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Normed Distances and Their Applications in Optimal Circuit Design

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Abstract. A new geometric method for optimal circuit design is presented. The method treats the optimal design problem through the concept of normed distances from a feasible point to the feasible region boundaries in a norm related to the probability distribution of the circuit parameters. The method treats directly the nonlinear feasible region boundaries without any region approximation. The normed distances are found through the solution of a nonlinear optimization problem. The sufficient optimality conditions for this optimization problem are established and an ordinary explicit formula for the normed distance is also derived. An iterative boundary search technique is used to solve the nonlinear optimization problem concerning the normed distances. The convergence of this technique is proved. Practical circuit examples are given to test the method.

Keywords: optimal circuit design, convexity based optimization, worst case design, design centering, yield optimization

1. Introduction

The goal of optimal circuit design is the circuit's robustness against the unavoidable statistical variations, which occur during the manufacturing process. These variations may cause some of the manufactured circuits to violate the design specification bounds. The percentage of the manufactured circuits, which satisfy the design specifications, is called the production yield. In general, optimal circuit design treats the problem of yield maximization prior to the manufacturing process. For example, design centering is an important optimal design problem, which seeks the optimal circuit parameter values that maximize the yield.

The methods dealing with the optimal circuit design can be classified into two approaches, in general. The first approach is based upon the Monte Carlo method for the estimation of the yield values during the optimization process (Singhal and Pinel, 1981; Hocevar et al., 1984; Styplinski and Oplaski, 1986; Yu et al., 1987; Zaabab et al., 1995). The Monte Carlo based methods require large computational effort and suffer from lack of reliable information for yield maximization. The second approach is classified as geometric approach (Bandler and Abdel-Malek, 1978; Bandler and Chen, 1988; Director et al., 1978; Abdel-Malek and Hassan, 1991; Abdel-Malek et al., 1997, 1999; Antreich et al., 1994; Sapatnekar et al., 1994; Seifi et al., 1999; Hassan and Rabie, 2000), and it treats the problem in a geometric manner under the assumption that the feasible region in the circuit parameter space where the design specifications are satisfied is bounded and convex. The simplicial approximation approach of Director et al. (1978) approximates the feasible region by a polytope, and then the center of the largest hypersphere that can be inscribed within the polytope is used as a design center. Bandler and Abdel-Malek (1978) used hyperboxes to describe the tolerance region. The feasible region is approximated by linearizing the constraints at the vertices of the hyperboxes. The yield is estimated through the evaluation of the hypervolume of the feasible part of the hyperboxes. The ellipsoidal technique of Abdel-Malek and Hassan (1991) approximates the feasible region by a hyperellipsoid, which is determined by generating a sequence of decreasing volume, different shape and center hyperellipsoids. The design center is the center of the final hyperellipsoid. Several significant modifications to the ellipsoidal technique have been made (Abdel-Malek et al., 1997, 1999; Hassan and Rabie, 2000). The Sapatnekar, Vaidya and Kang method (Sapatnekar et al., 1994), updates an initial polytope containing the feasible region by generating a set of tangential hyperplanes to the feasible region boundary. The center of the polytope is then found by maximizing a certain potential function. In this context, the concept of normed distances from a feasible point in the parameter space to a feasible region boundary has been frequently used to develop geometrical techniques for optimal circuit design (Brayton et al., 1980; Antreich et al., 1994; Seifi et al., 1999). These distances are measured in a well defined norm that corresponds to certain probability distribution of the circuit parameters. This probability distribution simulates the statistical variations that affect the circuit parameters during the manufacturing process. The normed distances are found, in general, through the solution of a nonlinear optimization problem. The solution process leads to the location of a certain boundary point. The distance from a given feasible point to the located boundary point in the norm used gives the required normed distance. The problem can be simplified by approximating the feasible region with a polytope (Brayton et al., 1980; Seifi et al., 1999). Antreich et al. (1994) treated the nonlinear optimization problem through an optimization package. However the solution process is facing some problems like numerical ill-conditioning, effects of simulator inaccuracies and the solution localization.

In this paper, a new method to obtain the normed distances is introduced and exploited in optimal circuit design. The method treats directly the nonlinear feasible region boundaries without any region approximation. The normed distances are found, in general, through the solution of a nonlinear optimization problem. The sufficient optimality conditions for this optimization problem are used to derive the conditions to be satisfied by the solution point which is in the same time a boundary point. An iterative boundary search technique is used to locate the required boundary (solution) point. Hence, the optimization problem can be solved and the normed distances can be found. The convergence properties of the boundary search technique are proved. Practical circuit examples are given to test the method. The paper is organized as follows: in Section 2 the fundamental concepts of the optimal circuit design problem are introduced. Also, the normed distances are defined through the solution of nonlinear optimization problem. In Section 3, the sufficient optimality conditions for the optimization problem concerning the normed distances are established. An ordinary explicit formula for the normed distance is also derived. A boundary search technique is presented in Section 4 to locate the boundary points required for finding the normed distances. Based on the normed distances, the design centering problem is formulated and treated in Section 5. Practical circuit examples are given in Section 6.

2. Normed distances and optimal circuit design

In optimal circuit design, the circuit is manipulated through some designable parameters $\underline{\boldsymbol{x}} \in R^n$. Circuit performance properties $\underline{\boldsymbol{p}} \in R^p$ are functions of circuit parameters. In general, the performance of a given parameter set has to be evaluated by numerical circuit simulations:

$$\underline{\boldsymbol{p}} : \underline{\boldsymbol{x}} \to \underline{\boldsymbol{p}}(\underline{\boldsymbol{x}}) \tag{1}$$

