Proceedings of 24th Conference
on Decision and Control
Ft. Lauderdale, FL • December 1985

A GENERALIZATION OF THE ERLANG FORMULA OF TRAFFIC ENGINEERING

V. Anantharam* and B.Gopinath**.

* Department of Electrical Engineering and Computer Science
and Electronics Research Laboratory
University of California, Berkeley.

** Bell Communications Research Inc., Morristown, New Jersey.

Summary

Consider a node in a communication network with n outgoing links grouped into k trunks of $n_1,...,n_k$ links respectively. $n_1+...+n_k = n$. Calls arrive in a Poisson stream of rate λ. The state of the node is specified by the number of idle links in each trunk. A **policy** is a rule by which a call, finding the node in some state, is assigned to one of the available links in one of the available outgoing trunks. The links are assumed to have exponential holding times with mean $\frac{1}{\mu}$, which are independent, and are independent of the arrival process. Further, a call assigned to trunk α, $1 \leqslant \alpha \leqslant k$ is immediately lost with probability $(1-\epsilon_\alpha)$---this feature models the nature of the links and the congestion downstream of the node along that route. A call is said to be **blocked** if all the outgoing links are busy when it arrives. It is known that the blocking probability is independent of the assignment policy. We give an explicit closed form formula for the blocking probability :

$$p_b = \frac{1}{\sum_{j_1=0}^{n_1} \cdots \sum_{j_k=0}^{n_k} \binom{n_1}{j_1} \cdots \binom{n_k}{j_k} (j_1+...+j_k)! (\frac{\mu}{\lambda_1})^{j_1} \cdots (\frac{\mu}{\lambda_k})^{j_k}}$$

where $\lambda_1 = \epsilon_1 \cdot \lambda, ..., \lambda_k = \epsilon_k \cdot \lambda$.

This generalizes the well-known Erlang formula of traffic engineering.

Introduction

To understand the behaviour of large networks of service systems it is necessary to first understand the behaviour of individual nodes in the system. One of the most common models for a node in a communication network which has a single incoming link and n outgoing links is the following. Customers or calls arrive in a Poisson stream of rate λ and are assigned immediately to any one of the available outgoing links, which becomes busy, and stays busy for an exponentially distributed holding time of mean $\frac{1}{\mu}$. These holding times are independent and are independent of the arrival process. Particularly in telephony, an issue of central importance is to understand when an incoming call may be lost because the desired connection to its destination cannot be made. A node is said to be **blocked** if all its outgoing links are busy. In this situation, one has the well-known Erlang formula for the steady state probability that the node is in the blocked state

$$p_b = \frac{1}{\sum_{i=0}^{n} \binom{n}{i} \cdot i! \cdot (\frac{\mu}{\lambda})^i} \tag{1.1}$$

Because of its practical significance, therefore, this is one of the most important and widely known formulas in the study of networks of interconnected service systems, such as computer communication networks, models of traffic flow, etc.

In their paper [1], Gopinath, Garcia and Varaiya, [GGV], analyze a more sophisticated model of a single node in a communication network. The outgoing links are grouped into k **trunks** of $n_1,...,n_k$ links respectively. $n_1+...+n_k = n$. Each of the links in a given trunk α has an associated loss probability $(1-\epsilon_\alpha)$, that is, a call assigned to the trunk α is immediately lost with the probability $(1-\epsilon_\alpha)$. The various loss probabilities $(1-\epsilon_\alpha)$ model the cumulative effects of the topology of the network and the congestion of other links and nodes downstream of the node under consideration. This is the crucial new feature in the proposed model. Calls arrive at the node in a Poisson strem of rate λ. Each arriving call is assigned to one of the outgoing trunks according to a **policy** U. To define this, first define the **state** of the node as the vector $(i_1,...,i_k)$ of the number of idle links in each trunk. $0 \leqslant i_1 \leqslant n_1, ..., 0 \leqslant i_k \leqslant n_k$. Thus the node can be in any one of $(n_1+1)....(n_k+1)$ states. For a state $I=(i_1,...,i_k)$ and $1 \leqslant \alpha \leqslant k$ such that $i_\alpha \neq 0$, let I_α denote the state

$$I_\alpha = (i_1,...,i_{\alpha-1}, i_\alpha - 1, i_{\alpha+1},...,i_k)$$

A policy U consists in specifying the numbers $U(I, I_\alpha)$ satisfying

$$U(I, I_\alpha) \geqslant 0$$

$$\sum_{1 \leqslant \alpha \leqslant k, i_\alpha \neq 0} U(I, I_\alpha) = 1 \quad \text{if} \quad I \neq (0,...,0)$$

Thus, when the node is in state I, and the policy is U, for $1 \leqslant \alpha \leqslant k$ with $i_\alpha \neq 0$, an incoming call is assigned to trunk α with probability $U(I, I_\alpha)$.

