GRAPH COLORING ALGORITHMS FOR FAST EVALUATION OF CURTIS DECOMPOSITIONS

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Abstract
Finding the minimum column multiplicity for a bound set of variables is an important problem in Curtis decomposition. To investigate this problem, we compared two graph-coloring programs: one exact, and another one based on heuristics which can give, however, provably exact results on some types of graphs. These programs were incorporated into the multi-valued decomposer MVGUD. We proved that the exact graph coloring is not necessary for high-quality functional decomposers. Thus we improved by orders of magnitude the speed of the column multiplicity problem, with very little or no sacrifice of decomposition quality. Comparison of our experimental results with competing decomposers shows that for nearly all benchmarks our solutions are best and time is usually not too high.

1. Generalized Curtis Decompositions
The decomposition method formulated by Curtis [10] has been widely used for multi-level realizations of single-output Boolean functions and gives better results than factorization. It found many applications in multi-level FPGA synthesis, VLSI design, Machine Learning and Data Mining. In an innovative approach to Finite State Machine design [7, 19, 16], the design starts from temporal logic constraints, from which a canonical representation of the non-deterministic machine is automatically created in the form: $F = F(x(t), x(t-1),..., x(t-r), y(t-1), y(t-2),..., y(t-v))$ (shown here with single input $x$ and single output $y$ for simplification), where $x(t-1)$ is the value of signal $x$ in the previous clock pulse. The function (relation) $F$ is next hierarchically decomposed [23, 6] using generalized Curtis decomposition. This relation has many inputs, outputs and terms, and is strongly unspecified; the problem is thus very complex.

The slow speed of the algorithms is the reason that the decomposition-based design methods are not yet as popularly used in EDA tools as they deserve, especially for efficient FSM design. Therefore, creating decomposers that are both efficient and effective is important. Two-level Curtis method decomposes function $F$ as follows: $F(X) = H(G(B), A)$ where functions $G$, $H$, $F$ can be all multi-output. Set of input variables $X$ is first partitioned to the set of bound variables $B$ and the set of free variables $A$ (sets $A$ and $B$ can overlap in the so-called "non-disjoint decompositions", then sets $A$ and $B$ are formally a cover of set $X$, we will keep the name partition for uniformity). The set of bound variables is called the bound set and the set of free variables is called the free set. There exist some methods to find set PAR of good partitions of set $X$ to sets $A_i$ and $B_i$ [22], but they are not a subject of this paper. Then each set $(A_i, B_i)$ is tested for decomposition. For some of these subsets $(A_i, B_i)$ there exist decompositions, for some other there are no Curtis decomposition. Some of the existing decompositions are also evaluated as better than others using certain cost functions. When the set $(A, B)$ is found that its corresponding decomposition is evaluated as having the smallest cost for all subsets of PAR, functions $G_j$ and $H_j$ are actually created from $F$. This two-level decomposition process is next recursively applied to new functions $G_i$ and $H_i$, until small functions $G_i$ and $H_i$ are created, that are not further decomposable (such functions are realized by CLBs or standard cells). Thus, the Curtis decomposition is multi-level, and each two-level stage should create the candidates for the next level decompositions, that will be as well decomposable as possible. Currently there exist no provably exact algorithms to find sufficiently good sets of partitions that would guarantee that the best two-level decomposition is not lost. Thus, if the presented below stages of the two-level decomposition were faster, the algorithms that generate larger sets PAR of pairs $(A_i, B_i)$ could be used, thus giving a higher chance of arriving at the minimum cost (i.e., best) decomposition. In Curtis decompositions, the primary goal is to minimize, in a number of steps, the total complexity of the hierarchical multi-level realization of a given function, relation or (non-deterministic) machine. In our case, by the best decomposition we understand one that minimizes the value of DFC. DFC or Decomposed Function Cardinality is the total cost of blocks, where the cost of a (binary) block with $n$ inputs and $m$ outputs is $2^n * m$, [23] (we use DFC because our research is mostly based on decompositions for VLSI layout generation and Machine Learning; most authors are interested in FPGAs and use the total number of Look-Up table blocks as the cost function; for instance, XC3000 CLB of Xilinx has a DFC cost of $2^5 = 32$).
The two-level Curtis decomposition algorithm can be summarized as follows.