Design specifications are characterized by specifying bounds on the circuit performance properties. Generally, these design specifications define a region in the parameter space called the feasible (constrained) region R_C and can be defined as:

$$R_C = \{\underline{\boldsymbol{x}} \in R^n \mid \underline{f}(\underline{\boldsymbol{x}}) \leq \underline{0}\} \tag{2}$$

where

$$\underline{\boldsymbol{f}} = (f_1, f_2, \ldots, f_m) : R^n \to R^m$$

For optimal circuit design, the circuit parameters are assumed to be randomly distributed with joint probability density function (pdf) $\phi(\underline{\boldsymbol{x}}, \underline{\boldsymbol{v}})$ where $\underline{\boldsymbol{v}}$ is the distribution parameters, e.g. various means, variances and correlation coefficients. Using the probability density function of the circuit parameters, the production yield can be defined as:

$$Y(\underline{\boldsymbol{v}}) = \int_{\boldsymbol{R_C}} \phi(\underline{\boldsymbol{x}}, \underline{\boldsymbol{v}})\, d\underline{x} \tag{3}$$

For design centering, the pdf is assumed to be dependent only on the nominal parameter vector $\underline{\boldsymbol{x}}^0 \in R^n$. The design centering problem can be formulated as:

$$\max_{\underline{\boldsymbol{x}}^0} \left[Y(\underline{\boldsymbol{x}}^0) = \int_{\boldsymbol{R_C}} \phi(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}^0)\, d\underline{x} \right] \tag{4}$$

With regards to practical requirements, circuit parameters are assumed to be normally distributed with pdf given by:

$$\phi(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}^0) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \underline{\boldsymbol{C}}}} \exp[-0.5(\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}^0)^T \underline{\boldsymbol{C}}^{-1} (\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}^0)] \tag{5}$$

where $\underline{\boldsymbol{x}}^0 \in R^n$ is the nominal parameter vector and $\underline{\boldsymbol{C}}$ is $(n \times n)$ covariance matrix which is positive definite. Other distributions like the unimodal are commonly approximated by normal pdf's. In particular, skew distributions are approximated by lognormal pdf's which are transformed into normal pdf's (Antreich et al., 1994).

There exists a correspondence between the level contours of a given probability density function and a particular norm (Brayton et al., 1980). For example, the level contours of the normal pdf given by (5) can be described using the l_2-norm such that:

$$\|\underline{\boldsymbol{x}}\| = \sqrt{\underline{\boldsymbol{x}}^T \underline{\boldsymbol{C}}^{-1} \underline{\boldsymbol{x}}} \tag{6}$$

where $\underline{\boldsymbol{C}}$ is the covariance matrix and $\underline{\boldsymbol{x}} \in R^n$. Also, the l_∞-norm can be used to describe the uniform distribution.

Assume that the circuit parameters are normally distributed, hence by using the pdf norm given by (6), the distance between the two points $\underline{\boldsymbol{x}}_1, \underline{\boldsymbol{x}}_2 \in R^n$ will be given by:

$$d(\underline{\boldsymbol{x}}_1, \underline{\boldsymbol{x}}_2) = \|\underline{\boldsymbol{x}}_1 - \underline{\boldsymbol{x}}_2\| = \sqrt{(\underline{\boldsymbol{x}}_1 - \underline{\boldsymbol{x}}_2)^T \underline{\boldsymbol{C}}^{-1} (\underline{\boldsymbol{x}}_1 - \underline{\boldsymbol{x}}_2)} \tag{7}$$

Let $\underline{\boldsymbol{x}}^0 \in R^n$ be a feasible point, i.e., $\underline{\boldsymbol{x}}^0 \in R_C$ where R_C is the feasible region, which is given by (2). Using the norm defined by the normal pdf, the distance between the point $\underline{\boldsymbol{x}}^0$ and the feasible region boundary ($f_i(\underline{\boldsymbol{x}}) = 0, i = 1, 2, \ldots, m$) is given by Brayton et al. (1980) and Antreich et al. (1994):

$$\beta_i = \min_{\underline{\boldsymbol{x}}} \{ d(\underline{\boldsymbol{x}}^0, \underline{\boldsymbol{x}}) \mid f_i(\underline{\boldsymbol{x}}) = 0 \} \tag{8}$$

The normed distance β_i is denoted in Antreich et al. (1994) as a worst case distance as it is the minimal distance from $\underline{\boldsymbol{x}}^0$ to violate the constraint boundary $f_i(\underline{\boldsymbol{x}}) = 0$ in the norm defined by the pdf. Several optimal design problems can be formulated by the aid of normed distances (see Antreich et al., 1994). For example, The design centering problem can be geometrically formulated by using the normed distances as follows:

$$\max_{\underline{\boldsymbol{x}}^0} \left(\min_{i=1,2,\ldots,m} \beta_i \right) \tag{9}$$

3. Mathematical basis for finding the normed distances

By using Eq. (7) and dropping the suffix i for simplicity, the distance β between the point $\underline{\boldsymbol{x}}^0$ and the hypersurface $f(\underline{\boldsymbol{x}}) = 0$ is formulated as:

$$\beta = \min_{\underline{\boldsymbol{x}}} \sqrt{(\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}^0)^T \underline{\boldsymbol{C}}^{-1} (\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}^0)} \tag{10}$$

such that

$$f(\underline{\boldsymbol{x}}) = 0$$

i.e., β will be the optimal value of the objective function of the constrained optimization problem (10).

The following proposition gives the conditions to be satisfied by the solution of problem (10). These conditions are derived on the basis of the second-order sufficient optimality conditions (Rao, 1984; Fletcher, 1990).

Proposition 3.1. $\underline{\boldsymbol{x}}^*$ *solves problem* (10) *if*:

(i) $f(\underline{\boldsymbol{x}}^*) = 0$ (11)

(ii) $$\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0 = \frac{\beta \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}}{\sqrt{\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}}} \tag{12}$$

where,

$$\underline{\boldsymbol{g}} = \underline{\nabla} f(\underline{\boldsymbol{x}}^*), \tag{13}$$

and

$$\beta = \left| \frac{\underline{\boldsymbol{g}}^T (\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)}{\sqrt{\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}}} \right| \tag{14}$$

(iii) *The matrix* $\underline{\boldsymbol{L}}$ *given by*

$$\underline{\boldsymbol{L}} = \left(\frac{1}{\beta} \underline{\boldsymbol{C}}^{-1} - \frac{1}{\sqrt{\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}}} \underline{\boldsymbol{H}} \right) \tag{15}$$

is positive definite where $\underline{\boldsymbol{H}}$ *is the Hessian matrix of* $\boldsymbol{f}(\underline{\boldsymbol{x}})$ *evaluated at* $\underline{\boldsymbol{x}}^*$.