If a call is assigned to a given link, the link immediately becomes busy, and remains busy for a holding or conversation time, which is an exponentially distributed random variable with mean $\frac{1}{\mu}$. The holding times are independent and are independent of the arrival process. At the end of the holding period the link becomes idle again.

This model is motivated by the fact that in a complicated network, a call leaving a node along a link in one of the outgoing trunks and proceeding from node to node towards its destination, may be lost if some of the nodes it encounters are blocked. This loss probability depends on the capacity and loading of the path the call is routed on, and as a first approximation can be taken as depending only on the initial link (different for the different outgoing trunks from the link), by the use of the different loss probabilities ($1-\epsilon_\alpha$). For further motivation one should read the preamble to the paper [GGV].

Note that the model for nodes proposed in the first paragraph is included in the refined model as a special case by ignoring the modelling flexibility provided by the varying loss probabilities $(1-\epsilon_\alpha)$.

Once again we are interested in quantifying the probability that a call is unable to make its desired connection. If the node is in the state (0, ...0), i.e., there are no available outgoing links, we say the node is **blocked.**

[GGV] prove the remarkable result that in the above model the probability of the node being in the blocked state is **independent** of the policy adopted to assign incoming calls to the outgoing trunks. Thus the law of overflow traffic is also independent of the assignment policy. In the case of two outgoing trunks (two types of outgoing links), they conjecture a formula for the blocking probability. Such a formula would have an importance comparable to that of the Erlang formula, when the refined model proposed by [GGV] is used in the node by node analysis of communication networks.

The purpose of this paper is to establish this formula, and also the generalization for arbitrary k :

$$p_b = \frac{1}{\sum_{j_1=0}^{n_1} \cdots \sum_{j_k=0}^{n_k} \binom{n_1}{j_1} \cdots \binom{n_k}{j_k} (j_1 + \cdots + j_k)! (\frac{\mu}{\lambda_1})^{j_1} \cdots (\frac{\mu}{\lambda_k})^{j_k}} \quad (1.2)$$

where $\lambda_1 = \epsilon_1 \cdot \lambda, \ldots, \lambda_k = \epsilon_k \cdot \lambda$.

Note that in the case k=1 this reduces to the Erlang formula discussed in the first paragraph.

Since explicitly evaluating the expression in equation (1.2) would require a large number of operations even in modest situations. it is useful to have an integral representation for p_b from which good approximations to the closed form solution can be made in practice. This can be arrived at through the gamma-function identity:

$$n! = \int_0^\infty t^n \exp(-t) dt$$

We show that

$$p_b = \frac{1}{\int_0^\infty (1 + \frac{t\mu}{\lambda_1})^{n_1} \cdots (1 + \frac{t\mu}{\lambda_k})^{n_k} \exp(-t) dt} \quad (1.3)$$

which integral may be approximated by one's favorite technique.

The organization of the paper is as follows: In Section 2. we give the Markov process description of the system under consideration. In Section 3., our idea for calculating p_b is carefully explained in the case of two outgoing trunks (k=2). We avoid writing out proofs of the propositions in the special case because there is no essential simplification in doing so. In Section 4. we handle the general problem with arbitrary k, giving complete proofs. Finally in Section 5., we go through the few steps needed to calculate the integral formula (1.3) from the combinatorial expression (1.2).

Markov process description.

The evolution of the state of the node is described by a continous time finite state Markov process with state space:

$$\{ I = (i_1 \ldots i_k) \text{ such that } 0 \leqslant i_1 \leqslant n_1, \ldots, 0 \leqslant i_k \leqslant n_k \}$$

Recall that, given state I and $1 \leqslant \alpha \leqslant k$ such that $i_\alpha \neq 0$, we let

$$I_\alpha = (i_1 \ldots i_{\alpha-1}, i_\alpha - 1, i_{\alpha+1}, \ldots, i_k)$$

