compensator-system plant loop has been shown in all of the papers on the compensator-based approach.

If x and w are not measurable, but the triple

$$\left\{\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} A & 0 \\ -YC_r & Z \end{pmatrix}, \begin{pmatrix} B \\ -YD_r \end{pmatrix}\right\} \tag{31}$$

is output feedback stabilizable, there exist matrices K_0 and K_2 such that the output feedback control

$$u=-K_0C\Delta x''-K_0(F+CX)\Delta x_c+K_2x_c \tag{32}$$

regulates $\Delta x''$ and Δx_c to zero. We note that

$$C\Delta x''+(F+CX)\Delta x_c=y-Du-(F+CX)x_c, \tag{33a}$$

or equivalently

$$u=-(I-K_0D)^{-1}(y+(K_2-F-CX)x_c). \tag{33b}$$

Hence, the feedback control design (32) uses only the measurable quantities in the servomechanism, namely y and x_c.

Moreover, the closed-loop matrix when (32) is used is given by

$$\begin{pmatrix} A-BK_0C & -BK_0(F+CX)-BK_2 \\ -YC_r-YD_rK_0C & Z-YD_rK_0(F+CX)-YD_rK_2 \end{pmatrix}. \tag{34}$$

Given K_0, and any arbitrary K_2', there exists K_2 such that

$$K_2'=K_0(F+CX)+K_2. \tag{35}$$

Thus, the feedback control (32) is equivalent to a feedback control for the pair (30) which is given by

$$\Delta u''-K_2\Delta x_c=-K_0C\Delta x''-K_2'\Delta x_c. \tag{36}$$

In other words, in the design of (32), Δx_c can be considered to be measurable.

Two options are available if the triple (31) cannot be stabilized by output feedback. Recognizing that $\Delta x''(t)$ is a function of both x and w and therefore not directly measurable, we are faced with the same problem we encounter in the observer-based approach. The first option calls for an observer which estimates both the states x and disturbance w. Design of this observer follows essentially the same procedures outlined in the observer-based approach. The estimation error dynamics are then governed by (17). Replacing x by $\hat{x}$ and w by $\hat{w}$ in (29) we generate the following observer feedback control

$$u=-K_1(\hat{x}-X\hat{w})+K_2x_c. \tag{37}$$

The closed-loop system behavior is dictated by

$$\begin{pmatrix} \Delta\dot{x}'' \\ \Delta\dot{x}_c \end{pmatrix}=\begin{pmatrix} A-BK_1 & -BK_2 \\ -YC_r-YD_rK_1 & Z-YD_rK_2 \end{pmatrix}\begin{pmatrix} \Delta x'' \\ \Delta x_c \end{pmatrix} + \begin{pmatrix} B \\ -YD_r \end{pmatrix}(K_1-K_1X)\begin{pmatrix} x-\hat{x} \\ w-\hat{w} \end{pmatrix} \tag{38}$$

and (17). Thus, the separation property holds and $\Delta x''\to 0$ and $\Delta x_c\to 0$ as $t\to\infty$. Note that x and w have to be estimated in order to stabilize the system which includes the servocompensator, an *a priori* dynamic element in the compensator-based approach. It is concluded that when output feedback is insufficient to stabilize the feedback system and observer feedback is considered as a candidate feedback design for stabilization, it may be conceptually more appealing to use the observer-based approach.

The remaining option is to use dynamic output feedback. A dynamic compensator

$$\Delta\dot{x}_d=P\Delta x_d+Q\Delta y \tag{39}$$

with

$$\Delta y=C\Delta x''+(F+CX)\Delta x_c \tag{40}$$

as input is constructed. Recall from (33) that Δy, an output of the open-loop system (24), (25), is a measurable quantity. The dynamic output feedback control then takes the form

$$\Delta u''=-K_d\Delta x_d-K_0\Delta y. \tag{41}$$

The design of (41) requires the choice of P, Q, K_d, K_0, and K_2 in (25) such that the closed-loop system described by (24), (25), (39), (40), and (41) is stabilized. Once again (41) is equivalent to a feedback control for the pair (30)

$$\Delta u''-K_2\Delta x_c=-K_d\Delta x_d-K_0C\Delta x''-K_2'\Delta x_c \tag{42}$$

where K_2' is defined in (35). Thus, Δx_c is indirectly fed back in the dynamic output feedback control scheme.

