NONLINEAR CONTROL OF A HELICOPTER BASED UNMANNED AERIAL VEHICLE MODEL

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Abstract. In this paper, output tracking control of a helicopter based unmanned aerial vehicle model is investigated. First, based on Newton-Euler equations, a dynamical model is derived by considering the helicopter as a rigid body upon which a set of forces and moments act. Second, we show that the model cannot be converted into a controllable linear system via exact state space linearization. In particular, for certain output functions, exact input-output linearization by state feedback results in unstable zero dynamics. Third, by neglecting weak couplings between forces and moments on the model, we apply input-output linearization to the approximate model for deriving an approximate control with positions and heading as the outputs . Given a bounded output trajectory, the tracking error of the original model by applying the approximate control is proved to be bounded. Finally, we prove by construction that the approximate model with the same outputs is differentially flat and hence the state and input can be expressed as functions of the outputs and their derivatives. This property is very useful for real-time trajectory generation. Based on natural time scale separation, we decompose the dynamics into two subsystems: inner and outer systems. A nonlinear controller, which is based on outer flatness property of the system, is proposed. Simulation results using output tracking controllers based on exact and approximate input-output linearization, and differentail flatness are presented. The controlled system with approximate control and outer flatness based control exhibits small control effort and stable behavior. The response of the controlled system using exact input-output linearization shows unstable pitch and roll dynamics which is shown as the non-minimum phase property of the system.

1. Introduction

Helicopter [24] is versatile in maneuverability and this makes helicopter based unmanned aerial vehicles (UAVs) indispensable for both civilian and military applications where human intervention, especially in restricted areas, is considered difficult or dangerous. Given a multi-agent, multi-objective UAV mission, control system design for a single UAV is a very complicated and challenging task. One natural way to reduce the complexity of system design is by compositional methods. Compositional methods attempt to solve a complex problem by decomposing the problem into a sequence of smaller problems of manageable complexity. In sophisticated flight management systems [10, 14], a single UAV flies from origin to destination while satisfying a large number of aerodynamic, scheduling, and environmental constraints by switching among a finite set of *control modes*, where each control mode essentially corresponds to a different output tracking controller. For example, for regulating at a fix location, *position and heading control mode* is used. In the case of sensor failure such as in the absence of position information from the global positioning system (GPS), *altitude and attitude control mode* is more desired to be used for stabilizing the vehicle since the on-board inertial navigation system (INS) would still be able to provide altitude and attitude information for control. The resulting hierarchical control strategy which involves the interaction of continuous and discrete dynamics can be modeled as a hybrid system [1] for system analysis and controller synthesis [17, 11]. In this paper, output tracking control of a helicopter model is investigated. The helicopter model is based on a UAV [13] being developed by the Berkeley Aerial Robot (BEAR) team at UC Berkeley.

Helicopter control [19, 24] requires the ability to produce moments and forces on the vehicle for two purposes: first, to produce equilibrium and thereby hold the helicopter in a desired trim state; and second, to produce accelerations and thereby change the helicopter's velocity, position and orientation. Like airplane control, helicopter control is accomplished primarily by producing moments about all three aircraft axes: roll, pitch, and yaw. The helicopter has in addition direct control over the vertical force on the aircraft, corresponding to

FIGURE 1. A group of UAVs are flying in formation to perform a searching task

its vertical take-off and landing (VTOL) capability. The engine power is controlled by a rotor speed governor to automatically manage the power. Helicopter flight dynamics are inherently unstable, particularly in hover.

Linearization by state feedback [7] has been successfully applied in control design for highly maneuverable aircraft such as VTOL aircraft [20] and conventional take-off and landing aircraft [32]. In this paper, we design an output tracking controller for a helicopter model based on input-output linearization. Our control design is constructed by first neglecting the coupling effect, then showing that the approximate control results in bounded tracking on the exact model. The idea of using approximate input-output linearization in helicopter control is motivated by the control design of planar VTOL in [5] and VTOL in [30]. In [5], approximate input-output linearization is applied by neglecting the coupling between rolling moment and lateral acceleration. A state transformation technique is used on constructing an output tracking control in [30]. In [29, 6], helicopter control design based on μ-synthesis and fuzzy logic are studied.

Differentially flat systems are systems in which all states and inputs can be expressed as functions of the outputs and their derivatives [33, 4]. They have the useful property that there is a one-to-one mapping between trajectories in output space and trajectories in state and input space. Instead of incorporating the dynamical equations as constraints, trajectory generation for differentially flat system can be computed by considering algebraic functions of the outputs and their derivatives as constraints. Thus, the complexity of computation can be reduced and the efficiency for computation can be enhanced. Differential flatness has been applied to approximate models of aircraft [18, 33] for trajectory generation. Trajectory plays a significant role in determining the performance of a closed-loop system especially under saturation. For details related to trajectory generation under the effect of control saturation, please refer to [8, 23].

In this paper, we first derive a helicopter dynamical model which is derived by considering the helicopter as a rigid body upon which a set of forces and moments act. In section 3, we prove that the model cannot be converted into a controllable linear system via exact state space linearization. In particular, for certain output functions, exact input-output linearization results in unstable zero dynamics. In section 4, by neglecting weak couplings between forces and moments on the model, we apply input-output linearization to the approximate model for deriving an approximate control with positions and heading as the outputs . Given a bounded output trajectory, the tracking error of the original model by applying the approximate control is proved to be bounded. Then, in section 5, we prove by construction that the approximate model with the same outputs is differentially flat and hence the state and input can be expressed as functions of the outputs and their derivatives. Based on geometric control theory, we decompose the dynamics into two subsystems: an inner and outer system. A nonlinear controller is proposed based on differential flatness of the outer system.

Finally, in section 6, simulation results using both output tracking controllers based on exact and approximate input-output linearization are presented for comparison. We conclude our work in section 7.

2. Helicopter Model

A model of a helicopter can be divided into four different subsystems, which are actuator dynamics, rotary wing dynamics, force and moment generation processes, and rigid body dynamics. The connections between subsystems, and state and control variables are defined in Figure 2. In this paper, due to the fact that the complete dynamics of a helicopter, taking into account flexibility of the rotors and fuselage, the dynamics of the engine and actuators is quite complex and somewhat unmanageable for the purpose of control, we consider a helicopter model as a rigid body incorporating with a force and moment generation process. For illustration, we use model data obtained from a model helicopter on which we will apply the proposed control law in real flight. However, the result is also applicable to other helicopters with similar force and moment generation processes.