Similarly, given $1 \leqslant \beta \leqslant k$ such that $i_\beta \neq n_\beta$, let

$$I^\beta = (i_1 \ldots i_{\beta-1}, i_\beta + 1, i_{\beta+1}, \ldots, i_k)$$

Then, for a fixed policy U, the transition rate matrix of the Markov process is given by:

$$R_U(I,J) = 0 \qquad \text{if} \quad J \neq I_\alpha \ , \ J \neq I^\beta \ J \neq I \quad (2.1)$$

$$R_U(I, I_\alpha) = \lambda_\alpha U(I, I_\alpha) \qquad \text{where} \quad \lambda_\alpha = \epsilon_\alpha \cdot \lambda \quad (2.2)$$

$$R_U(I, I^\beta) = (n_\beta - i_\beta)\mu \quad (2.3)$$

$$R_U(I,I) = -\sum_{1 \leqslant \alpha \leqslant k, i_\alpha \neq 0} \lambda_\alpha U(I, I_\alpha) - \sum_{1 \leqslant \beta \leqslant k, i_\beta \neq n_\beta} (n_\beta - i_\beta)\mu \quad (2.4)$$

Let the steady state distribution of this Markov process be $p_U(I)$. Note that $p_U[(0,\ldots,0)]$ is precisely the blocking probability p_b. Since we already know that the blocking probability is independent of the policy chosen, our method will be to calculate the formula for p_b by calculating the steady state distribution for a special, analytically convenient policy, which has the property of " decoupling " the transition rate matrix of the Markov process in a way that will become clear below.

The case k=2.

Our idea for calculating the Erlang formula for the blocking probability is most easily understood in the case k=2, so that, for clarity, we will first consider this case. However, the proofs of the claims we make in this section are no harder to establish for general k than they are for k=2, and will therefore be deferred to Section 4.

When k=2, the transition rate matrix is conveniently thought of by means of the state diagram of Fig. 1 .

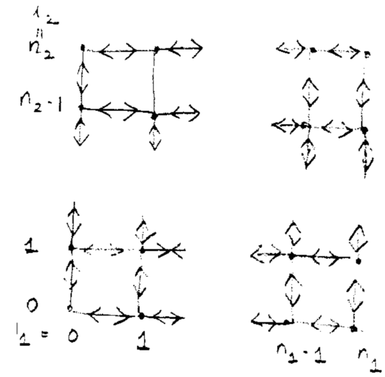

FIG 1.

Left to right and down to up transitions represent links becoming free, while transitions in the opposite directions occur when a new call arrives. The weights associated with the arrows in the state diagram can be inferred from the equations (2.1) ... (2.4) for the entries of the transition rate matrix R_U. Note that the right to left and up to down transitions have associated rates depending on the policy U.

A probability distribution p(I) on the state space is the stationary distribution $p_U(I)$ iff it satisfies the balance equations:

$$\sum_I p(I) R_U(I,J) = 0 \qquad \text{for each } J. \quad (3.0)$$

These translate, in the state diagram, into one balance equation per node, the balance being the equality of the total incoming rate to the total outgoing rate.

Note that, for a given policy U, the individual state to state transition rates need not be balanced, i.e., the rate on a left to right transition need not equal that on the corresponding right to left transition, and similarly the rate on an up to down transition need not equal that on the corresponding down to up transition. Our key observation is that there is a special policy, which is in fact unique, where these extra balance equations are satisfied by the equilibrium policy. As one would expect this makes the calculation of the closed form Erlang formula feasible.

Proposition

Consider the special policy for which

$$U(I, I_\alpha) = \frac{i_\alpha}{i_1 + i_2} \quad (3.1)$$

Let

$$p(i_1 i_2) = \frac{\binom{n_1}{i_1} \binom{n_2}{i_2} (\frac{\mu}{\lambda_1})^{i_1} (\frac{\mu}{\lambda_2})^{i_2} (i_1 + i_2)!}{\sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \binom{n_1}{j_1} \binom{n_2}{j_2} (\frac{\mu}{\lambda_1})^{j_1} (\frac{\mu}{\lambda_2})^{j_2} (j_1 + j_2)!} \quad (3.2)$$

Then we have :

$$p(I) U(I, I_\alpha) \lambda_\alpha = p(I_\alpha)(n_\alpha - (i_\alpha - 1))\mu \quad (3.3)$$

for each I and for each $1 \leqslant \alpha \leqslant 2$ such that $i_\alpha \neq 0$.

Further, p(I) is the steady state distribution defined in Section 2.

Note that

$$p_b = \frac{1}{\sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \binom{n_1}{j_1} \binom{n_2}{j_2} (\frac{\mu}{\lambda_1})^{j_1} (\frac{\mu}{\lambda_2})^{j_2} (j_1 + j_2)!} \quad (3.4)$$

Proof :

We will carry out the formal proof for the general case in Section 4. Here we only sketch the argument.