V. Conclusions

We have shown that in the design of linear multivariable servomechanisms, deviation variables play an important role. In both the observer-based and the compensator-based approach, internal stability and output regulation are found to be directly related to the dynamical behavior of the deviation variables. Although deviation variables have been utilized to analyze multivariable servomechanisms when an observer-based approach is used, they are introduced herein for the compensator-based approach. It is found that if output feedback control, such as (32), suffices to provide acceptable dynamic performance then the designer benefits from the compensator-based approach in view of its *a priori* choice of the dynamic element in the control loop. Perhaps this is why the design of classical PI servomechanisms followed a compensator-based approach. The disadvantages of this approach surface if output feedback alone cannot regulate the deviation variables to zero. The need to incorporate dynamic feedback for the composite plant and servocompensator makes the feedback design more complicated than that used in the observer-based approach.

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On the Stabilization of Nonlinear Systems

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***Abstract*—We extend the applicability of the global Q-parametrization method of controller design to a large class of unstable nonlinear plants. The main result is a two-step compensation theorem analogous to that of Zames for unstable linear plants—if P: $\mathcal{L}_{e2}\to\mathcal{L}_{e1}$ is a nonlinear (possibly unstable) plant and F_0 is *any* incrementally stable controller such that $P_1:=P(I-F_0(-P))^{-1}$ is incrementally stable, then the class of controllers F which yields an f.g. stable closed-loop system in the unity feedback configuration for P is globally parametrized by finite gain stable maps Q: $\mathcal{L}_{e1}\to\mathcal{L}_{e2}$ with $F=F_0+Q(I-P_1Q)^{-1}$.**

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I. Introduction

Our aim is to extend the Q-parametrization design theorem to nonlinear systems. The Q-parametrization theorem states that for a stable plant P, a compensator F yields a stable closed loop (see Fig. 1) if and only if $F = Q(I - PQ)^{-1}$ for some stable Q. This was proved in the linear case by Zames [11], and used in design by Desoer and Chen [3]. In the *nonlinear* case (where one requires in addition that the plant be incrementally stable) its roots go back to Desoer and Chan [4], and it has been stated explicitly by Desoer and Liu [5]. A consequence of this simple parametrization is that the I/O map is PQ. This has been exploited in [3] to provide an algorithm for compensator design in the case of rational transfer function matrices and in [8] to use optimization software to obtain efficient design for a closed-loop transfer function which is required to satisfy various complex *a priori* inequality constraints [8].

The Q-parametrization method requires that the plant be stable. This is a direct consequence of the algebraic nature of the result—in fact, it can be formulated in an abstract algebraic context [3], [6], [1]. Since there are applications where the plant is unstable (airplanes, chemical reactions, $\cdots$), it is of interest to extend the method to a larger class of plants. For linear plants, results in this direction have been obtained by Zames [12]. Zames considers the class of all plants which are stabilizable by *stable* compensators, i.e., the strongly stabilizable ones [10]. (This class includes the stable plants.) It is shown in [12] that by a two-step compensation scheme one may exploit the Q-parametrization results to design for the closed-loop transfer function for a strongly stabilizable plant.

This note may be considered the nonlinear version of [12]. After defining strong stabilizability suitably in the nonlinear context, we exhibit how design of the closed-loop system for a strongly stabilizable *nonlinear* plant may be carried out by a two-step scheme, where the latter uses the nonlinear Q-parametrization result. Some results pertaining to the robustness of stability of the closed loop, in the spirit of the model reference scheme results of [12], are also presented.

