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AN OPTIMAL STRATEGY FOR A CONFLICT RESOLUTION PROBLEM

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Summary

A problem of relevance to the design of multiple access protocols is the following : Let $X_1,...,X_N$ be i.i.d. random variables uniformly distributed in [0, 1]. We have to determine the largest X_i, $1 \leqslant i \leqslant N$, as follows : Given N, we pick a set A(1) $\subseteq$ [0, 1] and ask "Do you belong to A(1) ? ", whereupon each X_i responds with a yes or a no. Based on these responses we pick a set A(2) $\subseteq$ [0, 1] and ask " Do you belong to the set A(2) ? ", and so on. Is there an optimal strategy to choose the sets A(1), A(2), ... so as to minimize the expected number of questions required to determine the largest X_i ? Further, what is the infimum of the expected number of questions required ? Arrow et.al. prove the existence of a strategy that is optimal in the class of strategies where every set A(i) is of the form (a(i) 1], for some a(i) $\in$ [0, 1) . We show that this same strategy is also optimal in the larger class of strategies where the A(i) are allowed to be finite unions of right closed, left open intervals. Modulo measure theoretic technicalities this solves the general problem.

Introduction

In their paper [1], Arrow, Pesotchinsky and Sobel, [APS], have considered the following problem, which, for future reference, we shall call problem P.

P: Let $X_1,...,X_N$ be i.i.d. random variables uniformly distributed in [0, 1]. The aim is to decide which X_i, $1 \leqslant i \leqslant N$ is the largest, by means of **binary** questions, namely questions to which the response is a simple yes or no. We ask a question, and each X_i responds. Based on the responses we ask the next question, etc. Is there an optimal strategy for choosing the questions so as to minimize the expected number of questions it takes to determine which X_i is largest ? Further, what is the infimum expected number of questions we can achieve over all strategies ?

This problem has relevance to the design of multiple access protocols. Here the scenario is that there are N contenders each of whom has a message which he desires to transmit over a single bus. A fair scheme to ensure this is for each contender to decide his priority randomly, for example, according to the number specified by a random number generator taking values uniformly distributed in [0, 1]. Each contender has his own random number generator, and only he knows the number it has produced. To begin, based on his number, each contender sends a bit to a decision maker. If these bits are not enough to determine the leader, the decision maker requests a second bit, and so on. At any stage the only information available to the decisionmaker is the past responses, namely what has already been put down by the contenders. To resolve conflict as quickly as possible we would like to minimize the expected number of stages the decision maker has to go through. It is clear that any good solution to the problem considered in [APS] translates directly into a good solution for the above multiple access problem.

In [APS] the optimal strategy (and the corresponding expected number of questions) is found within the class of strategies of the following form : Given N, pick a number a(1) $\in$ [0, 1) and ask "Who has a number bigger than a(1) ? ". Depending on the responses, pick a number a(2) and ask " Who has his number bigger than a(2) ?", and so on. Let us call such questions **right-handed.** A question is right handed if it is of the type : " Who has his number belonging to the set A ? ", where A is of the type (a, 1], for some a $\in$ [0, 1). It is intuitively clear that it is possible to set up a dynamic programming recursion in this case, to determine the optimal right-handed strategy.

It is natural to try to investigate whether we can decrease the expected number of questions required when arbitrary binary questions are allowed. Such a question is of the type : Pick a set A $\subseteq$ [0, 1] and ask " Does your number belong to the set A ? ". Thus the most general strategy is one that picks a subset A(1) of [0, 1] and asks: "Does your number belong to A(1) ? ". Then, based on the responses it picks a subset A(2) and asks : " Does your number belong to the set A(2) ? ", and so on. Can we do any better with such general strategies as compared to the strategies considered in [APS] ?.

The fundamental difficulty with approaching this issue is that **there is no obvious way to set up a recursion.** The reader is welcome to experiment to disprove this statement.