Proof: Condition (i) of the proposition states that $\underline{\boldsymbol{x}}^*$ is a feasible point for problem (10).

By using the method of Lagrange multipliers, the lagrangian function will be:

$$L(\underline{\boldsymbol{x}}, \lambda) = \sqrt{(\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}^0)^T \underline{\boldsymbol{C}}^{-1} (\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}^0)} + \lambda f(\underline{\boldsymbol{x}}) \tag{16}$$

The first-order necessary conditions for the existence of a local minimizer at the feasible point $\underline{\boldsymbol{x}}^*$ will be (Rao, 1984; Fletcher, 1990):

$$\nabla_x L|_{\underline{\boldsymbol{x}}=\underline{\boldsymbol{x}}^*} = \underline{\boldsymbol{0}} \Rightarrow \frac{\underline{\boldsymbol{C}}^{-1}(\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)}{\sqrt{(\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)^T \underline{\boldsymbol{C}}^{-1} (\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)}} + \lambda \underline{\boldsymbol{g}} = \underline{\boldsymbol{0}} \tag{17}$$

Postmultiplying (17) by $\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}$ and rearranging we get:

$$\lambda = -\frac{\underline{\boldsymbol{g}}^T (\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)}{\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}\, \underline{\boldsymbol{g}} \sqrt{(\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)^T \underline{\boldsymbol{C}}^{-1} (\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)}} \tag{18}$$

Substitute for λ in (17) and rearranging, then:

$$\underline{\boldsymbol{C}}^{-1}(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0}) = \frac{\underline{\boldsymbol{g}}^{T}(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0})\underline{\boldsymbol{g}}}{\underline{\boldsymbol{g}}^{T}\underline{\boldsymbol{C}}\,\underline{\boldsymbol{g}}} \tag{19}$$

Premultiply both sides by $(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0})^{T}$, then:

$$(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0})^{T}\underline{\boldsymbol{C}}^{-1}(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0}) = \frac{[\underline{\boldsymbol{g}}^{T}(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0})]^{2}}{\underline{\boldsymbol{g}}^{T}\underline{\boldsymbol{C}}\underline{\boldsymbol{g}}} \tag{20}$$

Hence:

$$\beta = \sqrt{(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0})^{T}\underline{\boldsymbol{C}}^{-1}(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0})} = \frac{|\underline{\boldsymbol{g}}^{T}(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0})|}{\sqrt{\underline{\boldsymbol{g}}^{T}\underline{\boldsymbol{C}}\,\underline{\boldsymbol{g}}}} \tag{21}$$

It is to be noticed that if $f(\underline{\boldsymbol{x}}^{0}) < 0$ and $f(\underline{\boldsymbol{x}})$ is convex, then $\underline{\boldsymbol{g}}^{T}(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0}) > 0$, then:

$$\beta = \frac{\underline{\boldsymbol{g}}^{T}(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0})}{\sqrt{\underline{\boldsymbol{g}}^{T}\underline{\boldsymbol{C}}\underline{\boldsymbol{g}}}} \tag{22}$$

Without loss of generality we will assume that $f(\underline{\boldsymbol{x}}^{0}) < 0$ and we assume that $f(\underline{\boldsymbol{x}})$ is convex then β will be given by (22). By using Eqs. (19) and (22) we get:

$$\underline{\boldsymbol{C}}^{-1}(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0}) = \frac{\beta}{\sqrt{\underline{\boldsymbol{g}}^{T}\underline{\boldsymbol{C}}\,\underline{\boldsymbol{g}}}}\underline{\boldsymbol{g}} \tag{23}$$

Then:

$$(\underline{\boldsymbol{x}}^{*} - \underline{\boldsymbol{x}}^{0}) = \frac{\beta}{\sqrt{\underline{\boldsymbol{g}}^{T}\underline{\boldsymbol{C}}\underline{\boldsymbol{g}}}}\underline{\boldsymbol{C}}\underline{\boldsymbol{g}} \tag{24}$$

Hence condition (ii) of Proposition 3.1 is proved.

Condition (iii) of Proposition 3.1, will be derived by using the second-order sufficient condition (Rao, 1984; Fletcher, 1990),

$$\Delta\underline{\boldsymbol{x}}^{T}\left(\nabla_{x}^{2}L\big|_{\underline{\boldsymbol{x}}=\underline{\boldsymbol{x}}^{*}}\right)\Delta\underline{\boldsymbol{x}} > 0 \tag{25}$$

where,

$$\underline{\boldsymbol{g}}^{T}\Delta\underline{\boldsymbol{x}} = 0 \tag{26}$$

We first find the Hessian matrix of the lagrangian function, $(\nabla_x^2 L|_{\underline{x}=\underline{x}^*})$. By using Eq. (17), then:

$$\nabla_x^2 L\Big|_{\underline{x}=\underline{x}^*} = \frac{\underline{\boldsymbol{C}}^{-1}}{\sqrt{(\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)^T \underline{\boldsymbol{C}}^{-1} (\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)}} - \frac{[\underline{\boldsymbol{C}}^{-1}(\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)]\,[\underline{\boldsymbol{C}}^{-1}(\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)]^T}{[(\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)^T \underline{\boldsymbol{C}}^{-1} (\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)]^{\frac{3}{2}}} + \lambda \underline{\boldsymbol{H}} \tag{27}$$

where $\underline{\boldsymbol{H}}$ is the Hessian matrix of $f(\underline{\boldsymbol{x}})$ evaluated at $\underline{\boldsymbol{x}}^*$. By using Eqs. (21) and (22), the value of λ given by Eq. (18) will be:

$$\lambda = -\frac{1}{\sqrt{\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}\, \underline{\boldsymbol{g}}}} \tag{28}$$