First one checks that equations (3.1) do, in fact, define a policy. This is because

$$\sum_{\alpha=1,2} U(I,I_\alpha)=1 \quad \text{for each} \quad I \qquad (I \neq (0,0))$$

Next one checks that the equations (3.2) do, in fact, define a probability distribution on the state space. This comes from

$$\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} p(j_1 j_2)=1$$

Substituting (3.1) and (3.2) into (3.3) shows that the equations (3.3) are satisfied by the probability distribution (3.2), when the policy is (3.1). But if we go back to the formulae (2.1) .. (2.4) with U as in (3.1), we see that any probability distribution satisfying the equations (3.3) also satisfies the balance equations (3.0). (In terms ot the state transition diagram, if the individual state to state transition rates are balanced, the net rate of flow into any node must equal the net rate of flow out).

Since, for the policy (3.1), the process is clearly irreducible and finite state, it has a unique stationary distribution, which, by the above, must be given by the formula (3.2).

Since the blocking probability is the stationary probability of being in the state (0, 0) the formula (3.4) also holds *endpf* .

Notice that the equations (3.3) when read on the state transiton diagram mean precisely the balance of individual state to state transitions.

Notice also that the special policy (3.1) assigns calls to the outgoing trunks in proportion of the number of available free links in each trunk. Thus we call it the **proportional** policy.

The general case of k outgoing trunks.

Proposition :

Consider the special (proportional) policy U for which

$$U(I,I_\alpha)=\frac{i_\alpha}{i_1+\ldots+i_k} \qquad (4.1)$$

Let

$$p(I)=\frac{\binom{n_1}{i_1}\cdots\binom{n_k}{i_k}\left(\frac{\mu}{\lambda_1}\right)^{i_1}\cdots\left(\frac{\mu}{\lambda_k}\right)^{i_k}(i_1+\ldots+i_k)!}{\sum_{j_1=0}^{n_1}\cdots\sum_{j_k=0}^{n_k}\binom{n_1}{j_1}\cdots\binom{n_k}{j_k}\left(\frac{\mu}{\lambda_1}\right)^{j_1}\cdots\left(\frac{\mu}{\lambda_k}\right)^{j_k}(j_1+\ldots+j_k)!} \qquad (4.2)$$

Then we have :

$$p(I)\cdot U(I,I_\alpha)\cdot\lambda_\alpha=p(I_\alpha)\cdot(n_\alpha-(i_\alpha-1))\cdot\mu \qquad (4.3)$$

for each I and cach $1 \leqslant \alpha \leqslant k$ such that $i_\alpha \neq 0$.

Further, p(I) is the stationary distribution $p_U(I)$ defined in Section 2.

Note that:

$$p_b=\frac{1}{\sum_{j_1=0}^{n_1}\cdots\sum_{j_k=0}^{n_k}\binom{n_1}{j_1}\cdots\binom{n_k}{j_k}(j_1+\ldots+j_k)!\left(\frac{\mu}{\lambda_1}\right)^{j_1}\cdots\left(\frac{\mu}{\lambda_k}\right)^{j_k}} \qquad (4.4)$$

which is valid for **every** routing policy.

Proof :

The formulas (4.1) define a policy because

$$\sum_{1 \leqslant \alpha \leqslant k,\, i_\alpha \neq 0} U(I,I_\alpha) = 1$$

(4.2) defines a probability distribution on the state space, because

$$\sum_I p(I) = 1$$

Let I be a state and $1 \leqslant \alpha \leqslant k$ be such that $i_\alpha \neq 0$. Then we compute :

$$\frac{p(I_\alpha)}{p(I)} = \frac{\binom{n_1}{i_1}\cdots\binom{n_\alpha}{i_\alpha-1}\cdots\binom{n_k}{i_k}}{\binom{n_1}{i_1}\cdots\binom{n_\alpha}{i_\alpha}\cdots\binom{n_k}{i_k}}\cdot\tau$$

where

$$\tau = \frac{\left(\frac{\mu}{\lambda_1}\right)^{i_1}\cdots\left(\frac{\mu}{\lambda_\alpha}\right)^{i_\alpha-1}\cdots\left(\frac{\mu}{\lambda_k}\right)^{i_k}(i_1+\ldots+i_\alpha-1+\ldots+i_k)!}{\left(\frac{\mu}{\lambda_1}\right)^{i_1}\cdots\left(\frac{\mu}{\lambda_\alpha}\right)^{i_\alpha}\cdots\left(\frac{\mu}{\lambda_k}\right)^{i_k}(i_1+\ldots+i_\alpha+\ldots+i_k)!}$$

Thus

$$\frac{p(I_\alpha)}{p(I)} = \frac{i_\alpha}{n_\alpha-(i_\alpha-1)}\cdot\frac{\lambda_\alpha}{\mu}\cdot\frac{1}{(i_1+\ldots+i_k)!}$$
$$= \frac{U(I,I_\alpha)\cdot\lambda_\alpha}{(n_\alpha-(i_\alpha-1)\cdot\mu}$$

This verifies that the probability distribution of (4.2) satisfies the equations (4.3).