**STEP 1.** Find, using algorithms not described here, set PAR of "good pairs" of sets $(A_i, B_i)$ (set PAR is usually large, but still much
mu_opt or during or after coloring it is found that the current number of colors tested (it is obvious that the clique size is the lower bound of the multiplicity index) is (quasi)minimum. The multiplicity index is equal to the number of pairs of vertices from the bound set, they can be combined together (i.e., are compatible) if they constitute an incomplete tautology. A popular algorithm to find the minimum multiplicity index is to find a (quasi)minimum coloring of the incompleteness graph using some graph-coloring algorithm (graph coloring is an NP-complete problem, and even its provable-quality approximation version is NP-complete [13]).

Definition 2 If conditions 1) and 2) for coverings are satisfied and 3) Node “A” has at least one more edge than node “B”, then it is considered that the Curtis decomposition does not exist. If during the creation of the graph it is found that a partial graph already has a clique with more nodes than the previously found value of \( \mu \), then it is considered that the Curtis decomposition does not exist. The multiplicity index \( \mu \) is equal to the number of different colors used and it is not smaller than the chromatic number \( \chi \), \( \mu \geq \chi \) (\( \mu = \chi \) for exact coloring). All columns colored with the same color correspond to a mutually compatible set of cofactors.

If two columns are incompatible, which means if they cannot be combined to one column, there is an edge between the respective nodes. Because columns \( f_x \) and \( f_t \) are cofactors on the bound set, they can be combined together (i.e., are compatible) if they constitute an incomplete tautology \( f_x = f_t \). A popular algorithm to find the minimum multiplicity index is to find a (quasi)minimum coloring of the incompleteness graph using some graph-coloring algorithm (graph coloring is an NP-complete problem, and even its provable-quality approximation version is NP-complete [13]).

If during the creation of the graph it is found that a partial graph already has a clique with more nodes than the previously found value of \( \mu \), then it is considered that the Curtis decomposition does not exist. If during or after coloring it is found that the current number of colors or \( \mu \) exceeds the previous minimum value of the multiplicity index \( \mu_{opt} \), it is discarded, and the next set \( \{A_i, B_i\} \) is tested.

2. A New Algorithm DOM for Graph Coloring based on Domination Covering

Definition 1 A node “A” in an incompleteness graph covers some other node “B” in the graph if all of the following are satisfied: 1) Node “A” and node “B” have no common edge. 2) Node “A” has edges with all the nodes that node “B” has edges with. 3) Node “A” has at least one more edge than node “B”. When two nodes have a covering then both the nodes can be colored with the same color.

Definition 2 If conditions 1) and 2) for coverings are satisfied and node “A” has the same number of edges as node “B”, then it is called a pseudo-covering.

Theorem 1 If any node “A” in a graph covers any other node “B” in the graph, node “B” can be removed from the graph, and in a pseudo-covering any one of the nodes “A” or “B” can be removed.

Definition 3 A complete graph is one in which all pairs of vertices are connected.
In a complete graph, \( \text{total edges} = \frac{\text{nodes} \times (\text{nodes} - 1)}{2} \), where total edges is the sum of all the edges in the graph. In a Complete Graph no coverings or pseudo-coverings can be found and all nodes must have unique colors. Fig. 1d shows a complete graph with 4 nodes.

**Definition 4** A non-reducible graph is a graph that is not complete and has no covered or pseudo-covered node(s).

Graphs from Fig. 1a,b,c are reducible. The graph from Fig. 2a is non-reducible. Graphs from Fig. 2b,c are reducible, and graph from Fig. 2d is complete.

**Theorem 2** If a graph is reducible and can be reduced to a complete graph by successive removing of all its covered and pseudo-covered nodes, then Algorithm DOM finds the coloring with the minimum number of colors (the exact coloring).

Our approach to the Column Minimization Problem in Functional Decomposition is the following: For an arbitrary graph, it is assumed that the graph is reducible and the DOM algorithm is used. If it finds a solution by subsequent reduction and arrives at a complete graph without generating a non-reducible graph, we know that this solution is exact. If a non-reducible graph is generated, we color and remove a randomly selected node. The removal makes the graph reducible - in such case we have no proof of optimality, but still a good coloring is found if only few non-reducible graphs were consecutively converted to reducible graphs by the removal of nodes. Thus, if the characteristics of graphs of some class is that only few non-reducible graphs are created by DOM, this class is well-colorable by DOM. The number of removals is the upper bound on the difference \( \mu - \chi \).

