11, pp. 1017 -1020, 1987.

[25] B. S. Tsybakov, "Survey of USSR Contributions to Random Multiple-Access Communications", *IEEE Transactions on Information Theory*, Vol. IT-31, No. 2, pp. 143 -165, Mar. 1985.

[26] B. S. Tsybakov and V. A. Mikhailov, "Ergodicity of Slotted ALOHA System", *Problems of Information Transmission*, Vol. 15, pp. 73 -87, Mar. 1979.

[27] J. Walrand, *An Introduction to Queueing Networks*, Prentice Hall, 1988.