Next, we will verify that p(I) also satisfies the balance equations (3.0), which will prove that p(I) is the stationary distribution $p_U(I)$ for the process when the policy is as in (4.1). As a preliminary remark, we observe that for any state J and $1 \leqslant \alpha \leqslant k$ such that $j_\alpha \neq 0$ we have

$$(J_\alpha)^\alpha=J \qquad (4.5)$$

and for $1 \leqslant \beta \leqslant k$ such that $j_\beta \neq n_\beta$ we have

$$(J^\beta)_\beta=J \qquad (4.6)$$

Let us write down the balance equation corresponding to column J of the transition rate matrix R_U :

$$\sum_{1 \leqslant \alpha \leqslant k,\, j_\alpha \neq 0} p(J_\alpha)R_U(J_\alpha,J) + \sum_{1 \leqslant \beta \leqslant k,\, j_\beta \neq n_\beta} p(J^\beta)R_U(J^\beta,J) + p(J)R_U(J,J)=0 \qquad (4.7)$$

Note that

$$R_U(J_\alpha J) = R_U(J_\alpha,(J_\alpha)^\alpha) = (n_\alpha-(j_\alpha-1))\mu$$

from equations (2.3) and (4.5). We also have

$$R_U(J^\beta J) = R_U(J^\beta,(J^\beta)_\beta) = \lambda_\beta U(J^\beta,J)$$

from equations (2.2) and (4.6).

Using (2.4) and the above, equation (4.7) may be rearranged to read

$$\sum_{1 \leqslant \alpha \leqslant k,\, j_\alpha \neq 0} [\, p(J_\alpha)(n_\alpha-(j_\alpha-1)\mu - p(J)\lambda_\alpha U(J,J_\alpha) \,]$$
$$+\sum_{1 \leqslant \beta \leqslant k,\, j_\beta \neq n_\beta} [\, p(J^\beta)\lambda_\beta U(J^\beta J) - p(J)(n_\beta - j_\beta)\mu \,] = 0$$

which holds because each individual term vanishes by the equations (4.2).

We have verified that the probability distribution of (4.2) satisfies the balance equations (3.0) and is therefore the stationary distribution $p_U(I)$.

Finally note that the blocking probability is just the stationary probability of the state (0, ..., 0) to get (4.4). Equation (4.4) is the generalization of the Erlang formula we have been seeking.

An integral formula for the blocking probability.

In this section we will carry out the calculations which translate the formula (1.2) into the formula (1.3).

The starting point is the Gamma function identity;

$$n!=\int_0^\infty t^n \exp(-t)dt$$

Using this for $(j_1+\ldots+j_k)!$ in the formula (1.2) gives

$$p_b=\frac{1}{\sum_{j_1=0}^{n_1}\cdots\sum_{j_k=0}^{n_k}\binom{n_1}{j_1}\cdots\binom{n_k}{j_k}\left(\int_0^\infty t^{j_1+\ldots+j_k}\exp(-t)dt\right)\left(\frac{\mu}{\lambda_1}\right)^{j_1}\cdots\left(\frac{\mu}{\lambda_k}\right)^{j_k}}$$

Interchanging the summation and the integral

$$p_b=\frac{1}{\int_0^\infty\left(\sum_{j_1=0}^{n_1}\cdots\sum_{j_k=0}^{n_k}\binom{n_1}{j_1}\cdots\binom{n_k}{j_k}\left(\frac{t\mu}{\lambda_1}\right)^{j_1}\cdots\left(\frac{t\mu}{\lambda_k}\right)^{j_k}\right)dt}$$

Finally, observing that

$$(1+\frac{t\mu}{\lambda_\alpha})^{n_\alpha} = \sum_{i_\alpha=0}^{n_\alpha} \binom{n_\alpha}{i_\alpha} (\frac{t\mu}{\lambda_\alpha})^{i_\alpha}$$

gives

$$p_b = \frac{1}{\int_0^\infty (1+\frac{t\mu}{\lambda_1})^{n_1} \cdots (1+\frac{t\mu}{\lambda_k})^{n_k} exp(-t)dt}$$

as desired.

References

[1] B.Gopinath, J.M.Garcia and P.Varaiya "Blocking probability in a switching center with arbitrary routing policy", AT&T Bell Laboratories Technical Journal Vol 63, N0:5, May-June 1984, pg 709-720.