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An optimal strategy for a conflict resolution problem

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Abstract: Relevant to the design of multiple access protocols is the problem of finding the largest of N i.i.d. numbers $X_1, \ldots, X_N$ uniformly distributed over [0, 1] using the minimum number of questions of the following type. We pick a set $A(1) \subset [0, 1]$ and ask which $X_i \in A(1)$. Depending on the response, we pick another subset $A(2)$ and ask which $X_i \in A(2)$, and so on, until we identify the largest X_i. It is shown that the optimum sequence of questions must be of the type $A(k) = (a(k), 1]$; the best sequence $\{a(k)\}$ can then be determined by dynamic programming following the work of Arrow, Pesotchinsky and Sobel.

Keywords: Multiple access protocols, Binary questioning.

1. Introduction

In their paper [1], Arrow, Pesotchinsky and Sobel, considered the following problem:

(P) Let $X_1, \ldots, X_N$ be i.i.d. random numbers uniformly distributed on [0, 1]. The aim is to decide which X_i is the largest with the minimum expected number of *binary* questions, namely questions to which the response is a simple yes or no. We ask a question, and each X_i responds. Based on the responses we ask the next question, and so on, until the largest X_i is determined.

This problem is relevant to the design of multiple access protocols. Here there are N contenders each of which has a message that it desires to transmit over a single channel. A fair scheme to ensure this is for each contender to be assigned a random priority, for example, according a random number uniformly distributed on [0, 1], and give the channel to the leader, i.e., the contender with the highest priority. Each contender only knows the number assigned to it. To begin, based on its number, each contender sends a bit to a decision maker. If these bits are not enough to determine the leader, the decision maker requests a second bit, and so on. At any stage the only information available to the decision maker is the set of past responses. To determine the leader as quickly as possible we would like to minimize the expected number of stages the decision maker has to go through. It is clear that any good solution to the problem P in [1] translates directly into a good solution to this multiple access problem. For further discussion of multiple access problems see [2].

In [1] the optimal strategy (and the minimum expected number of questions) is found within the class of strategies of the following form: Given N, pick a number $a(1) \in [0, 1]$ and ask "Whose number is bigger than $a(1)$?". Depending on the responses, pick a number $a(2)$ and ask "Whose number is bigger than $a(2)$?", and so on. Call such questions *right-handed*. A question is right-handed if it is of the type: "Whose number belongs to the set A?", where A is of the form $(a, 1]$, for some $a \in [0, 1)$. It is straightforward to set up a dynamic programming recursion to determine the optimal right-handed strategy and this is done in [1].

It is natural to ask whether we can decrease the expected number of questions required when arbitrary binary questions are allowed. For such questions one picks an arbitrary (measurable) set $A \subset [0, 1]$ and asks "Does your number belong to the set A?". Thus the most general strategy is one that picks a subset $A(1)$ of [0, 1] and asks: "Does your number belong to $A(1)$?" Then, based on the response it picks a subset $A(2)$ and asks: "Does your number belong to the set $A(2)$?", and so on, until the leader is found. Can we do any better with such general strategies as compared to the strategies considered in [1]? The fundamental difficulty in answering this question is that there is no obvious way to set up a dynamic programming recursion. Our main result is that the added generality cannot help to reduce the minimum expected number of questions.

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Theorem. *The best right-handed strategy is also optimal in the class of all strategies.*

2. Proof of the theorem

The proof proceeds in two steps. We use the result of [1] that the expected number of questions required to determine the leader using the best right-handed strategy is strictly less than 2.5. We will show first, by induction on the number of contenders, that any strategy entails at least 2 questions on average to determine the leader. Using this, a 'bootstrapping' argument shows that any strategy whose first question is not right-handed requires on average more than 2.5 questions to resolve conflict. This suffices to establish the theorem.

Before proceeding we make a preliminary remark. Since every question is equivalent to its complement, we can assume without loss of generality that a question (more precisely, the corresponding set) contains 1. This will be implicit in the following.

Step 1. We first show that for any strategy K, $\mathrm{E}K \geq 2$, where $\mathrm{E}K$ denotes the expected number of questions required to resolve conflict under strategy K.

1. Consider the case of two contenders, $N = 2$. Suppose

$$\inf_K \mathrm{E}K = \Delta < 2.$$

If the first question of K is not right-handed the leader cannot be immediately determined, so K requires at least 2 questions on *every* sample path, in particular $\mathrm{E}K \geq 2$. (Note: We do not distinguish between sets which differ by zero measure; in particular, A is a right-handed if it differs by zero measure from a set of the form $(1-a, 1]$.)

We may therefore assume that K has right-handed first question $(1-a, 1]$. If the number of contenders answering yes to this first question is 0 or 2, we are left with a problem identical to the one we started with, and we need at least Δ more questions on average to resolve conflict. If only one of the contenders answers yes to the first question we are immediately through. Thus

$$\mathrm{E}K \geq 2a(1-a) + (1+\Delta)(1-2a(1-a)).$$

Observe that for any $a \in [0, 1]$ we have $2a(1-a) \leq \frac{1}{2}$, so

$$\mathrm{E}K \geq 1 + \tfrac{1}{2}\Delta.$$

Since this holds for any K, we have $\Delta \geq 1 + \frac{1}{2}\Delta$, or $\Delta \geq 2$.

2. Consider now the case of general N. Assume as induction hypothesis that, for any $m < N$, the expected number of questions to resolve conflict for any strategy is at least 2. We will show that for any strategy K with N contenders the same holds. Suppose, to the contrary that

$$\inf_K \mathrm{E}K = \Delta < 2.$$

Reasoning as before we may assume that the first question of K is right-handed and of the form $(1-a, 1]$. Three types of responses are possible to this first question.