Then substitute into (27) using (23) and (28):

$$\nabla_x^2 L\Big|_{\underline{x}=\underline{x}^*} = \frac{\underline{\boldsymbol{C}}^{-1}}{\beta} - \frac{\underline{\boldsymbol{g}}\,\underline{\boldsymbol{g}}^T}{\beta \underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}\, \underline{\boldsymbol{g}}} - \frac{1}{\sqrt{\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}}} \underline{\boldsymbol{H}} \tag{29}$$

By using (29) and (26), then:

$$\Delta \underline{\boldsymbol{x}}^T \left(\nabla_x^2 L\Big|_{\underline{x}=\underline{x}^*}\right) \Delta \underline{\boldsymbol{x}} = \Delta \underline{\boldsymbol{x}}^T \left[\frac{\underline{\boldsymbol{C}}^{-1}}{\beta} - \frac{1}{\sqrt{\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}\, \underline{\boldsymbol{g}}}} \underline{\boldsymbol{H}} \right] \Delta \underline{\boldsymbol{x}} \tag{30}$$

If the matrix $\underline{\boldsymbol{L}} = [\frac{\underline{\boldsymbol{C}}^{-1}}{\beta} - \frac{1}{\sqrt{\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}\, \underline{\boldsymbol{g}}}} \underline{\boldsymbol{H}}]$ is positive definite, then (30) will be positive and condition (25) will be satisfied. Hence, the proof is complete. □

Based on Proposition 3.1 the following remarks are made:

(i) From the second condition of the proposition, it is evident the gradient of the function $f(\underline{\boldsymbol{x}})$ evaluated at the solution point $\underline{\boldsymbol{x}}^*$ must be parallel to the vector $\underline{\boldsymbol{C}}^{-1}$ $(\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)$.
(ii) The expression for the normed distance given by (22), i.e. $\beta = \frac{\underline{\boldsymbol{g}}^T (\underline{\boldsymbol{x}}^* - \underline{\boldsymbol{x}}^0)}{\sqrt{\underline{\boldsymbol{g}}^T \underline{\boldsymbol{C}}\, \underline{\boldsymbol{g}}}}$ is similar to the ordinary expressions used by Brayton et al. (1980) and Seifi et al. (1999).

4. A search technique for the normed distances

In this section an iterative search technique will be used to solve the optimization problem (10). The technique exploits the work presented in Abdel-Malek et al. (1997) to generate a sequence of boundary points $\{\underline{\boldsymbol{x}}_k\}$ such that $f(\underline{\boldsymbol{x}}_k) = 0$. The sequence $\{\underline{\boldsymbol{x}}_k\}$ converges to the point $\underline{\boldsymbol{x}}^*$ which solves problem (10).

The technique starts with a point $\underline{\boldsymbol{x}}_1 \in R^n$ such that $f(\underline{\boldsymbol{x}}_1) = 0$. Starting from $\underline{\boldsymbol{x}}_1$, a point $\underline{\boldsymbol{x}}_C \in R^n$ is found and is given by:

$$\underline{\boldsymbol{x}}_C = \underline{\boldsymbol{x}}_1 + \gamma \frac{\underline{\boldsymbol{C}}\,\underline{\boldsymbol{g}}_1}{\sqrt{\underline{\boldsymbol{g}}_1^T \underline{\boldsymbol{C}}\,\underline{\boldsymbol{g}}_1}} \tag{31}$$

where

$$\left.\begin{aligned} \underline{\boldsymbol{g}}_1 &= \underline{\nabla} f(\underline{\boldsymbol{x}}_1) \\ \gamma &= \theta\beta_1, \quad \theta \in (0,1) \\ \text{and} \qquad \beta_1 &= \frac{\underline{g}_1^T(\underline{\boldsymbol{x}}_1 - \underline{\boldsymbol{x}}^0)}{\sqrt{\underline{\boldsymbol{g}}_1^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_1}} \end{aligned}\right\} \tag{32}$$

In general $f(\underline{\boldsymbol{x}}_C) \neq 0$. After $\underline{\boldsymbol{x}}_C$ is obtained a line search starting from $\underline{\boldsymbol{x}}_C$ in the $(\underline{\boldsymbol{x}}^0 - \underline{\boldsymbol{x}}_1)$ direction is carried out to find a point $\underline{\boldsymbol{x}}_2 \in R^n$ such that $f(\underline{\boldsymbol{x}}_2) = 0$. This iteration is repeated until convergence occurs and a point $\underline{\boldsymbol{x}}_f$ is found. An iteration of the technique can be given by the following steps:

$$\left.\begin{aligned} \underline{\boldsymbol{x}}_C &= \underline{\boldsymbol{x}}_k + \gamma \frac{\underline{\boldsymbol{C}}\,\underline{\boldsymbol{g}}_k}{\sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_k}} \\ \underline{\boldsymbol{x}}_{k+1} &= \underline{\boldsymbol{x}}_C + \mu_k(\underline{\boldsymbol{x}}^0 - \underline{\boldsymbol{x}}_k) \end{aligned}\right\} \tag{33}$$

$$f(\underline{\boldsymbol{x}}_k) = f(\underline{\boldsymbol{x}}_{k+1}) = 0$$

where μ_k is the step of the line search starting from $\underline{\boldsymbol{x}}_C$ in the $(\underline{\boldsymbol{x}}^0 - \underline{\boldsymbol{x}}_k)$ direction. Thus a sequence $\{\underline{\boldsymbol{x}}_k\}$ is generated where $\underline{\boldsymbol{x}}_k \in R^n$ and $f(\underline{\boldsymbol{x}}_k) = 0$. The generation of this sequence can be obtained by the following equation:

$$\underline{\boldsymbol{x}}_{k+1} = \underline{\boldsymbol{x}}_k + \gamma \frac{\underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_k}{\sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_k}} + \mu_k(\underline{\boldsymbol{x}}^0 - \underline{\boldsymbol{x}}_k) \tag{34}$$