To avoid measure theoretic technicalities, while still retaining the flavor of the problem, we restrict attention to subsets of [0, 1] that are finite unions of half open intervals of the type $(a, b]$ where $a \in [0, 1), b \in (0, 1]$ and $a \leqslant b$. Even then, setting up a manageable recursion, (which subsequently has to be analyzed to determine the optimal choice of sets) seems hard. Let us call the class of strategies where all questions are of the above type the class of **finite** strategies. The main result of this paper is the following.

Theorem.

Consider the problem P. For each N, there exists a strategy which is optimal in the class of finite strategies. In fact, this is precisely the one which is optimum in the smaller class of right-handed strategies. Thus there is nothing to be gained by allowing this added generality.

Proof of the Theorem.

The proof proceeds in two steps. The technique is motivated by the result of [APS] that the expected number of questions required to resolve conflict using the best right handed strategy is strictly less than 2.5. We will show first, by induction on the number of contenders, that any finite strategy entails at least 2 questions on average to resolve conflict. Using this we will bootstrap to show that any finite strategy whose first question is not right handed requires on average more than 2.5 questions to resolve conflict. Clearly this suffices to establish the theorem.

Before proceeding we make a preliminary remark. Since every question is equivalent to its complement, we can assume without loss of generality that a question contains the point 1. This will be implicit without comment in the following.

Step 1 : We first show that for any strategy K , EK $\geqslant$ 2, where EK denotes the expected number of questions required to resolve conflict under strategy K.

1. Consider the case of two contenders, N = 2. Let

$$\inf_K \mathrm{EK} = \Delta < 2$$

If the first question of K is not right handed the leader **cannot** be immediately determined. Thus for every element of the sample space ω, we require at least 2 questions, so **EK** $\geqslant$ 2.

We may therefore assume that K has right handed first question of measure a. If the number of contenders answering

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yes to the first question is 0 or 2, we are left with a problem identical to the one we started with, and we need at least Δ more questions on average to resolve conflict. If only one of the contenders answers yes to the first question we are immediately through. Thus we see that

$$EK \geqslant 2a(1-a) + (1+\Delta)(1-2a(1-a))$$

Observe that for any a $\in [0,1]$ we have $2a(1-a) \leqslant ½$. Thus

$$EK \geqslant 1 + \frac{\Delta}{2}$$

But the above holds for any K. Thus we have

$$\Delta \geqslant 1 + \frac{\Delta}{2}$$

i.e. $\Delta \geqslant 2$.

2. Consider now the case of general N. By inductive hypothesis, assume that we have proved, for any m $< N$, that the expected number of questions to resolve conflict for any strategy is at least 2. We will show that for any strategy K with N contenders the same holds. Suppose, on the contrary that

$$\inf_K EK = \Delta < 2$$

Reasoning as before we may assume that the first question of K is right handed. Let it have measure a. Three types of responses are possible to the first question.

(i) Each contender, or none of them responds yes to the question. In this case we are left with a problem identical to the one we started with and require at least Δ more questions to resolve conflict.

(ii) Exactly one contender responds yes to the question. Then we are immediately through. This event has probability $N(1-a)^{N-1}a$.

(iii) Anywhere from 2 to N-1 contenders respond yes to the question. Then by the inductive hypothesis, we require at least 2 more questions to resolve conflict. Note that once we have pinned down a nonzero number of contenders in a right handed set on the first question, it is pointless to allow subsequent questions to intersect the complement of this set.

Thus we have

$$EK \geqslant N(1-a)^{N-1}a + (1+\Delta)(1-N(1-a)^{N-1}a)$$

where, for the event (iii) we have used that $\Delta < 2$. Noting that for a $\in [0,1]$ we have $N(1-a)^{N-1}a \leqslant ½$, we have

$$EK \geqslant 1 + \frac{\Delta}{2}$$

But the above holds for any K. Thus we have

$$\Delta \geqslant 1 + \frac{\Delta}{2}$$

i.e. $\Delta \geqslant 2$.