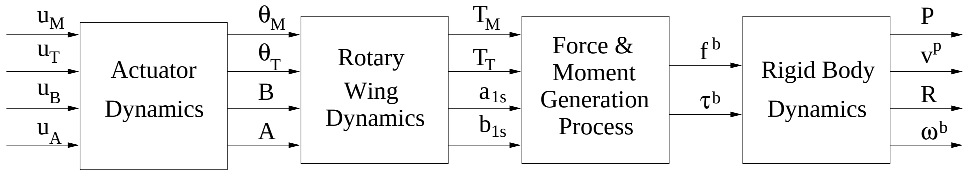

Figure 2. Helicopter dynamics

By regarding the helicopter as a rigid body as in [24], the equations of motion of a model helicopter can be derived by applying Newton-Euler equation.

2.1. **Rigid Body Dynamics.** Consider the helicopter depicted in Figure 3. The equations of motion for a model helicopter can be written with respect to the body coordinate frame, which is attached towards the center of mass of the model helicopter. The x axis is pointed to the body head and y axis goes to the right of the body. As shown in [21], the equations of motion for a rigid body subject to body force $f^b \in \mathbb{R}^3$ and torque $\tau^b \in \mathbb{R}^3$ applied at the center of mass and specified with respect to the body coordinate frame is given by the *Newton-Euler* equation in body coordinate, which can be written as

$$\begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} \dot{v}^b \\ \dot{\omega}^b \end{bmatrix} + \begin{bmatrix} \omega^b \times mv^b \\ \omega^b \times \mathcal{I}\omega^b \end{bmatrix} = \begin{bmatrix} f^b \\ \tau^b \end{bmatrix} \tag{2.1}$$

where $v^b \in \mathbb{R}^3$ is the body velocity vector, $\omega^b \in \mathbb{R}^3$ is the body angular velocity vector, $m \in \mathbb{R}$ specifies the mass, $I \in \mathbb{R}^{3\times 3}$ is an identity matrix, and $\mathcal{I} \in \mathbb{R}^{3\times 3}$ is an inertial matrix.

The position and velocity of the helicopter center of gravity are given by $P \in \mathbb{R}^3$ and $v^p = \dot{P} \in \mathbb{R}^3$, respectively, expressed to the spatial frame in North-East-Down orientation. Let $R \in SO(3)$ be the rotation matrix of the body axes relative to the spatial axes and $\omega^b \in \mathbb{R}^3$ be the body angular velocity vector. Given $e = [e_1\ e_2\ e_3]^T \in \mathbb{R}^3$, we define $\hat{e} \in so(3)$, the space of skew-symmetric matrices in $\mathbb{R}^{3\times 3}$, by

$$\hat{e} = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$$

Differentiating the orthogonality constraint $R^T R = I$, $\hat{\omega}^b = R^T \dot{R}$ as shown in [21]. We parameterize R by ZYX Euler angles with ϕ, θ and ψ about the x, y, z axes respectively.

$$\begin{aligned} R(\Theta) &= \exp(\hat{z}\psi)\exp(\hat{y}\theta)\exp(\hat{x}\phi) \\ &= \begin{bmatrix} c\theta c\psi & s\phi s\theta c\psi - c\phi s\psi & c\phi s\theta c\psi + s\theta s\psi \\ c\theta s\psi & s\phi s\theta s\psi + c\phi c\psi & c\phi s\theta s\psi - s\phi c\psi \\ -s\theta & s\phi c\theta & c\phi c\theta \end{bmatrix} \end{aligned} \tag{2.2}$$

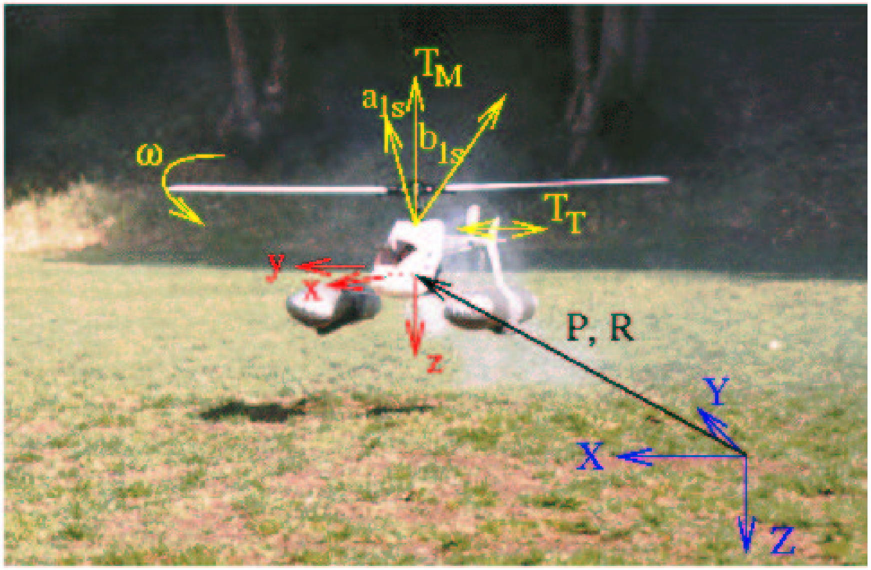

FIGURE 3. Coordinate frames for specifying rigid motions and forces acting on a helicopter.