Equation (34) can be written as:

$$\underline{\boldsymbol{x}}_{k+1} = F(\underline{\boldsymbol{x}}_k), \quad k = 1, 2, \ldots \tag{35}$$

where F is a transformation from R^n into itself. A fixed point $\underline{\boldsymbol{x}}_f \in R^n$ for this transformation F is defined as Josi et al. (1985):

$$\underline{\boldsymbol{x}}_f = F(\underline{\boldsymbol{x}}_f) \tag{36}$$

Lemma 4.1. *The fixed point* $\underline{\boldsymbol{x}}_f$ *of the transformation* F *given by* (34) *and* (35) *satisfies*:

$$\underline{\boldsymbol{x}}_f - \underline{\boldsymbol{x}}^{\mathbf{0}} = \frac{\beta \underline{\boldsymbol{C}}\, \underline{\boldsymbol{g}}_f}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} \tag{37}$$

where

$$\beta = \frac{\underline{\boldsymbol{g}}_f^T(\underline{\boldsymbol{x}}_f - \underline{\boldsymbol{x}}^0)}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} \tag{38}$$

Proof: The fixed point $\underline{\boldsymbol{x}}_f$ satisfies $\underline{\boldsymbol{x}}_f = F(\underline{\boldsymbol{x}}_f)$, hence using (34):

$$\underline{\boldsymbol{x}}_f = \underline{\boldsymbol{x}}_f + \gamma \frac{\underline{\boldsymbol{C}}\, \underline{\boldsymbol{g}}_f}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} + \mu_f(\underline{\boldsymbol{x}}^0 - \underline{\boldsymbol{x}}_f) \tag{39}$$

Then

$$\frac{\gamma}{\mu_f} \frac{\underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} + (\underline{\boldsymbol{x}}^0 - \underline{\boldsymbol{x}}_f) = \underline{\mathbf{0}} \tag{40}$$

Multiply both sides by $\underline{\boldsymbol{g}}_f^T$ we get:

$$\frac{\gamma}{\mu_f} \frac{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} + \underline{\boldsymbol{g}}_f^T(\underline{\boldsymbol{x}}^0 - \underline{\boldsymbol{x}}_f) = 0 \tag{41}$$

or

$$\frac{\gamma}{\mu_f} = \frac{\underline{\boldsymbol{g}}_f^T(\underline{\boldsymbol{x}}_f - \underline{\boldsymbol{x}}^0)}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} = \beta \tag{42}$$

Substitute into (40), hence:

$$(\underline{\boldsymbol{x}}_f - \underline{\boldsymbol{x}}^0) = \frac{\beta}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f. \tag{43}$$

It is to be noticed that the fixed point $\underline{\boldsymbol{x}}_f$ satisfies the first and the second conditions of Proposition 3.1 in the previous section. This implies that the fixed point of the transformation F given by (34) and (35) is a KKT point for problem (10) i.e. it satisfies the Kaursk- Kuhn-Tucker necessary conditions (Bazaraa et al., 1993; Fletcher, 1990) □

Lemma 4.2. *Assume that the constraint function $f(x)$ is convex, then for the generated sequence of boundary points in the boundary search technique given by* (33), *the value of the line search step μ_k is bounded by the relation*

$$\mu_k \geq \gamma \frac{\sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_{\boldsymbol{k}}}}{\underline{\boldsymbol{g}}_k^T(\underline{\boldsymbol{x}}_k - \underline{\boldsymbol{x}}^0)} \tag{44}$$

where γ is given by (32)

Proof: From (34), the change of the position vector at the kth iteration is given by

$$\underline{\Delta \boldsymbol{x}}_k = \gamma \frac{\underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_k}{\sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_k}} + \mu_k(\underline{\boldsymbol{x}}^0 - \underline{\boldsymbol{x}}_k) \tag{45}$$

Multiply both sides by $\underline{\boldsymbol{g}}_k^T$ we get:

$$\underline{\boldsymbol{g}}_k^T \underline{\Delta \boldsymbol{x}}_k = \gamma \sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_k^T} + \mu_k \underline{\boldsymbol{g}}_k^T(\underline{\boldsymbol{x}}^0 - \underline{\boldsymbol{x}}_k) \tag{46}$$

Thus we have

$$\frac{\underline{\boldsymbol{g}}_k^T \underline{\Delta \boldsymbol{x}}_k}{\underline{\boldsymbol{g}}_k^T(\underline{\boldsymbol{x}}_k - \underline{\boldsymbol{x}}^0)} = \gamma \frac{\sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_k}}{\underline{\boldsymbol{g}}_k^T(\underline{\boldsymbol{x}}_k - \underline{\boldsymbol{x}}^0)} - \mu_k \tag{47}$$

As $f(\underline{\boldsymbol{x}})$ is convex and assuming that $f(\underline{\boldsymbol{x}}^0) < 0$, then $\underline{\boldsymbol{g}}_k^T(\underline{\boldsymbol{x}}_k - \underline{\boldsymbol{x}}^0) > 0$ and $\underline{\boldsymbol{g}}_k^T \underline{\Delta \boldsymbol{x}}_k \leq 0$. Hence, (47) implies that

$$\mu_k \geq \gamma \frac{\sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_{\boldsymbol{k}}}}{\underline{\boldsymbol{g}}_k^T(\underline{\boldsymbol{x}}_k - \underline{\boldsymbol{x}}^0)}. \tag{48}$$

At the fixed point of the boundary search technique we have $\underline{\Delta \boldsymbol{x}}_k = \boldsymbol{0}$, hence (48) is satisfied as equality. This case agrees with Eq. (42).