The following explains how DOM colors a reducible graph. [1.] Fig. 1(a) shows an Incompatibility Graph. Nodes 2 and 7 are covered by node 1, so in Fig. 1(b) nodes 2 and 7 are removed and it is remembered that they were covered by node 1. [2.] Next, in Fig. 1(b) node 5 is removed as it is covered by node 4, and it is remembered that node 4 covers node 5. [3.] After removing node 7 the resulting graph shown in Fig. 1(d) is a complete graph. [4.] In Fig. 1(e), each node in the Complete Graph is given a unique color. [5.] In Fig. 1(f) the covered nodes are colored with the same color as the covering node. The color assignments are: Color A \{1, 2, 7\}, Color B \{3\}, Color C \{4, 5\}, Color D \{6\}. Fig. 1(e) shows the completely colored graph. Four colors were used which is the minimum required for this graph. (\( \mu = \chi \), exact solution was found).

Example showing how DOM colors a non-reducible graph. [1.] An incompatibility graph is shown in Fig. 2(a). This graph is not reducible. [2.] As the first step the graph is checked for coverings, but no coverings are found in this graph, so the first node (random) is removed from the graph, which is node 1, and it is assigned a minimum possible color which in this case is color A. [3.] This results in a new graph, shown in Fig. 2(b). In this graph node 4 covers node 2 and node 6. So node 2 and node 6 are removed from the graph, and it is remembered that node 4 covers node 2 and node 6. [4.] On removing node 2 and node 6, in the resulting graph shown in Fig. 2(c) nodes 3 and 5 have a pseudo-covering so the first one of these nodes which is node 3 is removed, and then node 4, 5, and 7 form a complete graph. The complete graph is shown in Fig. 2(d). [5.] Now nodes are colored with the minimum possible color, and each covered node is given the same color as the node which covered it. The coloring is shown in Fig. 2(e). Three colors were used to color the graph, which is the minimum required for this graph. The color assignments are: Color A \{1, 3, 5\}, Color B \{7\}, Color C \{2, 4, 6\}.

In the example like this no proof of exact solution can be given but only few consecutive graphs (here, only the initial graph) were non-reducible, so the solution is of a good quality (here, \( \mu \) differs by not more than one color from \( \chi \)). We showed experimentally elsewhere that DOM departs from the exact minimum for large random graphs, but we will demonstrate experimentally here that it will perform well in most cases on real-life benchmarks from decomposition. One weak point of the algorithm is when no coverings or pseudo-coverings are found at any stage of the coloring, then a node is selected and assigned a minimum possible color. If the coloring of this node is a bad choice, it will result in a solution which is not minimal. But experiments show that such complicated graphs (worst case graphs) will rarely, if ever, occur during the Column Minimization steps of decomposition, which is our application. When \( \chi \) is equal or slightly higher than the size of the maximum clique, which is the case in our graphs, the results are very good.

The strong point of DOM is that it can find the minimum solution without backtracking in all cases when the graph which results after checking for coverings is a complete graph. Thus this program will be effective in finding the minimum solution in all
reducible graphs. Observe that if, for instance, $\mu = 7$ is found and the $\chi = 6$, then still only 3 wires ($3^2 > 7$) are needed for output of block $G$, so the only loss of non-exact coloring is one column of don’t cares, instead of two columns of don’t cares in case of exact coloring with $\mu = \chi$. Let us make a point that in both cases the number of new wires (functions) created is the minimum and the same (here, 3).

### 3. An Evaluation of DOM on the Column Multiplicity Problem

In this section we will evaluate the importance of an Exact Graph Coloring in Curtis Decompositions. Our aim is to investigate if an Exact Graph Coloring is required in Functional Decomposition and if it leads to better results on the graphs that are created from practical function benchmarks. We instantiated three algorithms into MVGUD, a Greedy Clique Partitioning (CLIP), the Dominance Graph Coloring (DOM) and the Exact Graph Coloring (EXOC). The decomposer was run with different numbers of variables in the Bound Set on two kinds of benchmarks: MCNC benchmarks for circuits (presented below), and Machine Learning Benchmarks (from the Wright Labs Database) for data from Machine Learning, Pattern Recognition and Knowledge Discovery in Data Bases.

A comparison of DOM and EXOC was first done on randomly generated graphs, for varying number of nodes and varying percentage of edges (not shown because of a lack of space). Conclusions were reached about how well DOM and EXOC perform on the different kinds of graphs. Tests were done to characterize the kind of graphs that are generated in decomposition with regard to the number of nodes in the graph and the percentage of edges in the graph in order to see if the same conclusions hold for the graphs generated during Functional Decomposition $F = H(G(B,A))$.