where $x = [1\ 0\ 0]^T$, $y = [0\ 1\ 0]^T$, $z = [0\ 0\ 1]^T$ and $c\theta$, $s\theta$ are abbreviations for $\cos(\theta)$ and $\sin(\theta)$ respectively, and similarly for the other terms. By differentiating R with respect to time, we have the state equations of the Euler angles, $\Theta = [\phi\ \theta\ \psi]^T$, which are

$$\begin{aligned}\dot{\Theta} &= \Psi(\Theta)\omega^b \\ &= \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix} \omega^b\end{aligned}$$

where $t\theta$ is an abbreviation for $\tan(\theta)$. In the ZYX Euler angle parameterization of rotation matrix, there are singularities at $\theta = \pm\pi/2$. For the following discussion, we assume that the trajectory of helicopter does not pass through the singularities. If the trajectory is required to pass through the singularities, we can simply switch to another chart parameterizing the rotation matrix. By using the fact that $v^p = Rv^b$, we can rewrite the motion equations of a rigid body as

$$\begin{bmatrix} \dot{P} \\ \dot{v}^p \\ \dot{\Theta} \\ \dot{\omega}^b \end{bmatrix} = \begin{bmatrix} v^p \\ \frac{1}{m}R(\Theta)f^b \\ \Psi(\Theta)\omega^b \\ \mathcal{I}^{-1}(\tau^b - \omega^b \times \mathcal{I}\omega^b) \end{bmatrix} \tag{2.3}$$

2.2. **Force and Moment Generation Processes.** The helicopter system can be considered as a lumped model consisting of a main rotor, a tail rotor, a horizontal stabilizer, a vertical stabilizer and a fuselage, which are denoted with subscripts M, T, H, V, F, respectively.

In the following, we express the external wrench, a force/moment pair, exerted on the helicopter. The force experienced by the helicopter is the resultant force of the thrust generated by the main and tail rotors, damping forces from the horizontal and vertical stabilizer, aerodynamic force due to fuselage, and gravitational force. The torque is composed of the torques generated by the main rotor, tail rotor and fuselage, and moment generated by the forces as defined in Figure 4.

Assume that the helicopter is operated at low speeds and hence the drag contributed from the horizontal, vertical stabilizers, and the fuselage can be ignored. Therefore, the terms with subscripts H, V and F are discarded. As shown in [16], the external wrench can be written as:

$$
\begin{aligned}
f^b &= \begin{bmatrix} X_M \\ Y_M + Y_T \\ Z_M \end{bmatrix} + R^T(\Theta) \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \\
\tau^b &= \begin{bmatrix} R_M \\ M_M + M_T \\ N_M \end{bmatrix} + \begin{bmatrix} Y_M h_M + Z_M y_M + Y_T h_T \\ -X_M h_M + Z_M l_M \\ -Y_M l_M - Y_T l_T \end{bmatrix}
\end{aligned}
$$

The forces and torques generated by the main rotor are controlled by T_M, a_{1s} and b_{1s}, in which T_M is the main rotor thrust, a_{1s} and b_{1s} are the longitudinal and lateral tilts of the tip path plane of the main rotor with respect to the shaft, respectively. The tail rotor is considered as a source of pure lateral force Y_T and anti-torque Q_T, which are controlled by T_T, tail rotor thrust. The forces and torques can be expressed as

$$X_M = -T_M \sin a_{1s} \tag{2.4}$$

$$Y_M = T_M \sin b_{1s} \tag{2.5}$$

$$Z_M = -T_M \cos a_{1s} \cos b_{1s} \tag{2.6}$$

$$Y_T = -T_T \tag{2.7}$$

$$R_M \simeq \frac{\partial R_M}{\partial b_{1s}} b_{1s} - Q_M \sin a_{1s} \tag{2.8}$$

$$M_M \simeq \frac{\partial M_M}{\partial a_{1s}} a_{1s} + Q_M \sin b_{1s} \tag{2.9}$$

$$N_M \simeq -Q_M \cos a_{1s} \cos b_{1s} \tag{2.10}$$

$$M_T = -Q_T \tag{2.11}$$

The moments generated by the main and tail rotor can be calculated by using the constants, $\{l_M, y_M, h_M, h_T, l_T\}$, as defined in Figure 4. In above, we approximate the rotor torque equations by $Q_i \simeq C_i^Q T_i^{1.5} + D_i^Q$ for $i = M, T$. The details of generation of the rotor torques, Q_M, Q_T, one can obtain by applying the equations as shown in [16]. The system parameters are given in Appendix A.1.

2.3. **System equations.** We assume that all the states can be measured accurately. In order to present the system in an input-affine form, we assume that the inputs of the about nonlinear system are the derivatives of T_M, T_T, a_{1s} and b_{1s}. Define $P = [p_x\ p_y\ p_z]^T$, $v^p = [v_x^p\ v_y^p\ v_z^p]^T$, and $\omega^b = [\omega_x^b\ \omega_y^b\ \omega_z^b]^T$. The system equations can be rewritten as

$$\dot{x} = f(x) + \sum_{i=1}^{4} g_i w_i \tag{2.12}$$

where $f : \mathbb{R}^{16} \to \mathbb{R}^{16}$ and $g_i \in \mathbb{R}^{16}$ can be expressed as

$$
f(x) = \begin{bmatrix} v^p \\ \frac{1}{m} R(\Theta) f^b \\ \Psi(\Theta)\omega^b \\ \mathcal{I}^{-1}(\tau^b - \omega^b \times \mathcal{I}\omega^b) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; g_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; g_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; g_3 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; g_4 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
$$

In above, $x = [p_x\ p_y\ p_z\ v_x^p\ v_y^p\ v_z^p\ \phi\ \theta\ \psi\ \omega_x^b\ \omega_y^b\ \omega_z^b\ T_M\ T_T\ a_{1s}\ b_{1s}]^T \in X \subset \mathbb{R}^{16}$ and $w = [w_1, w_2, w_3, w_4] \in \mathbb{R}^4$ are defined as the state vector, with dimension $n = 16$, and auxiliary input vector, with dimension $m = 4$, respectively. It can be easily seen that $f(x)$, g_i are smooth vector fields.

