which, in general, is not zero. Without this term, the system (A.8)–(A.10) has an equilibrium at the origin $\tilde{x}_m = 0, \tilde{x} = 0, \tilde{\theta} = 0$, suitable for a local stability analysis by linearization. After this analysis, the extent of local stability properties should be tested by reintroducing the forcing term $-\gamma w_* e_*$ as a perturbation. The linearization of (A.8)–(A.10) around the origin leaves (A.8) unchanged while (A.9)–(A.10) becomes

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A_* & bw_*^T \\ -\gamma(w_* h^T + e_* C) & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -\gamma w_* h_m^T \end{bmatrix} \tilde{x}_m. \tag{A.12}$$

Since (A.8) does not depend on $\tilde{x}$ or $\tilde{\theta}$ and is u.a.s., the local stability properties of the zero solution of (A.8)–(A.10) are determined by the properties of the system

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\theta}} \end{bmatrix} = \left(\begin{bmatrix} A_* & bw_*^T \\ -\gamma w_* h^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\gamma e_* C & 0 \end{bmatrix} \right) \begin{bmatrix} \tilde{x} \\ \tilde{\theta} \end{bmatrix}. \tag{A.13}$$

If e_* is small, the term $\gamma e_* C$ can be treated as a perturbation and stability analysis is performed without it, that is, on the system (1.1).

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REFERENCES

[1] B. D. O. Anderson, "Exponential stability of linear equations arising in adaptive identification," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 83–88, 1977.
[2] B. D. O. Anderson, R. Bitmead, C. R. Johnson, Jr., and R. Kosut, "Stability theorems for the relaxation of the strictly positive real condition in hyperstable adaptive schemes," in *Proc. 23rd IEEE Conf. Decision Contr.*, Las Vegas, NV, 1984.
[3] K. J. Astrom, "Analysis of rohrs counter examples to adaptive control," in *Proc. 22nd IEEE Conf. Decision Contr.*, San Antonio, TX, 1983.
[4] S. Boyd and S. Sastry, "Necessary and sufficient conditions for parameter convergence in adaptive control," in *Proc. 1984 Amer. Contr. Conf.*, San Diego, CA, 1984.
[5] J. K. Hale, *Ordinary Differential Equations*. Molaban, FL: Krieger, 1980.
[6] P. A. Ioannou and P. V. Kokotovic, *Adaptive Systems with Reduced Models* (Springer-Verlag Series, Lecture Notes in Control and Information Sciences, Vol. 47). New York: Springer-Verlag, 1983.
[7] ——, "Robust redesign of adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 202–211, 1984.
[8] P. V. Kokotovic and B. D. Riedle, "Instabilities and stabilization of an adaptive system," in *Proc. 1984 Amer. Contr. Conf.*, San Diego, CA, 1984.
[9] R. L. Kosut, R. Johnson, and B. D. O. Anderson, "Conditions for local stability and robustness of adaptive control systems," in *Proc. 22nd IEEE Conf. Decision Contr.*, San Antonio, TX, 1983.
[10] J. Krause, M. Athans, S. Sastry, and L. Valavani, "Robustness studies in adaptive control," in *Proc. 22nd IEEE Conf. Decision Contr.*, San Antonio, TX, 1983.
[11] A. P. Morgan, and K. S. Narendra, "On the uniform asymptotic stability of certain linear nonautonomous differential equations," *SIAM J. Contr. Optimiz.*, vol. 15, pp. 5–24, 1977.
[12] K. S. Narendra and L. S. Valavani, "Stable adaptive controller design-direct control," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 570–583, 1978.
[13] K. S. Narendra and A. Annaswamy, "Robust adaptive control," in *Proc. Identificat. Contr. Workshop Flexible Space Structures*, San Diego, CA, 1984.
[14] C. E. Rohrs, L. Valavani, M. Athans, and G. Stein, "Analytical verification of undesirable properties of direct model reference adaptive control algorithms," in *Proc. 20th IEEE Conf. Decision Contr.*, San Diego, CA, 1981.
[15] ——, "Robustness of adaptive control algorithms in the presence of unmodeled dynamics," in *Proc. 21st IEEE Conf. Decision Contr.*, Orlando, FL, 1982.