II. Preliminaries

$a := b$ means "a denotes b" Let $\mathbb{R}:=$ the field of real numbers. $\mathbb{Z}:=$ the integers. We consider systems whose inputs, outputs, etc., are defined on $\mathsf{T} \subseteq \mathbb{R}$, typically $\mathsf{T} = \mathbb{R}_+$ or $\mathsf{T} = \mathbb{Z}_+$. For V any normed space let $\mathsf{F} := \{f: \mathsf{T} \to V\}$, with norm $\|\cdot\|_{\mathsf{F}}$. Typically, $V = \mathbb{R}^n$. For any $\tau \in \mathsf{T}$ and $f \in \mathsf{F}$, let $f_\tau \in \mathsf{F}$ be defined by

$$f_\tau(t) = f(t) \quad \text{if } t \leqslant \tau; \qquad f_\tau(t) = 0 \quad \text{if } t > \tau.$$

Let $\|f\|_\tau := \|f_\tau\|_{\mathsf{F}}$ and P_τ: $\mathsf{F} \to \mathsf{F}$ be such that $P_\tau f = f_\tau$. Using usual operations of addition and scalar multiplication, we may define vector spaces of the type

$$\mathscr{L}_e := \{ f \in \mathsf{F} | \forall \tau \in \mathsf{T},\ \|f\|_\tau < \infty \}.$$

Let H: $\mathscr{L}_{e1} \to \mathscr{L}_{e2}$. We say H is *causal* iff $\forall \tau \in \mathsf{T}$ we have $P_\tau H P_\tau = P_\tau H$. All maps encountered in this paper will be causal.

Let H: $\mathscr{L}_{e1} \to \mathscr{L}_{e2}$ be causal; we say H is *finite-gain stable* (f.g. stable) iff $\exists \gamma(H) < \infty$ such that

$$\|Hx\|_\tau \leqslant \gamma(H)\|x\|_\tau, \qquad \forall \tau \in \mathsf{T},\ \forall x \in \mathscr{L}_{e1}.$$

Let H: $\mathscr{L}_{e1} \to \mathscr{L}_{e2}$ be f.g. stable, then we say H is *incrementally stable* (inc. stable) iff $\exists \tilde{\gamma}(H) < \infty$ such that

$$\|Hx - Hy\|_\tau \leqslant \tilde{\gamma}(H)\|x - y\|_\tau \qquad \forall \tau \in \mathsf{T}\ \forall x, y \in \mathscr{L}_{e1}.$$

We consider feedback systems of the type shown in Fig. 1. The input (output, error) space of such a system, denoted U,(Y, E), is the Cartesian product of the spaces of the individual inputs (outputs, errors, respectively). We say that a feedback system is *well posed* iff it defines causal (closed-loop) maps: H_{YU}: $U \to Y$ and H_{EU}: $U \to E$. [For the system ${}^1\mathcal{S}(P,F)$ in Fig. 1, H_{YU}: $(u_1, u_2) \mapsto (y_1, y_2)$ and H_{EU}: $(u_1, u_2) \mapsto (e_1, e_2)$.] We *assume* throughout that the systems we consider are well posed.

We say a system is *finite gain stable* iff H_{YU} and H_{EU} are f.g. stable maps.

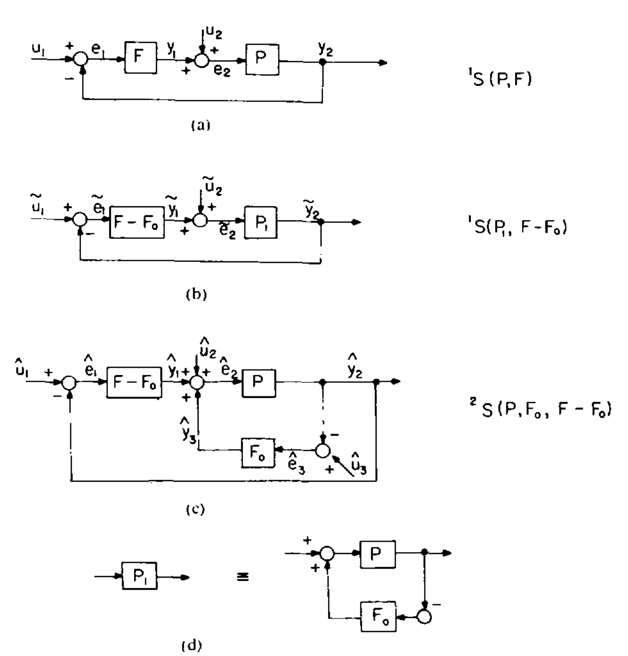

Fig. 1. (a) Defines the system ${}^1\mathcal{S}(P,F)$. (b) Defines the system ${}^1\mathcal{S}(P_1, F - F_0)$. (c) Defines the system ${}^2\mathcal{S}(P, F_0, F - F_0)$. (d) Interprets the relation between P_1 and P and F_0.