In the following proposition we will show that the boundary search technique will converge only to a fixed point which satisfies the third condition of Proposition 3.1. Hence, the technique will converge only to a point that satisfies the second-order sufficient conditions of problem (10). □

Proposition 4.1. *Starting from a point $\underline{\boldsymbol{x}}_1 \in R^n$ in the neighborhood of the fixed point $\underline{\boldsymbol{x}}_f$ where $f(\underline{\boldsymbol{x}}_1) = 0$, and assuming that the constraint function $f(\underline{\boldsymbol{x}})$ is quadratic and convex in this neighborhood such that the matrix $\underline{\boldsymbol{C}\boldsymbol{H}}$ is positive definite where $\underline{\boldsymbol{H}}$ is the Hessian matrix of $f(\underline{\boldsymbol{x}})$, then the sequence $\{\underline{\boldsymbol{x}}_k\}$ generated by Eq.* (34) *converges to the fixed point $\underline{\boldsymbol{x}}_f$ which is the solution of the optimization problem* (10).

Proof: Banach's condition principle (Josi et al., 1985) states that if a mapping F is a contraction mapping on a complete metric space (X, d) into itself, then F has a unique fixed point in X say $\underline{\boldsymbol{x}}_f$. Moreover, if $\underline{\boldsymbol{x}}_1$ is an arbitrary point in X and the sequence $\{\underline{\boldsymbol{x}}_k\}$ is defined by:

$$\underline{\boldsymbol{x}}_{k+1} = F(\underline{\boldsymbol{x}}_k), \quad k = 1, 2, \ldots$$

then

$$\lim_{k\to\infty} \underline{\boldsymbol{x}}_k = \underline{\boldsymbol{x}}_f$$

Under the assumption of a quadratic behavior of the function $f(\underline{\boldsymbol{x}})$ in the neighborhood of the fixed point $\underline{\boldsymbol{x}}_f$, then:

$$\underline{\boldsymbol{g}}_k = \underline{\boldsymbol{H}}\,\underline{\boldsymbol{x}}_k + \underline{\boldsymbol{\eta}}, \quad \underline{\boldsymbol{\eta}} \in R^n \tag{49}$$

Substitute into (34), then

$$\underline{\boldsymbol{x}}_{k+1} = \left[\underline{\boldsymbol{I}} - \mu_k \underline{\boldsymbol{I}} + \frac{\gamma \underline{\boldsymbol{C}}\,\underline{\boldsymbol{H}}}{\sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_k}}\right]\underline{\boldsymbol{x}}_k + \mu_k \underline{\boldsymbol{x}}^0 + \frac{\gamma \underline{\boldsymbol{C}}\underline{\eta}}{\sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_k}} \tag{50}$$

where $\underline{\boldsymbol{I}}$ is the identity matrix. Using the result of Lemma 4.2, we may assume that in the neighborhood of the fixed point $\underline{\boldsymbol{x}}_f$ the value of $(-\mu_k)$ is approximated by:

$$-\mu_k \approx -\gamma \frac{\sqrt{\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_{\boldsymbol{k}}}}{\underline{\boldsymbol{g}}_k^T (\underline{\boldsymbol{x}}_k - \underline{\boldsymbol{x}}^0)} \approx -\gamma \frac{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_f}}{\underline{\boldsymbol{g}}_f^T (\underline{\boldsymbol{x}}_f - \underline{\boldsymbol{x}}^0)} \approx -\frac{\gamma}{\beta}, \tag{51}$$

and the value of $\underline{\boldsymbol{g}}_k^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_k$ is almost constant in a small neighborhood of $\underline{\boldsymbol{x}}_f$, thus (50) can be written as:

$$\underline{\boldsymbol{x}}_{k+1} = \left[\underline{\boldsymbol{I}} - \frac{\gamma}{\beta}\underline{\boldsymbol{I}} + \frac{\gamma \underline{\boldsymbol{C}}\,\underline{\boldsymbol{H}}}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_f}}\right]\underline{\boldsymbol{x}}_k + \frac{\gamma}{\beta}\underline{\boldsymbol{x}}^0 + \frac{\gamma\, \underline{\boldsymbol{C}}\underline{\eta}}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_f}} \tag{52}$$

or

$$\underline{\boldsymbol{x}}_{k+1} = \underline{\boldsymbol{B}}\,\underline{\boldsymbol{x}}_k + \underline{\boldsymbol{b}} \tag{53}$$

where

$$\underline{\boldsymbol{B}} = \left[\underline{\boldsymbol{I}} - \frac{\gamma}{\beta}\underline{\boldsymbol{I}} + \frac{\gamma \underline{\boldsymbol{C}}\,\underline{\boldsymbol{H}}}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_f}}\right] \tag{54}$$

and

$$\underline{\boldsymbol{b}} = \frac{\gamma}{\beta}\underline{\boldsymbol{x}}^0 + \frac{\gamma \underline{\boldsymbol{C}}\,\underline{\eta}}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}}\,\underline{\boldsymbol{g}}_f}} \tag{55}$$

Let

$$F(\underline{\boldsymbol{x}}_1) = \underline{\boldsymbol{B}}\,\underline{\boldsymbol{x}}_1 + \underline{\boldsymbol{b}}$$
$$F(\underline{\boldsymbol{x}}_2) = \underline{\boldsymbol{B}}\,\underline{\boldsymbol{x}}_2 + \underline{\boldsymbol{b}}$$

Then

$$\|F(\underline{\boldsymbol{x}}_1) - F(\underline{\boldsymbol{x}}_2)\| = \|\underline{\boldsymbol{B}}\|\,\|\underline{\boldsymbol{x}}_1 - \underline{\boldsymbol{x}}_2\| \tag{56}$$

For F to be a contraction mapping, then $\|\underline{\boldsymbol{B}}\| < 1$, i.e., the spectral radius of $\underline{\boldsymbol{B}}$ should be less than 1 (Gill et al., 1991), or the eigenvalues of the $\underline{\boldsymbol{B}}$ matrix are inside the unit circle.