Whether the method used by MVGUD to calculate DFC is a good evaluation of the cost of the decomposed multi-valued blocks is not discussed here, but since the DFC is used for a comparison between different methods of calculating the Column Multiplicity in Decomposition, within the same decomposer, the method of calculating the DFC does not matter for the purpose of evaluating algorithms for calculating column multiplicity. What matters is that the same method is used for all the algorithms that are compared. The goal of the testing is to see if an Exact Graph Coloring is necessary to calculate the Column Multiplicity in Functional Decomposition, and if the DFC can be improved in case that MVGUD is run with EXOC, in comparison to when it is run with DOM or with CLIP. MVGUD was tested with 2 - 5, and more variables in the Bound Set (only some results are reported here because of space).

**Notations Used in the Tables.** The following is an explanation of the Notations used in the Tables in this section: **Bench**: Name of the Benchmark function; i: Number of inputs of the Benchmark; o: Number of outputs of the Benchmark; c: Number of cubes in the Benchmark; D: Decomposed function cardinality of the decomposed function; AL: Name of Algorithm used in MVGUD (E=EXOC, D=DOM, C=CLIPS); n: Multi-valued blocks in the decomposed function; NP: Number of passes, or number of times the function to calculate the column multiplicity was called; TC: Total Colors, iterative sum of colors generated for each pass; AC: Average Colors = TC/NP; T(s): User time in seconds.

A comparison was first made by running MVGUD with 2 variables in the Bound Set for EXOC, DOM and CLIP. Comparisons were made with respect to the DFC, the number of two-input gates in the final decomposed function, and the time taken by MVGUD to decompose the function. DOM provides a smaller DFC in 6 cases and CLIP provides a smaller DFC in 5 cases, while EXOC provides a tie with the best solution in 6 of the cases. Hence EXOC does not provide any real improvement, on the other hand it is much slower than CLIP and DOM, which is to be expected. The reason for this...
kind of results is that since in this experiment there are 2 variables in
the Bound Set, in most cases the size of the Incompatibility Graph
at different levels of the decomposition is 4, which is a small graph,
and it is known that for small graphs both CLIP and DOM usually
generate the best solution as well. Hence we conclude that for two
variables in the Bound Set it is not worthwhile having an Exact
Graph Coloring to calculate the Column Multiplicity in MVGUD,
and a good heuristic algorithm to calculate the Column Multiplicity
is sufficient.

Table 1 shows the result for 4 variables in the Bound Set. $Av
E\%$ was calculated to see how dense or sparse the graphs
generated during the decomposition are. This was calculated in
the following way: For any graph with number of nodes = $n$, the
total possible edges for this graph (100% edges) will be equal
to $n \times (n-1)/2$. Hence if the number of edges in the graph is equal
to $e$, then the edge_percent = $(e \times 100)/$total possible edges. This
will give the edge_percent in a graph with $n$ nodes and $e$ edges.
Since the decomposer calls the function to calculate the Column
Multiplicity a number of times, the $Av E\%$ was calculated by adding
the edge_percent for each graph each time the function to find Column
Multiplicity was called, and then dividing this total by the num-
ber of times the function to calculate the Column Multiplicity was
called. EXOC(E), DOM(D) and CLIP (C) generate the same re-
results in all the cases in terms of DFC and number of CLB’s.
The reason for the slow times of MVGUD with EXOC can be explained
as follows: when MVGUD is run with 5 variables (not shown) in
the Bound Set, in most cases the average number of nodes in the
graph is 32 and the edge percentage is always high with the highest
being 77% and the lowest being 46.1%. This means that the graphs
generated during decomposition were nearly always (since this is
an average) dense graphs. It was found experimentally on random
graphs that for dense graphs EXOC takes a long time to find the Ex-
act solution, hence we have such slow times for EXOC. Whenever
DOM does not generate an exact solution, it is usually 1 or 2 colors
away from the Exact solution and rarely more than that, and this
being on randomly generated graphs. Now considering that there
were 5 variables in the Bound Set, then the Incompatibility graph
will have 32 nodes, and for a Curtis decomposition to exist, if a col-
oring of the graph with 16 colors or less is found then one exists.
In a Table for 5 variables in the column for Average colors $AC$ we
would find that the largest average color is 7.65 for the benchmark
sao2. But this means that these graphs generated during decompo-
sition, had low chromatic numbers, which were much less than 16.
So even if DOM or CLIP generate a solution that is 2 or 3 colors
away, the solution will be accepted as a Curtis Decomposition be-
cause it will still be less than 16. The same reasoning applies to
Table 1 where a comparison is made with 4 variables in the Bound
Set. Hence we conclude that for 4 or 5 or greater number of vari-
bles in the Bound Set an Exact Graph Coloring does not produce
better Curtis Decompositions, and having a good heuristic algo-