III. Main Results

We first state the Q-parametrization theorem for *nonlinear* systems. For a proof see [5].

Theorem 1 (Global Parametrization of I/O Maps): Consider the system ${}^1\mathcal{S}(P, F)$ shown in Fig. 1, where P: $\mathscr{L}_{e2} \to \mathscr{L}_{e1}$, F: $\mathscr{L}_{e1} \to \mathscr{L}_{e2}$. Assume ${}^1\mathcal{S}(P, F)$ is well posed. If P is *inc. stable*, then

$$ {}^1\mathcal{S}(P, F) \text{ is f.g. stable} \tag{3.1}$$

if and only if for some f.g. stable Q: $\mathscr{L}_{e1} \to \mathscr{L}_{e2}$

$$F = Q(I - PQ)^{-1}. \tag{3.2}$$

Furthermore, in terms of P and F, with $u_2 = 0$ we have

$$Q = F(I + PF)^{-1} = H_{e_1 u_1} \tag{3.3}$$

$$H_{y_2 u_1} = PQ. \tag{3.4}$$

Remark: From (3.3), $(I - PQ)^{-1} = (I + PF)$: $\mathscr{L}_{e1} \to \mathscr{L}_{e1}$. Further, (3.3) shows that the f.g. stability of ${}^1\mathcal{S}(P, F)$, ($H_{e_1u_1}$ in particular) requires that Q be f.g. stable.

Our next result is a partial extension of the nonlinear Q-parametrization theorem to f.g. stable, but not necessarily inc. stable plants. It is essentially a restatement of the small gain theorem (see e.g., [7]) but has useful design implications. We omit the (easy) proof.

Theorem 2 (Robustness of Stability): Consider the feedback system ${}^1\mathcal{S}(P_b, F)$, where P_b is inc. stable and $F = Q(I - P_bQ)^{-1}$ for some f.g. stable Q. Consider a perturbation of P_b:

$$P = P_b + \Delta P, \qquad \text{with } \Delta P \text{ f.g. stable.} \tag{3.5}$$

Then

$$\gamma(\Delta P)\gamma(Q) < 1 \tag{3.6}$$

$$\Rightarrow {}^1\mathcal{S}(P, F) \text{ is f.g. stable.} \tag{3.7}$$

Remarks:

1) Thus, for an f.g. stable, but not necessarily incrementally stable

nonlinear plant P, design would proceed by first finding a norm-close incrementally stable approximation P_b and designing for $'S(P_b, F)$ with the constraint $\gamma(P - P_b)\gamma(Q) < 1$ imposed on Q.

2) For a weakly nonlinear f.g. stable plant P, i.e., one having a norm-close *stable linear* approximation P_b, linear design methods could be applied with the constraint $\gamma(P - P_b)\gamma(Q) < 1$ on Q.

We next proceed in the spirit of [12] to develop a two-step scheme for the design of closed-loop systems involving a class of plants larger than the incrementally stable ones.

Definition: A nonlinear plant P: $\mathscr{L}_{e2} \to \mathscr{L}_{e1}$ is said to be *strongly stabilizable* iff there is an inc. stable F_0: $\mathscr{L}_{e1} \to \mathscr{L}_{e2}$, such that with [see Fig. 1(d)]

$$P_1 = P(I - F_0(-P))^{-1} \tag{3.8}$$

we have

$$'S(P, F_0) \text{ is f.g. stable} \tag{3.9}$$

$$P_1 \text{ is inc. stable.} \tag{3.10}$$

Obviously, any incrementally stable plant P is strongly stabilizable.

The proof of the following theorem, which deals with strongly stabilizable nonlinear plants, may be found in Appendix I.