Step 2 : The final step is to use the above to show that if K is any finite strategy for which the first question, $A \subsetneq [0,1]$, is not right handed, then $EK \geqslant 2.5$. We directly consider the case of general N. Let A^c denote the complement of A.

(i) Suppose that each contender or none responds yes to the first question. Then, restricting to the set of the first question or its complement, we are left with a problem identical to the one we started with, so that, by the analysis of Step 1. we see that we need at least 2 more questions, on average to resolve conflict.

(ii) If the number of contenders responding yes to the first question is anywhere from 1 to N-1, we postulate the following genie :

(a) The genie tells us which of the sets A and A^c contains the leader.

(b) If A contains the leader the genie tells us the value of the leader among the contenders whose values are in A^c, and the identities of the contenders whose values are in A and which exceed the leading contender in A^c.

(c) Similarly, if A^c contains the leader, the genie tells us the value of the leader among the contenders in A, and the identities of the contenders whose values are in A^c and which exceed the leading contender in A.

As usual, what we mean by postulating a genie is that we permit ourselves to use different strategies on events on which the genie gives us different answers. Clearly we can do no better without the genie than we can with it.

If A contains the leader, the genie leaves us with the problem of determining the leader among the contenders in A which exceed the leading contender in A^c, and these contenders are **independently and uniformly distributed** on the portion of A which exceeds the leader in A^c. Similar remarks apply when the leader is in A^c.

Thus, except on the event where the leader is in A and the second best contender is in A^c or vice versa, which event we denote Γ, we require, by Step 1. above, at least two more questions on average to determine the leader. On Γ, in any case, we require at least one more question to be through. Thus if we can prove that the measure of Γ is at most ½ we will have proved the Theorem.

We first establish notation. Let meas (X) denote the measure of X, for $X \subseteq [0,1]$. If A consists of M disjoint intervals, we can think of [0,1] as broken up into right closed intervals

$$A^c{}_M, A_M, A^c{}_{M-1} \cdots A^c{}_1, A_1$$

with $A = \bigcup_{i=1}^{M} A_i$, $A^c = \bigcup_{i=1}^{M} A^c{}_i$, $0 member A^c{}_M$, $1 member A_1$, and $meas\ (A^c{}_j)$, $meas\ (A_j) > 0\ forall j = 1,...,M$ except possibly $meas\ (A^c{}_M) = 0$.

We define two functions F and F^c on [0,1] by

$$F(x) = meas\ (A \cap (x,1])$$

$$F^c(x) = meas\ (A^c \cap (x,1])$$

Notice that $F(x) + F^c(x) = 1-x$.

Finally, define functions S and D (mnemonics for same and different respectively) by

$$S(x) = F(x)\ 1(x member A) + F^c(x)\ 1(x member A^c)$$

$$D(x) = F(x)\ 1(x member A^c) + F^c(x)\ 1(x member A)$$

Thus $S(x) + D(x) = 1-x$.

The reader is invited to verify that

$$meas(\Gamma) = \int_{x=0}^{1} N(N-1)x^{N-2}\, D(x)\, dx$$

and

$$1 - meas(\Gamma) = \int_{x=0}^{1} N(N-1)x^{N-2}\, S(x)\, dx$$

If we define

$$P(x) = \int_{x}^{1} S(y) - D(y)\, dy$$

we can easily prove that $P(x) \geqslant 0, forall x member [0,1]$, and since

$$\int_{x=0}^{1} x^{N-2}[S(x) - D(x)]\, dx = -\int_{x=0}^{1} x^{N-2}\frac{d}{dx}P(x)\, dx = \int_{x=0}^{1} P(x)\frac{d}{dx}x^{N-2}dx \geqslant 0$$

we have shown that $meas(\Gamma) \leqslant ½$ and are finally done $endpf$.

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Reference

[1] Kenneth J. Arrow, Leon Pesotchinsky and Milton Sobel. "On partitioning of a sample with binary questions in lieu of collecting observations.".Technical report No. 295. Centre for Research on Organizational Efficiency. Stanford University.September 1979.