rithm to find the Column Multiplicity or even a greedy algorithm to
find the Column Multiplicity is good enough.

By the results of the testing we can definitively say that we have
proved that an Exact Graph Coloring is not required to find the
Column Multiplicity where Curtis Decompositions are considered.
Exact Graph Colorings only take up more time and fails to produce
any significant change in the results. This is true with respect to
both Circuit Benchmarks and Machine Learning Benchmarks (not
presented here). Also the results shown raise the question that in
cases where CLIP did not generate the same total numbers of col-
ors as EXOC, why did the DFC not improve when we used EXOC?

The only possible answer to this question is that the decomposi-
tions generated by CLIP were still acceptable decompositions, even
if they use non minimum numbers of colors which in turn means
that these graphs generated during the decomposition process must
be having low chromatic numbers. This provides a very valuable
insight into the kinds of graphs that are generated during the de-
composition process: the graphs generated during the decomposi-
tion process are definitely of a different nature than random graphs.
This is because it is known [24] that 98% of real-life functions is
decomposable; thus their graphs have small $\chi$ (in contrast, only
1% of randomly generated functions are decomposable). The result
that random graphs are much more difficult than graphs originating
from real-life problem is known from other EDA areas [9], but was
not known for functional decomposition.

As found in experiments EXOC was unable to provide a better DFC
for the Machine Learning Benchmarks. In order to see the total
numbers of colors generated by DOM, EXOC and CLIP on the
same graphs, which were generated during the process of Func-
tional Decomposition, the following experiment was performed:
MVGUD was made to run with all three algorithms EXOC, CLIP,
and DOM calculating the Column Multiplicity, and only the results
of one of them was accepted and the results from the other two
was discarded. The count of the colors was kept for all three Algo-
rithms, thus demonstrating how EXOC, CLIP, and DOM compare
with respect to the total number of colors generated on the same
graphs, only now these graphs have been generated from practi-
cal function Benchmarks. Table 2 is a summary of the results of
46 program runs. Table 2 shows how DOM and CLIP compare
with respect to the number of times that the total number of colors
generated by DOM and CLIP are the same as the total number of
colors generated by EXOC, and the number of times the total colors
generated by DOM and CLIP were not exact and by how much.

In Table 2, the row Exact stands for the case when the total
total numbers of colors generated by DOM and CLIP was the same as
the total colors generated by EXOC. Error 1 stands for the case in
which the total numbers of colors generated by DOM and CLIP
were one color away from the total numbers of colors generated
by EXOC, and so on, till Error 6. Corresponding to these rows,
the column Nu gives the number of times, and column % is equal
to Nu/TotalNumber/ProgramRuns x 100. [2] As can be seen from
the Table 2, DOM performs extremely well, and CLIP does
not perform so well. DOM thus proves to be a very good heuristic
algorithm. [3] Table 3 is a total of the rows of Table 2 for DOM
and CLIP.

4. General Comparison of MVGUD with other Decom-
posers and Conclusions

Table 4 shows the result of comparison of MVGUD and other
decomposers on some benchmarks (recall that in contrast to others,
we do not have a fixed number of inputs to a block). Observe that
DFC allows to compare (only approximately) decomposers that de-
compose to various types and sizes of blocks. In Table, all the func-
tions in the table are binary and are taken from the set of MCNC
benchmarks. S1 is the binary decomposer from Freiberg (Stein-
bach). TRADE (MI) is an earlier decomposer developed at Port-
land State University [25]; MISII (MI) at University of California,
can find relatively quickly hierarchical decompositions of smaller complexity. DFC of our solutions is in most cases smaller than that for other decomposers.

Thus, we created a highly efficient and effective decomposer, and, paraphrasing O. Coudert, [9] we showed experimentally that - "Column Minimization for Curtis Decomposition is Easy."