Theorem 3 (Two-Step Compensation): Let the nonlinear plant P: $\mathscr{L}_{e2} \to \mathscr{L}_{e1}$ be strongly stabilizable, and let F_0: $\mathscr{L}_{e1} \to \mathscr{L}_{e2}$, be inc. stable such that with

$$P_1 = P(I - F_0(-P))^{-1} \tag{3.8}$$

the conditions (3.9) and (3.10) hold. Then [see Fig. 1(b) and (c)],

$$'S(P, F) \text{ is f.g. stable for some } F: \mathscr{L}_{e1} \to \mathscr{L}_{e2} \tag{3.11}$$

$\Leftrightarrow$

$$'S(P_1, F - F_0) \text{ is f.g. stable for some } F - F_0: \mathscr{L}_{e1} \to \mathscr{L}_{e2} \tag{3.12}$$

$\Leftrightarrow$ there is a f.g. stable Q such that

$$F - F_0 = Q(I - P_1 Q)^{-1} \tag{3.13}$$

yields

$$^2S(P, F_0, F - F_0) \text{ is f.g. stable.} \tag{3.14}$$

Remarks:

1) The claims of this theorem are highly nonobvious and interesting from a design viewpoint. It says that *any* causal *nonlinear* controller F stabilizing a strongly stabilizable nonlinear plant P can be obtained by a two-step process. First, we use *any* incrementally stable F_0 which yields (3.9) and (3.10) for $P_1 := P(I - F_0(-P))^{-1}$. Then, using Q-parametrization, we design the compensator $(F - F_0)$ for the inc. stable P_1; see (3.13).

2) The theorem gives a global parametrization of *all* stabilizing compensators F for a strongly stabilizable nonlinear plant. Obviously, one needs to build only $'S(P, F)$ with

$$F = F_0 + Q(I - P_1 Q)^{-1}$$

where Q ranges over the f.g. stable maps from $\mathscr{L}_{e1}$ to $\mathscr{L}_{e2}$.

3) Since we are handling nonlinear systems we have to be very careful about signs: in general we *cannot* write

$$I - F_0(-P) = I + F_0 P.$$

This equality holds if F_0 is odd, i.e., if

$$F_0(x) = -F_0(-x) \qquad \forall x \in \mathscr{L}_{e1}.$$

4) This is a powerful generalization of the corresponding linear theorem of [12].

IV. Summary

Here are some of the interesting problems and issues raised by our results.

1) Given a causal map P: $\mathscr{L}_{e2} \to \mathscr{L}_{e1}$, how does one "factor" it as $P = \check{P}P_s$ where P_s is causally invertible and $\check{P}$ is the minimal "bad" part of P in some appropriate sense? By analogy with fractional representation theory for linear plants (both lumped and distributed), $\check{P}$ would correspond, very loosely, to the "unstable zeros" of P, which, as is well known, impose a fundamental limitation on the class of achievable input–output maps in linear problems [2], [9], [13].

2) How does one recognize if a given P is strongly stabilizable? For linear time-invariant lumped systems an extremely elegant and easily verifiable characterization is available in [10] the necessary and sufficient condition for strong stabilizability being that the blocking zeros and the poles of P on the positive real axis satisfy a "parity interlacing property." Is a comparably efficient characterization possible for nonlinear maps? Further, arguing by analogy with the linear case, is it possible to show that the class of strongly stabilizable plants is generic in an appropriate sense?

3) Is there any easy way to find some F_0 which works to strongly stabilize a given map P? Note that it is sufficient that *any* inc. stable F_0 satisfying (3.9) and (3.10) be available—design requirements may be met subsequently by the use of the Q-parametrization method.

We believe that the answers to questions such as these are of fundamental importance to the understanding of the behavior of nonlinear feedback systems from the I/O point of view.