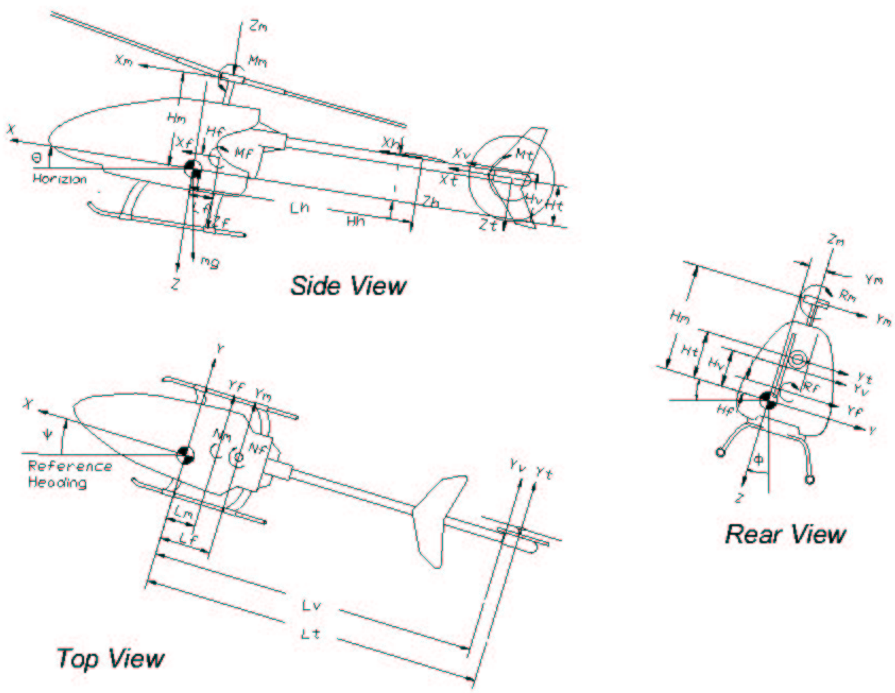

FIGURE 4. The free-body diagram of a helicopter in flight. (Figure courtesy of D. H. Shim)

As defined in [24], a helicopter is said to be in trim if all the forces, aerodynamic and gravitational, and aerodynamic moments acting on the helicopter about the center of gravity are in balance. Hence, by solving the nonlinear equations of body wrench, which are $f^b = 0$ and $\tau^b = 0$, one can solve for the system trim condition. The trim condition is irrespective of the values of P, v^p, ψ. In the following, we define x_0 in trim condition, in which $p_x = 0$, $p_y = 0$, $p_z = 0$, $v_x^p = 0$, $v_y^p = 0$, $v_z^p = 0$, $\psi = 0$, $T_M = 47.97$, $T_T = 2.42$, $a_{1s} = -0.018$, $b_{1s} = 0.0061$, $\phi = 0.044$, $\theta = 0.018$, and hence $\omega^b = 0$ and $w_i = 0$ for $i = 1, \ldots, 4$.

3. EXACT LINEARIZATION BY STATE FEEDBACK

In this section, we show that the model cannot be converted into a controllable linear system via exact state space linearization. In addition, for certain output functions, exact input-output linearization results in unstable zero dynamics.

Consider the square system (*i.e.*, system with as many inputs as outputs) multi-input multi-output (MIMO) nonlinear control system described by

$$\sum : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + gw \\ y & = & h(x) \end{array} \right. \tag{3.1}$$

with $g = [g_1 \; g_2 \; g_3 \; g_4]$ and $y_j = h_j(x) \; for \; j = 1, \ldots, 4$. $h_1(x), \cdots, h_4(x)$ are smooth functions on $\mathbb{R}^{16}$. Define strict relative degree γ_j at $x_0 \in X$ with respect to output y_j as an integer such that

$$\left\{ \begin{array}{ll} L_{g_i} L_f^l h_j(x) \equiv 0 & \forall x \in X, 0 \leq l \leq \gamma_j - 2, \; \forall i \in \{1, \ldots, m\} \\ L_{g_i} L_f^{\gamma_j - 1} h_j(x_0) \neq 0 & \end{array} \right.$$

Collecting these calculation, we have

$$\begin{aligned}
\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} &= \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_m} h_m \end{bmatrix} + \begin{bmatrix} L_{g_1} L_f^{\gamma_1 -1} h_1 & \cdots & L_{g_m} L_f^{\gamma_1 -1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{\gamma_m -1} h_m & \cdots & L_{g_m} L_f^{\gamma_m -1} h_m \end{bmatrix} w \\
&:= b(x) + A(x) w
\end{aligned} \tag{3.2}$$

where $A(x)$ is called the decoupling matrix. If $A(x)$ is invertible at every point in X, then the static-state feedback given by $w = (A(x))^{-1}[-b(x) + v]$ will result in a closed-loop system that is decoupled from input v to output y. This decoupled and input-output linearized system is given by

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \tag{3.3}$$

If the matrix $A(x)$ is singular, we cannot use a static state feedback to linearized the nonlinear system, and we have to look for a dynamic state feedback to achieve linearization by state feedback.

Definition 3.1. (Vector Relative Degree [7].) The system (3.1) is said to have vector relative degree $\{\gamma_1, \ldots, \gamma_m\}$ at $x_0 \in X$ if

$$\begin{cases} L_{g_i} L_f^l h_j(x) \equiv 0 & \forall x \in X, 0 \le l \le \gamma_j - 2, \ \forall i \in [1, m] \\ L_{g_i} L_f^{\gamma_j -1} h_j(x_0) \ne 0 & \end{cases}$$

for $j = 1, \ldots, m$ and the matrix $A(x_0)$ is nonsingular.

Define the distributions,

$$\begin{aligned}
G_0 &= \operatorname{span}\{g_1, \ldots, g_m\} \\
G_1 &= \operatorname{span}\{g_1, \ldots, g_m, ad_f g_1, \ldots, ad_f g_m\} \\
&\vdots \\
G_i &= \operatorname{span}\{ad_f^k g_j : 0 \le k \le i, 1 \le j \le 4\}
\end{aligned}$$

for $i = 1, \ldots, n-1$. The conditions under which there exist outputs $h_1, \ldots, h_m$ such that the MIMO system has vector relative degree and furthermore is such that $\gamma_1 + \cdots + \gamma_m = n$ are given as the following lemma:

Lemma 3.2. (Full State MIMO Linearization [7].) *Suppose the matrix $g(x_0)$ has rank m. Then, there exist m functions $\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$ such that the system*

$$\begin{aligned}
\dot{x} &= f(x) + g(x) w, \\
y &= \lambda(x),
\end{aligned}$$

has vector relative degree $\{\gamma_1, \ldots, \gamma_m\}$ with

$$\gamma_1 + \cdots + \gamma_m = n$$

iff

(1) *For each $0 \le i \le n-1$ the distribution G_i has constant dimension in a neighborhood X of x_0.*
(2) *The distribution G_{n-1} has dimension n.*
(3) *For each $0 \le i \le n-2$ the distribution G_i is involutive.*

By applying the above lemma, we have the following result regarding full state MIMO linearization of the system equations (2.12).