The matrix $\underline{\boldsymbol{C}}\,\underline{\boldsymbol{H}}$ is symmetric and positive definite and can be diagonalized such that:

$$\underline{\boldsymbol{C}}\,\underline{\boldsymbol{H}} = \underline{\boldsymbol{Q}}^T \underline{\boldsymbol{\Delta}}_\lambda \underline{\boldsymbol{Q}}, \quad \underline{\boldsymbol{Q}}^T \underline{\boldsymbol{Q}} = \underline{\boldsymbol{I}} \tag{57}$$

where $\underline{\boldsymbol{\Delta}}_\lambda$ is a diagonal matrix whose entries are the eigenvalues of the matrix $\underline{\boldsymbol{C}}\,\underline{\boldsymbol{H}}$. Substituting into (54) we get:

$$\underline{\boldsymbol{B}} = \underline{\boldsymbol{Q}}^T \left[\underline{\boldsymbol{I}} - \frac{\gamma}{\beta}\underline{\boldsymbol{I}} + \frac{\gamma\,\underline{\boldsymbol{\Delta}}_\lambda}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}}\,\underline{\boldsymbol{g}}_f}} \right] \underline{\boldsymbol{Q}} \tag{58}$$

The eigenvalues of the $\underline{\boldsymbol{B}}$ matrix are given by:

$$\lambda_i^B = 1 - \frac{\gamma}{\beta} + \frac{\gamma \lambda_i}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}}\underline{\boldsymbol{g}}_f}}, \quad i = 1, 2, \ldots, n \tag{59}$$

where λ_i is the corresponding eigenvalue of the $\underline{\boldsymbol{C}}\,\underline{\boldsymbol{H}}$ matrix. Hence, for F to be a contraction mapping, then:

$$\left|\lambda_i^B\right| < 1, \quad i = 1, 2, \ldots, n \tag{60}$$

or

$$-1 < \left(1 - \frac{\gamma}{\beta} + \frac{\gamma \lambda_i}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}}\,\underline{\boldsymbol{g}}_f}}\right) < 1 \tag{61}$$

i.e.,

$$0 < \gamma < \frac{2}{\frac{1}{\beta} - \frac{\lambda_i}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}}} \tag{62}$$

Hence there exists a range for γ to converge if:

$$\frac{1}{\beta} - \frac{\lambda_i}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} > 0 \tag{63}$$

This condition is satisfied if the fixed point $\underline{\boldsymbol{x}}_f$ satisfies also the third condition of Proposition 3.1, i.e., the matrix:

$$\underline{\boldsymbol{L}} = \left(\frac{1}{\beta} \underline{\boldsymbol{C}}^{-1} - \frac{\underline{\boldsymbol{H}}}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} \right) \tag{64}$$

is positive definite. Hence the matrix:

$$\underline{\boldsymbol{C}}\,\underline{\boldsymbol{L}} = \left(\frac{1}{\beta} \underline{\boldsymbol{I}} - \frac{\underline{\boldsymbol{C}}\,\underline{\boldsymbol{H}}}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} \right) \tag{65}$$

can be diagonalized such that:

$$\underline{\boldsymbol{C}}\,\underline{\boldsymbol{L}} = \underline{\boldsymbol{Q}}^T \left(\frac{1}{\beta} \boldsymbol{I} - \frac{\Delta_\lambda}{\sqrt{\boldsymbol{g}_f^T \underline{\boldsymbol{C}} \boldsymbol{g}_f}} \right) \boldsymbol{Q}, \quad \underline{\boldsymbol{Q}}^T \boldsymbol{Q} = \boldsymbol{I} \tag{66}$$

and its eigenvalues are positive such that:

$$\lambda_i^{CL} = \frac{1}{\beta} - \frac{\lambda_i}{\sqrt{\underline{\boldsymbol{g}}_f^T \underline{\boldsymbol{C}} \underline{\boldsymbol{g}}_f}} > 0, \quad i = 1, 2, \ldots, n$$

where λ_i is the corresponding eigenvalue of the the matrix $\underline{\boldsymbol{C}}\,\underline{\boldsymbol{H}}$ matrix. This completes the proof. □

5. Design centering using the normed distances

As previously stated, the design centering problem seeks the nominal vector of the circuit parameters $\underline{\boldsymbol{x}}^0 \in R^n$, which maximizes the yield. The design centering problem is geometrically formulated by using the normed distances as follows (Antreich et al., 1994):

$$\max_{\underline{\boldsymbol{x}}^0} \left(\min_{i=1,2,\ldots,m} \beta_i \right) \tag{67}$$

where, β_i, $i = 1, 2, \ldots, m$ is the nomed distance from $\underline{\boldsymbol{x}}^0$ to the constraint boundary $f_i(\underline{\boldsymbol{x}}) = 0$ in the norm defined by the normal probability distribution (7).

The design centering problem (67) can be solved by transforming it into a nonlinear programming problem by introducing an additional variable z and the problem will be:

$$\begin{aligned} &\max_{\underline{\boldsymbol{x}}^0, z} z \\ &\text{such that} \\ &z - \beta_i \leq 0, \quad i = 1, 2, \ldots, m \end{aligned} \tag{68}$$

To deal with problem (68), the normed distances (β_i, $i = 1, 2, \ldots, m$) from the feasible point $\underline{\boldsymbol{x}}^0$ to the feasible region boundaries must be found. To find the value of β_i a line search from $\underline{\boldsymbol{x}}^0$ in the $\underline{\boldsymbol{C}\boldsymbol{g}_i}$ direction, (where $\underline{\boldsymbol{g}_i} = \underline{\nabla f_i(\underline{\boldsymbol{x}}^0)}$) can be made to locate a point $\underline{\boldsymbol{x}}_1 \in R^n$ on the boundary $\overline{f_i}(\underline{\boldsymbol{x}}) = 0$. The direction $\underline{\boldsymbol{C}\boldsymbol{g}_i}$ maximizes the linearization of the function $f_i(\underline{\boldsymbol{x}})$ in the norm defined by the $\underline{\boldsymbol{C}}$ matrix (Rao, 1984). This enables to start the boundary search technique (33) with a point $\underline{\boldsymbol{x}}_1$ near to the required boundary point (Abdel-Malek et al., 1997). Also, for the processing of the search technique (33), a suggested value of γ is given by: $\gamma = \theta\beta_1, \theta \in (0, 1)$ where β_1 is given by (32). This suggested value is a good choice of γ, since from (62) it is clear that the upper bound of the value of γ is greater than 2β. The value of θ used in the following examples is 0.5.