On Stabilization and the Existence of Coprime Factorizations

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Abstract—**Let $\mathcal{H}$ be an integral domain and $\mathfrak{F}$ its quotient field. We show there are plants over $\mathfrak{F}$ which have no stable coprime factorizations, but can be stabilized in a stable closed loop. This answers a question posed by Vidyasagar, Schneider, and Francis.**

Our purpose is to develop an example which answers a question posed by Vidyasagar, Schneider, and Francis [4]. Admittedly the example is somewhat artificial from a system theoretic standpoint, but it should prove of assistance in subsequent investigations.

I. The Problem

Let $\mathcal{H}$ be an integral domain and $\mathfrak{F}$ its quotient field. If $P \in \mathfrak{F}^{n \times m}$ and $C \in \mathfrak{F}^{m \times n}$ are such that the closed-loop system (Fig. 1) is stable, then the input to error matrix

$$H_{eu} = \begin{bmatrix} (I_n + PC)^{-1} & -P(I_m + CP)^{-1} \\ C(I_n + PC)^{-1} & (I_m + CP)^{-1} \end{bmatrix} \tag{1.1}$$

is such that $H_{eu} \in \mathcal{H}^{(n+m) \times (n+m)}$ (by definition $\mathcal{H}$ consists of stable scalar plants—see [4]). It may be readily proved that the matrix $\left[\begin{smallmatrix} C & 0 \\ 0 & P \end{smallmatrix}\right]$ always has both a right coprime fractional representation and a left coprime fractional representation when the closed loop is stable [4].

The question posed in [4] is the following. *Is it always necessary that C and P individually have coprime factorizations when the closed loop is stable?* An answer is of great importance because the bane of fractional representation theory is the question of existence of coprime representations.

We construct an example which answers the question in the negative. The key idea is that our integral domain $\mathcal{H}$ is not a unique factorization domain.

II. The Example

Let $\mathcal{H} = \mathbb{Z}[-5] \approx \mathbb{Z}[x]/(x^2 + 5)$ where $\mathbb{Z}$ is the ring of integers, $\mathbb{Z}[x]$ the ring of polynomials in x with integral coefficients, $(x^2 + 5)$ is the ideal generated by $x^2 + 5 \in Z[x]$, and / means we form the quotient ring. (For all undefined terms see any good book on algebra, e.g., [1]–[3].) Then $\mathcal{H}$ is an integral domain because the ideal $(x^2 + 5)$ is prime. It is also not a unique factorization domain, as can be verified from

$$6 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 2.3. \tag{2.1}$$

Our example is scalar. Let $p = (1 + \sqrt{-5})/2 \in \mathfrak{F}$ and $C = (1 - \sqrt{-5})/-2 \in \mathfrak{F}$. We verify that

$$H_{eu} = \begin{bmatrix} -2 & 1 + \sqrt{-5} \\ -(1 - \sqrt{-5}) & -2 \end{bmatrix} \in H^{2 \times 2}. \tag{2.2}$$

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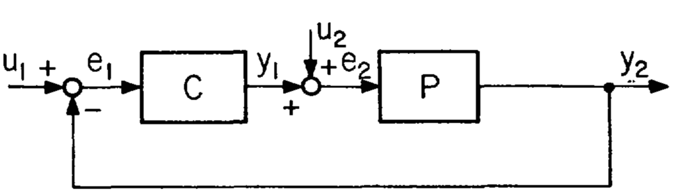

Fig. 1.

Also, from [4, Theorem 4.1] it follows that, if we let

$$N=\begin{bmatrix}\frac{1-\sqrt{-5}}{-2} & 0\\ 0 & \frac{1+\sqrt{-5}}{2}\end{bmatrix}\begin{bmatrix}-2 & 1+\sqrt{-5}\\ 1-\sqrt{-5} & 2\end{bmatrix}$$

$$=\begin{bmatrix}1-\sqrt{-5} & 3\\ 3 & -1-\sqrt{-5}\end{bmatrix}$$

and

$$D=\begin{bmatrix}-2 & 1+\sqrt{-5}\\ 1-\sqrt{-5} & -2\end{bmatrix}$$

then the pair (N, D) gives a right coprime fractional representation of $\left[\begin{smallmatrix}C & 0\\ 0 & P\end{smallmatrix}\right]$.