Theorem 3.3. *Consider the system equations (2.12). There do not exist any* m *functions* $\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$ *defined on* X*, such that the system has vector relative degree* $\{\gamma_1, \ldots, \gamma_m\}$ *at* x_0 *with* $\sum_{k=1}^{m} \gamma_k = n$
Proof: *It can be computed that the distributions have constant dimensions near* x_0*, which are* $\{4, 8, 12, 14, 16, \cdots, 16\}$*. The distribution* G_{15} *has dimension 16. However, the distribution* G_2 *and* G_3 *fail to be involutive. By applying Lemma 3.2, we complete the proof.*

Thus, the above theorem suggests that it is impossible to find a set of outputs such that the model can be converted into a controllable linear system via exact state space linearization.

However, if the MIMO system has relative degree $\gamma_1 + \cdots + \gamma_m < n$, then we can write a normal form [25] for the equations. In particular, consider the following set of outputs

$$\{p_x,\ p_y,\ p_z,\ \phi,\ \theta,\ \psi\}, \tag{3.4}$$

which can be used to form various control mode and obtained from sensors such as GPS and INS. For the square system, there are $C_4^6 = 15$ possible input-output pairs. We define $k_1, \cdots, k_4 \in \{1, \ldots, 6\}$ as the indices of the output functions selected from the combinations. To perform exact input-output linearization, we pick the j^{th} output y_j of the system equation and differentiate it with respect to time. For all $j = 1, \ldots, 6$, one can check that one has to differentiate every of the outputs 3 times before encountering one of the inputs, *i.e.,*

$$y_j^{(3)} = L_j^3 h_j + \sum_{i=1}^{4} L_{g_i} L_f^2 h_j\ w_i. \tag{3.5}$$

Given $k_1, \cdots, k_4$, the input-output system can be written as

$$\begin{bmatrix} y_{k_1}^{(3)} \\ \vdots \\ y_{k_4}^{(3)} \end{bmatrix} = \begin{bmatrix} L_f^3 h_{k_1} \\ \vdots \\ L_f^3 h_{k_4} \end{bmatrix} + A_{k_1 k_2 k_3 k_4}(x) \begin{bmatrix} w_1 \\ \vdots \\ w_4 \end{bmatrix} \tag{3.6}$$

where the decoupling matrix is defined by

$$A_{k_1 k_2 k_3 k_4}(x) := \begin{bmatrix} L_{g_1} L_f^2 h_{k_1} & \cdots & L_{g_4} L_f^2 h_{k_1} \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^2 h_{k_4} & \cdots & L_{g_4} L_f^2 h_{k_4} \end{bmatrix} \tag{3.7}$$

Then, we have the following proposition to show that the decoupling matrix for each possible combination is nonsingular for all $x \in X$.

Proposition 3.4. *Given system equations (2.12) and any* m *output functions of (3.4), the decoupling matrix* $A_{k_1 k_2 k_3 k_4}(x)$ *of the input-output system has full rank for all* $x \in X$ *for all possible combinations of* $k_1, \cdots, k_4 \in \{1, \ldots, 6\}$*.*
Proof: *See Appendix A.2.*

Since $A_{k_1 k_2 k_3 k_4}(x)$ has full rank and $L_{g_i} h_j \equiv 0$ and $L_{g_i} L_f h_j \equiv 0$ for all $x \in X$, by the definition, the system has vector relative degree $\{3, 3, 3, 3\}$ for all $x \in X$. It follows that $A_{k_1 k_2 k_3 k_4}(x)$ is nonsingular and $A_{k_1 k_2 k_3 k_4}^{-1}(x)$ exists for all $x \in X$. Thus, the state feedback control law

$$\begin{bmatrix} w_1 \\ \vdots \\ w_4 \end{bmatrix} = A_{k_1 k_2 k_3 k_4}^{-1}(x) \begin{bmatrix} -L_f^3 h_{k_1} + v_1 \\ \vdots \\ -L_f^3 h_{k_4} + v_4 \end{bmatrix} \tag{3.8}$$

yields the linear closed loop system

$$\begin{bmatrix} y_{k_1}^{(3)} \\ \vdots \\ y_{k_4}^{(3)} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_4 \end{bmatrix} \tag{3.9}$$

This feedback law makes the input-output map linear, but has the unfortunate side-effect of making some dynamics unobservable. In order to guarantee the internal stability of the system, it is not sufficient to look at input-output stability, we must also show that all internal (unobservable) modes of the system are stable as well.

If a system has relative degree $\gamma = \gamma_1 + \cdots + \gamma_p < n$, then we can write a normal form for the equations (3.1) by choosing as coordinates

$$\begin{array}{llll} \xi_1^1 = h_1(x), & \xi_2^1 = L_f h_1(x), & \ldots, & \xi_{\gamma_1}^1 = L_f^{\gamma_1 - 1} h_1(x), \\ \xi_1^2 = h_2(x), & \xi_2^2 = L_f h_2(x), & \ldots, & \xi_{\gamma_2}^2 = L_f^{\gamma_2 - 1} h_2(x), \\ \vdots & & & \\ \xi_1^m = h_m(x), & \xi_2^m = L_f h_m(x), & \ldots, & \xi_{\gamma_m}^m = L_f^{\gamma_m - 1} h_m(x). \end{array} \tag{3.10}$$