Appendix

Proof of (3.11) ⇒ (3.12): Suppose we apply $\tilde{u}_1 \in \mathscr{L}_{e1}$ and $\tilde{u}_2 \in \mathscr{L}_{e2}$ as inputs to $'S(P_1, F - F_0)$. Let $\tilde{e}_1$ and $\tilde{e}_2$ be the resulting errors. We define

$$u_1 := \tilde{u}_1 \tag{A1}$$

$$e_1 := \tilde{e}_1 \tag{A2}$$

$$e_2 := (I - F_0(-P))^{-1}\tilde{e}_2 \tag{A3}$$

$$u_2 := \tilde{u}_2 + F_0(-P_1)\tilde{e}_2 - F_0(\tilde{u}_1 - P_1\tilde{e}_2). \tag{A4}$$

Step 1: If u_1 and u_2 [as defined in (A1) and (A4)] are applied as inputs to $'S(P, F)$, the errors e_1 and e_2 [as defined in (A2) and (A3)] will satisfy the summing node equations of $'S(P, F)$. This follows from direct substitution.

Step 2: We will now show that the f.g. stability of $'S(P, F)$ implies that of $'S(P_1, F - F_0)$.

By the assumed f.g. stability of $'S(P, F)$ and Step 1, we know there are constants $K_1 < \infty$, $K_2 < \infty$, such that $\forall(\tilde{u}_1, \tilde{u}_2) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2}$ and $\forall \tau \in T$

$$\|e_1\|_\tau \leq K_1(\|u_1\|_\tau + \|u_2\|_\tau) \tag{A5}$$

$$\|e_2\|_\tau \leq K_2(\|u_1\|_\tau + \|u_2\|_\tau) \tag{A6}$$

where (e_1, e_2, u_1, u_2) are defined in terms of $(\tilde{e}_1, \tilde{e}_2, \tilde{u}_1, \tilde{u}_2)$ by (A1)–(A4).

Also, from (A4) and the assumed inc. stability of F_0 we have,

$$\forall(\tilde{u}_1, \tilde{u}_2) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \quad \text{and} \quad \forall \tau \in T$$

$$\|u_2\|_\tau \leq \|\tilde{u}_2\| + \tilde{\gamma}(F_0)\|\tilde{u}_1\|_\tau. \tag{A7}$$

Finally by (A3) we obtain [see Fig. 1(a)]

$$\tilde{e}_2 = (I - F_0(-P))e_2 = e_2 - F_0(-P)e_2 + F_0(u_1 - Pe_2) - F_0 e_1$$

and the assumed inc. stability of F_0 gives

$$\forall(\tilde{u}_1, \tilde{u}_2) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \quad \text{and} \quad \forall \tau \in T$$

$$\|\tilde{e}_2\|_\tau \leq \|e_2\|_\tau + \tilde{\gamma}(F_0)\|u_1\|_\tau + \gamma(F_0)\|e_1\|_\tau. \tag{A8}$$

We use (A5) and (A7) in (A2) to get

$$\|e_1\|_\tau = \|\tilde{e}_1\|_\tau \leq K_1(\|u_1\|_\tau + \|\tilde{u}_2\|_\tau + \tilde{\gamma}(F_0)\|\tilde{u}_1\|_\tau)$$

which, by (A1), gives

$$\forall(\tilde{u}_1, \tilde{u}_2) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \quad \text{and} \quad \forall \tau \in T$$

$$\|\tilde{e}_1\|_\tau \leq K_1[1 + \tilde{\gamma}(F_0)]\|\tilde{u}_1\|_\tau + K_1\|\tilde{u}_2\|_\tau. \tag{A9}$$

From (A5), (A6) and (A8) we have

$$\|\tilde{e}_2\|_\tau \leqslant [K_2 + \tilde{\gamma}(F_0) + K_1\gamma(F_0)]\|u_1\|_\tau + [K_2 + K_1\gamma(F_0)]\|u_2\|_\tau$$

which from (A1) and (A7) gives

$$\forall(u_1, u_2) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \quad \text{and} \quad \forall\tau \in \mathrm{T}$$
$$\|\tilde{e}_2\|_\tau \leqslant ([K_2 + K_1\gamma(F_0)][1 + \tilde{\gamma}(F_0)] + \tilde{\gamma}(F_0))\|\tilde{u}_1\|_\tau + [K_2 + K_1\gamma(F_0)]\|\tilde{u}_2\|_\tau. \quad \text{(A10)}$$

From (A9) and (A10) we see that $H_{\tilde{e}\tilde{u}}$: $(\tilde{u}_1, \tilde{u}_2) \mapsto (\tilde{e}_1, \tilde{e}_2)$ is f.g. stable. Since $\tilde{y}_1 = \tilde{e}_2 - \tilde{u}_2$ and $\tilde{y}_2 = \tilde{u}_1 + \tilde{e}_1$, it follows that $H_{\tilde{y}\tilde{u}}$ is also f.g. stable. Thus, $'S(P_1, F - F_0)$ is f.g stable whenever $'S(P, F)$ is.