We will now show that p has no stable coprime factorization (similar arguments work for c). First, we have to find all possible fractional representations for p. Let

$$P=\frac{1+\sqrt{-5}}{2}=\frac{a+b\sqrt{-5}}{c+d\sqrt{-5}}\qquad a,\ b,\ c,\ d\in\mathbb{Z}. \tag{2.3}$$

Multiplying we get the conditions

$$\begin{aligned}c+d&=2b\\ c-5d&=2a\end{aligned} \tag{2.4}$$

which on simplification yield

$$\begin{aligned}2a-2b+6d&=0\\ c&=2a+5d.\end{aligned} \tag{2.5}$$

It is easy to see that all possible solutions in integers of the above equations are given by

$$(a,\ b,\ c,\ d)=\alpha(1,\ 1,\ 2,\ 0)+\beta(3,\ 0,\ 1,\ -1)\qquad \alpha,\ \beta\in\mathbb{Z}. \tag{2.6}$$

Indeed, given any solution (a, b, c, d) let $\alpha = b$ and $\beta = -d$. Thus, all possible fractional representations of p are given by

$$p=\frac{(\alpha+3\beta)+\alpha\sqrt{-5}}{(2\alpha+\beta)-\beta\sqrt{-5}} \tag{2.7}$$

where $\alpha, \beta \in \mathbb{Z}$, at least one nonzero.

Next, we prove that no such representation can be coprime over $\mathcal{H}$. Suppose the contrary. Then there exist integers u, v, ω, x such that

$$(u+v\sqrt{-5})((\alpha+3\beta)+\alpha\sqrt{-5})+(\omega+x\sqrt{-5})((2\alpha+\beta)-\beta\sqrt{-5})=1. \tag{2.8}$$

Multiplying out gives the conditions

$$\begin{bmatrix}\alpha & \alpha+3\beta & -\beta & 2\alpha+\beta\\ \alpha+3\beta & -5\alpha & 2\alpha+\beta & 5\beta\end{bmatrix}\begin{bmatrix}u\\ v\\ \omega\\ x\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}. \tag{2.9}$$

This has an immediate module theoretic interpretation. To establish contradiction, we wish to prove that the submodule of $\mathbb{Z}^2$ generated over $\mathbb{Z}$ by the column vectors of the 2×4 matrix above can never contain $\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]$, whatever the values of α, β.

Let us multiply the above equation on the left by the $\mathbb{Z}$-unimodular matrix $\left[\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right]$. Since

$$\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix} \tag{2.10}$$

we have the equivalent problem about the integer matrix

$$\begin{bmatrix}\alpha & \alpha+3\beta & -\beta & 2\alpha+\beta\\ 3\beta & -6\alpha-3\beta & 2\alpha+2\beta & 4\beta-2\alpha\end{bmatrix}, \tag{2.11}$$

i.e., about the submodule spanned by its columns, over $\mathbb{Z}$.

By forming $\mathbb{Z}$-linear combinations of the columns of the above matrix, one can check that the vectors $\left[\begin{smallmatrix}\alpha\\ 3\beta\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\beta\\ -2(\beta+\alpha)\end{smallmatrix}\right]$ generate the submodule we are interested in. Thus, an equivalent problem is to show that it is impossible to find integers m and n such that

$$m\begin{bmatrix}\alpha\\ 3\beta\end{bmatrix}+n\begin{bmatrix}\beta\\ -2(\beta+\alpha)\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}. \tag{2.12}$$

If α and β have a nontrivial common divisor, this is clearly impossible. So we assume $\gcd(\alpha, \beta) = 1$. Now, the only choices of m and n making the first row zero are $m=k\beta$ $n=-k\alpha$ with some $k \in \mathbb{Z}$. If $k \neq \pm 1$, it appears as a common factor of the second row, so once again it is impossible to satisfy (2.12). So assume $k = \pm 1$. We get

$$\pm(3\beta^2+2\alpha\beta+2\alpha^2)=1.$$

We may rewrite this as

$$2\beta^2+\alpha^2+(\alpha+\beta)^2=\pm 1. \tag{2.13}$$

But it is clearly impossible to find integers α and β which satisfy (2.13). Thus, we have proved the impossibility of satisfying (2.12). It follows that p has no coprime fractional representation.

III. Conclusion (The Answer)

If $\mathcal{H}$ is permitted to be an arbitrary integral domain as in [4], it is possible to stabilize plants which have no coprime factorizations.

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References

[1] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra.* Reading, MA: Addison-Wesley, 1969.
[2] T. W. Hungerford, *Algebra.* New York: Springer-Verlag, 1974, GTM 73.
[3] N. Jacobson, *Basic Algebra I.* San Francisco, CA: Freeman, 1974.
[4] M. Vidyasagar, H. Schneider, and B. A. Francis, "Algebraic and topological aspects of feedback stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 880–894, Aug. 1982.