As shown in [25], these ξ_i^j qualify as a partial set of coordinates. Furthermore, one can complete the basis by choosing $n-\gamma$ more functions $\eta_1(x), \eta_2(x), \ldots, \eta_{n-\gamma}(x)$. Define $\xi = [\xi_1^1, \ldots, \xi_{\gamma_1}^1, \xi_1^2, \ldots, \xi_{\gamma_2}^2, \ldots, \xi_1^m, \ldots, \xi_{\gamma_m}^m]^T$ and $\eta = [\eta_1, \eta_2, \ldots, \eta_{n-\gamma}]^T$. In these ξ, η coordinates the system equations (3.1) have the following normal form

$$\begin{array}{llll} y_1 = \xi_1^1, & y_2 = \xi_1^2, & \ldots, & y_m = \xi_1^m, \\ \dot{\xi}_1^1 = \xi_2^1, & \dot{\xi}_1^2 = \xi_2^2, & \ldots, & \dot{\xi}_1^m = \xi_2^m, \\ \vdots & \vdots & \vdots & \vdots \\ \dot{\xi}_{\gamma_1 - 1}^1 = \xi_{\gamma_1}^1, & \dot{\xi}_{\gamma_2 - 1}^2 = \xi_{\gamma_2}^2, & \ldots, & \dot{\xi}_{\gamma_m - 1}^m = \xi_{\gamma_m}^m, \end{array}$$
$$\begin{bmatrix} \dot{\xi}_{\gamma_1}^1 \\ \dot{\xi}_{\gamma_2}^2 \\ \vdots \\ \dot{\xi}_{\gamma_m}^m, \end{bmatrix} = b(\Phi(\xi, \eta)) + A(\Phi(\xi, \eta))w, \tag{3.11}$$
$$\dot{\eta} = q(\xi, \eta) + p(\xi, \eta)w$$

where $\Phi : x \mapsto (\xi, \eta)$ is the diffeomorphism mapping from x into the normal form coordinates. Note that $p \in \mathbb{R}^{n-\gamma \times m}$, $q \in \mathbb{R}^{n-\gamma}$.

In order to analyze the internal stability of the system, zero dynamics of the system should be examined. The zero dynamics of a nonlinear system are the internal dynamics of the system subject to the constraint that the outputs and all derivatives of the outputs are set to zero for all time, *i.e.,* $\xi \equiv 0$. This can be done by using the control law with $v_1 = \cdots = v_4 = 0$ and initializing the system with the trim condition $(\xi_0, \eta_0) = \Phi^{-1}(x_0)$. Hence, the zero dynamics can be written as

$$\dot{\eta} = q(0, \eta) + p(0, \eta) A(\Phi(0, \eta))^{-1} b(\Phi(0, \eta)).$$

For illustration, two modes of operation will be explained.

In **position and heading control mode**, $\{p_x, p_y, p_z, \psi\}$ are chosen as outputs, one can easily verify that the zero dynamics may be parameterized by $\eta = [\phi, \omega_x^b, \theta, \omega_y^b]^T$, and the linearized zero dynamics at equilibrium point has eigenvalues $\pm 16.4528i$ and $\pm 12.1123i$. The linearization is inconclusive.

We define a nonsingular matrix T which transforms the state transition matrix of the linearized zero dynamics into block diagonal form. By applying $\eta^* = T^{-1}\eta$, the nonlinear dynamics are locally decoupled. From the phase portrait as shown in Figure 5, one can conclude that the nonlinear system is not asymptotically stable, since the equilibrium point is surrounded by a family of periodic orbits. In position and heading control mode, the helicopter system is non-minimum phase.

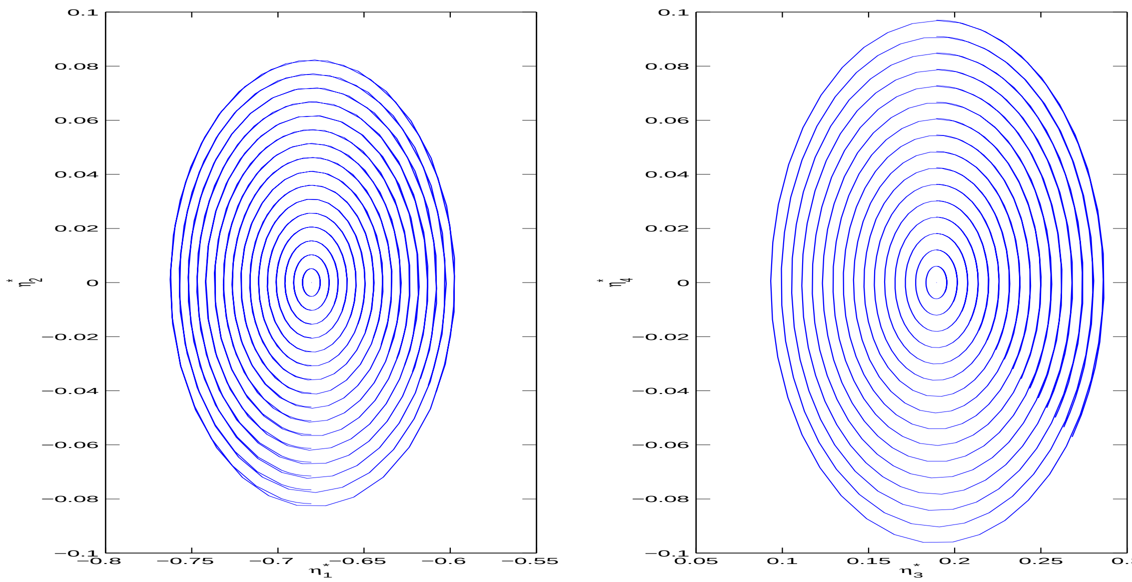

FIGURE 5. Phase portrait of zero dynamics in position and heading control mode

In **attitude and altitude control mode**, $\{\phi, \theta, \psi, p_z\}$ are chosen as outputs. It can be easily verified that the zero dynamics becomes

$$\begin{aligned} \ddot{p}_x &= 0.1757 \\ \ddot{p}_y &= -0.4335, \end{aligned}$$

where the states p_x,p_y and their first derivatives become unobservable. The zero dynamics is unstable and hence the system is non-minimum phase. In this mode, the helicopter is constantly drifting in the $X - Y$ plane while attitude and altitude are held constantly.

4. Approximate Linearization by State Feedback

In this section, approximate linearization technique is applied to the helicopter system. First, the system equations (2.3) is rewritten as

$$\ddot{P} = \frac{1}{m} R(\Theta) \begin{bmatrix} -T_M \sin a_{1s} \\ T_M \sin b_{1s} - T_T \\ -T_M \cos a_{1s} \cos b_{1s} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \tag{4.1}$$

$$\dot{\Theta} = \Psi(\Theta)\omega^b \tag{4.2}$$

$$\dot{\omega} = \mathcal{I}^{-1}(\tau^b - \omega^b \times \mathcal{I}\omega^b). \tag{4.3}$$

The reason of the failure of exact linearization is due to the existence of couplings between rolling (pitching) moments and lateral (longitudinal) acceleration. Those couplings are introduced due to the presence of a_{1s}, b_{1s} and T_T. As shown before, if position and heading control mode is chosen the internal dynamics are not regulated and exhibit an unstable behavior.