Proof of (3.12) ⇒ (3.11): It follows the same lines as the above proof. We apply $u_1 \in \mathscr{L}_{e1}$ and $u_2 \in \mathscr{L}_{e2}$ to $'S(P, F)$. Let e_1 and e_2 be the resulting errors. We define

$$\tilde{u}_1 := u_1$$
$$\tilde{e}_1 := e_1$$
$$\tilde{e}_2 := (I - F_0(-P))e_2$$
$$\tilde{u}_2 := u_2 - F_0(-P)e_2 + F_0(u_1 - Pe_2).$$

Then, assuming the f.g. stability of $'S(P_1, F - F_0)$ we establish, as above, the f.g. stability of $'S(P, F)$.

Proof of (3.12) ⇒ (3.13): Suppose we apply $\hat{u}_1 \in \mathscr{L}_{e1}$, $\hat{u}_2 \in \mathscr{L}_{e2}$, and $\hat{u}_3 \in \mathscr{L}_{e1}$ as inputs to $^2S(P, F_0, F - F_0)$. Let $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ be the resulting errors.

We define

$$\tilde{e}_1 := \hat{e}_1 \quad \text{(A11)}$$
$$\tilde{e}_2 := (I - F_0(-P))\hat{e}_2 \quad \text{(A12)}$$
$$\tilde{u}_1 := \hat{u}_1 \quad \text{(A13)}$$
$$\tilde{u}_2 := \hat{u}_2 + F_0(\hat{u}_3 - P\hat{e}_2) - F_0(-P)\hat{e}_2. \quad \text{(A14)}$$

Step 1: By direct substitution we see that if $\tilde{u}_1$ and $\tilde{u}_2$ [as defined in (A13) and (A14)] are applied as inputs to $'S(P_1, F - F_0)$ the errors $\tilde{e}_1$ and $\tilde{e}_2$ [as defined in (A11) and (A12)] will satisfy the summing node equations of $'S(P_1, F - F_0)$.

Step 2: We will now show that the f.g. stability of $'S(P_1, F - F_0)$ implies that of $^2S(P, F_0, F - F_0)$.

By the assumed f.g. stability of $'S(P_1, F - F_0)$ and Step 1, we know there are constants $M_1 < \infty$ and $M_2 < \infty$ such that $\forall(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \times \mathscr{L}_{e1}$ and $\forall\tau \in \mathrm{T}$

$$\|\tilde{e}_1\|_\tau \leqslant M_1[\|\tilde{u}_1\| + \|\tilde{u}_2\|_\tau] \quad \text{(A15)}$$
$$\|\tilde{e}_2\|_\tau \leqslant M_2[\|\tilde{u}_1\| + \|\tilde{u}_2\|_\tau] \quad \text{(A16)}$$

where $(\tilde{e}_1, \tilde{e}_2, \tilde{u}_1, \tilde{u}_2)$ are defined in terms of $(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{u}_1, \hat{u}_2, \hat{u}_3)$ by (A11)–(A14).

From (A14) and the assumed inc. stability of F_0, we have:

$$\forall(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \times \mathscr{L}_{e1} \quad \text{and} \quad \forall\tau \in \mathrm{T}$$
$$\|\tilde{u}_2\|_\tau \leqslant \|\hat{u}_2\|_\tau + \tilde{\gamma}(F_0)\|\hat{u}_3\|_\tau. \quad \text{(A17)}$$

From (A12) and the f.g. stability of P_1, we have

$$\forall(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \times \mathscr{L}_{e1} \quad \text{and} \quad \forall\tau \in \mathrm{T}$$
$$\|\hat{e}_3\|_\tau \leqslant \|\hat{u}_3\|_\tau + \gamma(P_1)\|\tilde{e}_2\|_\tau. \quad \text{(A18)}$$