6. Rractical examples

6.1. Design of a two-section transmission line

The given approach of design centering by using the normed distances is applied in the optimal design of a two-section 10 : 1 quarter wave transmission line (Bandler and Chen, 1988). The constraint functions are given by the reflection coefficient sampled at 11 frequencies normalized with respect to 1 GHz: {0.5, 0.6, ..., 1.5}. The magnitude of the reflection coefficient at these frequencies should not exceed 0.55. The characteristic impedences Z_1, Z_2 are the designable parameters. The initial design is given by (1.71, 4.36) which is a feasible point. The designable parameters are assumed to be normally distributed with covariance matrix given by

$$\underline{\boldsymbol{C}} = \frac{1}{4}\begin{bmatrix} 5.6252 & 7.8234 \\ 7.8234 & 16.6957 \end{bmatrix} \tag{69}$$

The design centering process is performed by transforming problem (67) into a nonlinear programming problem (given by (68)), and is solved by the aid of the optimization routines supplied by the IMSL package. A FORTRAN routine which implements the boundary search technique proposed in Section 4 is used. The solution of the design centering problem (67) was (2.396, 4.9911), which is considered as the design center. To evaluate this design center, the initial and the final yield values corresponding to the initial design and the proposed design center are estimated via the Monte Carlo method. The Monte Carlo method is performed using 500 sample points assuming that the circuit parameters are normally

Table 1. Results of the optimal design of the transmission line.

Covariance matrix	Initial yield (%)	Final yield (%)
$\underline{C}$	29.8	33.8
$\underline{C}/4$	42.6	78
$\underline{C}/16$	64.6	99.6

distributed with covariance matrices of different scales of the matrix $\underline{C}$ given by (69). The results are given in Table 1.

6.2. *Design of a CMOS operational amplifier*

A CMOS operational amplifier (Gray and Meyer, 1984; Sapatnekar et al., 1994; Hassan and Rabie, 2000) is optimally designed using the given approach. The operational amplifier is shown in figure 1. The widths w_1, w_5, w_6, w_7, w_8 of the transistors M_1, M_5, M_6, M_7, M_8 in addition to the biasing current I_B and the compensation capacitor C_c are considered as the design parameters. The feasible region is defined by the following constraints

$$\text{Low frequenc gain} \ \geq \ 98 \text{ dB}$$
$$\text{Gain–Bandwidth product} \ \geq \ 17 \text{ Mrad/sec}$$
$$\text{Power dissipation} \ \leq 0.65 \text{ mw}$$
$$\text{Area} = w_1 + w_5 + w_6 + w_7 + w_8 \leq 510 \ \mu\text{m}$$

All the physical and geometrical constants necessary for the calculations are given in Gray and Meyer (1984). The initial values for the circuit parameters are (100 μm, 100 μm, 100 μm, 100 μm, 100 μm, 20 μA, 5 pf). The designable parameters are assumed to

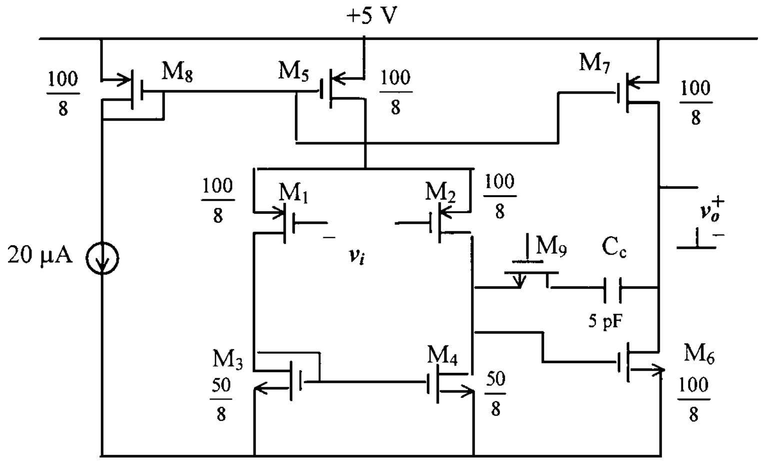

Figure 1. A CMOS operational amplifier.

Table 2. Results of the optimal design of the CMOS operational amplifier.

Covariance matrix	Initial yield (%)	Final yield (%)
$\underline{C}$	16.5	84.8
$\underline{C}/2$	27.8	99.7

be normally distributed with covariance matrix $\underline{\mathbf{C}} = \text{diag}\,(12, 12, 12, 12, 12, 1.5, 1.5)$. The same procedures implemented in the previous example are used here. The solution of the design centering problem (67) was (102.3692 μm, 87.59888 μm, 98.69896 μm, 83.7556 μm, 102.718 μm, 19.13198 μA, 2.71685 pf), which is considered as the design center. To evaluate this design center yield values are estimated for both the initial design and the final design via the Monte Carlo method with 1000 sample points. The results are given in Table 2.

7. Conclusions

In this paper a new method for optimal circuit design is presented. The method treats the optimal design problem through the concept of the normed distances from a feasible point to the feasible region boundaries in a norm related to the probability distribution of the circuit parameters. The evaluation of the normed distances is based on the solution of a nonlinear optimization problem. The new method put the mathematical basis for the solution of this nonlinear optimization problem in addition to derivation of an ordinary explicit formula to evaluate the normed distances. The solution process depends on the location of certain boundary points on the feasible region boundary. These boundary points are located by implementing a convergent boundary search technique. The new method is applied to optimally design practical circuit examples. The results show the effectiveness of this method in optimal design of circuits.

Nomenclature

ϕ	Joint probability density function of the circuit parameters.
$\underline{\boldsymbol{\nu}}$	Vector of distribution parameters.
β	Normed distance.
λ	Largange multiplier parameter.
γ	Constant.
θ	Constant.
$\underline{\eta}$	n-dimensional vector.
$\underline{\boldsymbol{\Delta}}_\lambda$	$(n \times n)$ diagonal matrix.

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