(a) Each contender, or none of them responds yes to the question. In this case we are left with a problem identical to the one we started with and require at least Δ more questions to resolve conflict.

(b) Exactly one contender responds yes to the question. Then we are immediately through. This event has probability $N(1-a)^{n-1}a$.

(c) Anywhere from 2 to $N-1$ contenders respond yes to the question. By the induction hypothesis, we then require at least 2 more questions to resolve conflict.

Thus we have

$$\mathrm{E}K \geq N(1-a)^{N-1}a + (1+\Delta)\left[1 - N(1-a)^{N-1}a\right],$$

where, for the event (c) we used $\Delta < 2$. Since $N(1-a)^{n-1}a \leq \frac{1}{2}$ for $a \in [0, 1]$, this gives

$$\mathrm{E}K \geq 1 + \tfrac{1}{2}\Delta.$$

This holds for any K, and so $\Delta \geq 1 + \frac{1}{2}\Delta$, or $\Delta \geq 2$.

Step 2. The final step is to use the result above to show that $\mathrm{E}K \geq 2.5$ for any strategy K for which the first question, $A \subset [0, 1]$, is not right-handed. We directly consider the case of general N. Let A^c denote the complement of A.

(1) Consider the event where either every contender or no contender responds yes to the first question, i.e., every X_i is in A or in A^c. Then we are left with a problem identical to the one we

started with restricted to the set A or A^c, and by Step 1 above we need at least 2 more questions on average to resolve conflict. Thus on this event we need on average at least 3 questions to resolve conflict.

(2) Consider the complementary event where the number of contenders responding yes to the first question is between 1 and $N-1$. We postulate the following genie:

- The genie tells us which of the sets A and A^c contains the leader.
- If A contains the leader the genie tells us the value of the leader among the contenders whose values are in A^c, and the identities of the contenders whose values are in A and which exceed the leading contender in A^c.
- Similarly, if A^c contains the leader, the genie tells us the value of the leader among the contenders in A, and the identifies of the contenders whose values are in A^c and which exceed the leading contender in A.

By postulating a genie we mean that we permit ourselves to use different strategies on events on which the genie gives us different answers. Clearly we can do no better without the genie than we can with it.

If A contains the leader, the genie leaves us with the problem of determining the leader among the contenders in A that exceed the leading contender in A^c, and these contenders are independently and uniformly distributed on the portion of A which exceeds the leader in A^c. Similar remarks apply when the leader is in A^c.

Thus, except on the event where the leader is in A and the second best contender is in A^c or vice versa, which event we denote Γ, we require, by Step 1 above, at least two more questions on average to determine the leader. On the other hand if the genie is absent, then we require at least two questions on every sample in Γ. Thus if we can prove that the measure of Γ is at most $\frac{1}{2}$ we will have proved the Theorem. Note: We do not distinguish between sets which differ by zero measure; in particular a question A is right-handed if A differs by zero measure from a set of the form $(a, 1]$.

Let $\mu(X)$ denote the measure of X, for $X \subset [0, 1]$. Define two functions F and F^c on $[0, 1]$ by

$$F(x) = \mu(A \cap (x, 1]),$$
$$F^c(x) = \mu(A^c \cap (x, 1]).$$

Notice that $F(x) + F^c(x) = 1 - x$. Next define functions S and D (mnemonics for same and different respectively) by

$$S(x) = F(x)1(x \in A) + F^c(x)1(x \in A^c),$$
$$D(X) = F(x)1(x \in A^c) + F^c(x)1(x \in A).$$

Then $S(x) + D(x) = 1 - x$. Now

$$\mu(\Gamma) = \sum_{i \neq j} \int_0^1 P\{X_k < x \text{ for } k \neq i, j, \; X_i \in A^c \cap [x, x + \mathrm{d}x), \; X_j \in A \cap (x, 1]\}$$
$$+ \sum_{i \neq j} \int_0^1 P\{X_k < x \text{ for } k \neq i, j, \; X_i \in A \cap [x, x + \mathrm{d}x), \; X_j \cap A^c \cap (x, 1]\},$$

so that

$$\mu(\Gamma) = \int_0^1 N(N-1)x^{N-2}D(x)\,\mathrm{d}x.$$

One can now easily check that

$$1 - \mu(\Gamma) = \int_0^1 N(N-1)x^{N-2}S(x)\,\mathrm{d}x.$$

If we define

$$P(x) = \int_x^1 S(y) - D(y)\,\mathrm{d}y,$$

we can easily prove that $P(x) \geq 0$, for $x \in [0, 1]$, and since

$$\int_{x=0}^1 x^{N-2}[S(x) - D(x)]\,\mathrm{d}x$$
$$= -\int_{x=0}^1 x^{N-2}\frac{\mathrm{d}}{\mathrm{d}x}P(x)\,\mathrm{d}x$$
$$= \int_{x=0}^1 P(x)\frac{\mathrm{d}}{\mathrm{d}x}x^{N-2} \geq 0$$

we have shown that $\mu(\Gamma) \leq \frac{1}{2}$ and the proof is complete.

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References

[1] K.J. Arrow, L. Pesotchinsky and M. Sobel, On partitioning of a sample with binary questions in lieu of collecting observations, Technical Report No. 295, Center for Research on Organizational Efficiency, Stanford University (Sep. 1979).

[2] Special Issue on Random-Access Communications, *IEEE Trans. Inform. Theory* **31** (2) (March 1985).