Here, we propose to approximately linearize the system by neglecting the coupling terms. This can be done by assuming that $a_{1s}, b_{1s}, T_T/T_M$ are small. Therefore, equation (4.1) can be modeled as follows

$$\ddot{P}_m = \frac{1}{m} R \begin{bmatrix} 0 \\ 0 \\ -T_M \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \tag{4.4}$$

while keeping equations (4.2) and (4.3) the same. In what follows, we demonstrate the idea of approximate linearization in position and heading control mode.

We differentiate outputs $P_m = [p_{xm}\ p_{ym}\ p_{zm}]^T$ on (4.4) and ψ on (4.2) until at least one input appears in each output equation, and we get the final equations in the form:

$$\begin{bmatrix} p_{xm}^{(3)} \\ p_{ym}^{(3)} \\ p_{zm}^{(3)} \\ \psi^{(3)} \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} + \begin{bmatrix} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & * & * \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

Since the decoupling matrix has rank 2, decoupling of the system cannot be achieved by static state feedback. Here, we propose to use dynamic decoupling[2] . Following the algorithm, we have to add an integrator to w_1, since column one in the decoupling matrix has more than one nonzero elements. Hence, the first three rows in the decoupling matrix are all zeros, and we have to differentiate the first three outputs again. Then, we can see that the decoupling matrix has the same form as in previous iteration. By following the algorithm, it requires to add another integrator to w_1, and continue differentiating the first three outputs. And, the iteration ends, since the decoupling matrix finally has full rank. The extended system is in the following form:

$$\begin{bmatrix} p_{xm}^{(5)} \\ p_{ym}^{(5)} \\ p_{zm}^{(5)} \\ \psi^{(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} b^P \\ \\ b^\psi \end{bmatrix}}_{b_e} + \underbrace{\begin{bmatrix} A^P \\ \\ A^\psi \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} \ddot{w}_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}}_{u}$$

of which the vector relative degree is $\{5, 5, 5, 3\}$.

We can rewrite the true system in normal form (ξ, η) of modeled system. Define $\xi_1^1 = p_{xm},\ \xi_1^2 = p_{ym},\ \xi_1^3 = p_{zm},\ \xi_1^4 = \psi$, $\xi_1^P = [\xi_1^1\ \xi_1^2\ \xi_1^3]^T$, $\xi_1^\psi = \xi_1^4$ and we have

$$\left.\begin{array}{rcl} \dot{\xi}_1^P & = & \xi_2^P \\ \dot{\xi}_2^P & = & \xi_3^P + h(x) \\ \dot{\xi}_3^P & = & \xi_4^P \\ \dot{\xi}_4^P & = & \xi_5^P \\ \dot{\xi}_5^P & = & b^P + A^P u \\ & & \\ \dot{\xi}_1^\psi & = & \xi_2^\psi \\ \dot{\xi}_2^\psi & = & \xi_3^\psi \\ \dot{\xi}_3^\psi & = & b^\psi + A^\psi u \end{array}\right\} \tag{4.5}$$

in which

$$h(x) = \frac{1}{m} R(\Theta) T_M \begin{bmatrix} -\sin a_{1s} \\ \sin b_{1s} - T_T/T_M \\ \cos a_{1s} \cos b_{1s} - 1 \end{bmatrix}. \tag{4.6}$$

In above, the linearized model system does not contain any unobservable (zero) dynamics and hence is minimum phase, since the sum of Kronecker indices $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\sum_{j=1}^4 \gamma_j = 18$ is equal to the order of the extended system $n_e = 16 + 2$. As defined in [5], the system is said to be slightly non-minimum phase, since the true system is non-minimum phase but the approximate system is minimum phase.

We can then apply the tracking control law designed for the model system to the true system, which is

$$\begin{aligned} u &= -A_e^{-1} b_e \\ &+ A_e^{-1} \begin{bmatrix} y_{d1}^{(\gamma_1+1)} - \alpha_1^1 e_1^{(\gamma_1)} - \cdots - \alpha_{\gamma_1+1}^1 e_1 \\ \vdots \\ y_{d4}^{(\gamma_4+1)} - \alpha_1^4 e_4^{(\gamma_4)} - \cdots - \alpha_{\gamma_4+1}^4 e_4 \end{bmatrix} \end{aligned}$$

where $e_i^{(j)} = \xi_{j+1}^i - y_{di}^{(j)}$ for $j = 0, \ldots, \gamma_i$ and $i = 1, \ldots, 4$ and the polynomials

$$s^{\gamma_i+1} + \alpha_1^i s^{\gamma_i} + \cdots + \alpha_{\gamma_i+1}^i \tag{4.8}$$

chosen Hurwitz. The following theorem provides a bound for the performance of this control when applied to the true system.

Theorem 4.1. *Given that the desired trajectory and its first $\gamma_i - 1$ derivatives are bounded, the states of the system (4.5) are bounded and the tracking errors satisfy*

$$\begin{aligned} \|e^1\| &= \|\xi_1^1 - y_{d1}\| \le k\epsilon \\ &\vdots \\ \|e^4\| &= \|\xi_1^4 - y_{d4}\| \le k\epsilon \end{aligned} \tag{4.9}$$

where $\epsilon = \max(|a_{1s}|, |b_{1s}|, |T_T/T_M|)$ and k is bounded.