And since (A12) gives us

$$\hat{e}_2 = (I - F_0(-P))^{-1}\tilde{e}_2 = \left[I + F_0(-P)(I - F_0(-P))^{-1}\right]\tilde{e}_2 = [I + F_0(-P_1)]\tilde{e}_2$$

we have, by the f.g. stability of F_0 and P_1

$$\forall(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \times \mathscr{L}_{e1} \quad \text{and} \quad \forall\tau \in \mathrm{T}$$
$$\|\hat{e}_2\|_\tau \leqslant [1 + \gamma(F_0)\gamma(P_1)]\|\tilde{e}_2\|_\tau. \quad \text{(A19)}$$

Using (A15) and (A17) in (A11) gives

$$\|\hat{e}_1\|_\tau = \|\tilde{e}_1\|_\tau \leqslant M_1[\|\tilde{u}_1\|_\tau + \|\hat{u}_2\|_\tau + \tilde{\gamma}(F_0)\|\hat{u}_3\|_\tau].$$

So, by (A13) we have

$$\forall(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \times \mathscr{L}_{e1} \quad \text{and} \quad \forall\tau \in \mathrm{T}$$
$$\|\hat{e}_1\|_\tau \leqslant M_1[\|\hat{u}_1\|_\tau + \|\hat{u}_2\|_\tau + \tilde{\gamma}(F_0)\|\hat{u}_3\|_\tau]. \quad \text{(A20)}$$

From (A19) and (A16) we have

$$\|\hat{e}_2\|_\tau \leqslant [1 + \gamma(F_0)\gamma(P_1)]\cdot M_2\cdot[\|\tilde{u}_1\|_\tau + \|\tilde{u}_2\|_\tau].$$

So, using (A13) and (A17) gives

$$\forall(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \times \mathscr{L}_{e1} \quad \text{and} \quad \forall\tau \in \mathrm{T}$$
$$\|\hat{e}_2\|_\tau \leqslant M_2[1 + \gamma(F_0)\gamma(P_1)][\|\hat{u}_1\|_\tau + \|\hat{u}_2\|_\tau + \gamma(F_0)\|\hat{u}_3\|_\tau]. \quad \text{(A21)}$$

Finally, (A18) and (A16) gives

$$\|\hat{e}_3\|_\tau \leqslant \|\hat{u}_3\|_\tau + M_2\gamma(P_1)[\|\tilde{u}_1\|_\tau + \|\tilde{u}_2\|_\tau]$$

which, from (A13) and (A18) gives

$$\forall(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \mathscr{L}_{e1} \times \mathscr{L}_{e2} \times \mathscr{L}_{e1} \quad \text{and} \quad \forall\tau \in \mathrm{T}$$
$$\|\hat{e}_3\|_\tau \leqslant M_2\gamma(P_1)[\|\hat{u}_1\|_\tau + \|\hat{u}_2\|_\tau + \tilde{\gamma}(F_0)\|\hat{u}_3\|_\tau] + \|\hat{u}_3\|_\tau. \quad \text{(A22)}$$

From (A20)–(A22) we see that $H_{\hat{e}\hat{u}}$: $(\hat{u}_1, \hat{u}_2, \hat{u}_3) \mapsto (\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is f.g. stable. Since $\hat{y}_1 = \hat{e}_2 - \hat{u}_2 - F_0\hat{e}_3$, $\hat{y}_2 = \hat{u}_3 - \hat{e}_3$, $\hat{y}_3 = F_0\hat{e}_3$ and F_0 is f.g. stable, we see that $H_{\hat{y}\hat{u}}$ is f.g. stable. Thus, $^2S(P, F_0, F - F_0)$ is f.g. stable whenever $'S(P_1, F - F_0)$ is.

Proof of (3.13) ⇒ (3.12): Specializing $^2S(P, F_0, F - F_0)$ by setting $\hat{u}_3 = 0$ gives $'S(P_1, F - F_0)$. Clearly f.g. stability of the former implies that of the latter.

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