Proof: For the i^{th} component of h, it can be easily shown that

$$\begin{aligned} \|h_i\| &\le \frac{T_M}{m} \|r_i\| \left\| \begin{bmatrix} -\sin a_{1s} \\ \sin b_{1s} - T_T/T_M \\ \cos a_{1s} \cos b_{1s} - 1 \end{bmatrix} \right\| \\ &\le \underbrace{\left| \frac{T_M}{m} \right| \sqrt{14}}_{K} \epsilon \end{aligned}$$

where r_i is the i^{th} row of R and the following facts are used: $|\sin x| \le |x|, \cos x \le |1 - x|$ for $x \in (-\pi/2, \pi/2]$. For illustration of bounded tracking, we consider ξ_1^1 only in the following proof, and the result applies to $\xi_1^2, \xi_1^3, \xi_1^4$ as well. Define an error vector as

$$e^1 = \begin{bmatrix} \xi_1^1 \\ \vdots \\ \xi_5^1 \end{bmatrix} - \begin{bmatrix} y_{d1} \\ \vdots \\ y_{d1}^4 \end{bmatrix} \tag{4.10}$$

Then the closed loop system can be expressed as

$$\begin{bmatrix} \dot{e}_1^1 \\ \dot{e}_2^1 \\ \dot{e}_3^1 \\ \dot{e}_4^1 \\ \dot{e}_5^1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -\alpha_0^1 & -\alpha_1^1 & \cdots & -\alpha_4^1 \end{bmatrix}}_{A_1} \begin{bmatrix} e_1^1 \\ e_2^1 \\ e_3^1 \\ e_4^1 \\ e_5^1 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \epsilon K \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\zeta}$$

We first show that e^1 is bounded. To this end, consider as Lyapunov function for the above error system $V = {e^1}^T P e^1$ where $P > 0$ is chosen so that $A_1^T P + P A_1 = -I$. This can be done since $\dot{e}^1 = A_1 e^1$ is stable. Taking the derivative of V along the trajectory, we find

$$\begin{aligned} \dot{V} &= -\|e^1\|^2 + 2{e^1}^T P \zeta \\ &\le -\|e^1\|^2 + \epsilon 2 \|e^1\| \bar{\sigma}(P) K \\ &\le -\|e^1\|^2 + \epsilon \|e^1\| K \end{aligned}$$

Thus, $\dot{V} < 0$ whenever $\|e^1\| \geq \epsilon K$ which implies $\|e^1\|$ is bounded and, hence, $\|\xi^1\|$ is bounded. Furthermore, we can conclude that the tracking error will be $O(\epsilon)$. □

In above, we have demonstrated the idea of applying approximate linearization in position and heading control mode. However, one can apply the same principle to show that there exist another set of outputs, such as $\{p_x, p_y, p_z, \beta\}$, called **position and side slip angle mode**, which approximately linearize the system. In particular, if the controller tries to keep slide slip angle, β, to zero, the system is said to be operated in **coordinated flight mode**. Since the desired side-ward movement is zero, and the desired heading is aligned with the tangent of the trajectory projected onto X-Y plane, *i.e.,* $\psi = \text{atan2}(v^p_y, v^p_x)$. Thus, to minimize the drag force introduced by the fuselage, coordinated flight mode is the most desirable mode of operation.

5. Trajectory Generation based on Differential Flatness

In the previous section, we have shown the idea of using approximate linearization to derive a tracking controller. However, one can use the same framework to design a trajectory generator. In this section, we prove by construction that the approximate model with the same outputs is differentially flat and hence the state and input can be expressed as functions of the outputs and their derivatives. This property is very useful for real-time trajectory generation[18, 33] in hierarchical control system as described in [10].

A system is said to be differentially flat, if there exist output functions, called flat outputs, such that all the states and inputs can be expressed as analytic functions of the flat outputs and a finite number of their derivatives. To be precise, we use the following definition.

Definition 5.1. (Differentially Flat System[4, 33]) Given a system $\dot{x} = f(x, u)$ has states $x \in \mathbb{R}^n$, and inputs $u \in \mathbb{R}^m$, the system is said to be differentially flat if there exist outputs $y \in \mathbb{R}^m$ of the form $y = F_y(x, u, \dot{u}, \ldots, u^{(p)})$ such that, $x = F_x(y, \dot{y}, \ldots, y^{(q)}),\ u = F_u(y, \dot{y}, \ldots, y^{(q)})$. □

In general, there doesn't exist any systematic characterization of flat systems and the search for flat outputs is usually difficult. However, as stated in [4], dynamically feedback linearizable systems [7] by an endogenous compensator are differentially flat.

As shown above, the helicopter model is approximately linearized, hence it suggests that the approximate mode is differentially flat with respect to the outputs. This property can be used to generate an approximate state-input trajectory, which should be close to the true state-input trajectory as long as the assumptions are valid, of the exact system. Differential flatness has been applied to approximate models of aircraft [18] for trajectory generation. On the other hand, this property can also be used for the generation of output trajectory while taking state and input constraints into consideration. Therefore, as shown in [33], the trajectory can be generated for optimizing a specified cost function but being subject to algebraic constraints without including any dynamical constraint.

In following, we present a constructive proof for showing that the model system with positions and heading as outputs is differentially flat.

Theorem 5.2. *Consider the system equations(4.4)(4.2)(4.3) with output chosen to be* $\{p_{xm}, p_{ym}, p_{zm}, \psi\}$*. The resulting system is differentially flat on sets where* $\phi \neq \pm\pi/2$ *and* $\theta \neq \pm\pi/2$.

Proof: *To show that all the states can be derived by the outputs and their derivatives, first we rewrite (4.4) as*

$$\underbrace{\begin{bmatrix} \ddot{p}_{xm} \\ \ddot{p}_{ym} \\ \ddot{p}_{zm} \end{bmatrix}}_{\ddot{P}_m} = e^{\hat{z}\psi} e^{\hat{y}\theta} e^{\hat{x}\phi} \begin{bmatrix} 0 \\ 0 \\ -T_M/m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \tag{5.1}$$

and the equation becomes

$$e^{-\hat{z}\psi}\begin{bmatrix}\ddot{p}_{xm}\\ \ddot{p}_{ym}\\ \ddot{p}_{zm}-g\end{bmatrix} = e^{\hat{y}\theta}e^{\hat{x}\phi}\begin{bmatrix}0\\0\\-T_M/m\end{bmatrix} \tag{5.2}$$

By taking norm on both sides, T_M can be obtained as

$$T_M = m\sqrt{(\ddot{p}_{xm})^2+(\ddot{p}_{ym})^2+(\ddot{p}_{zm}-g)^2} \tag{5.3}$$

and hence $\dot{T}_M, \ddot{T}_M, T_M^{(3)} = \ddot{w}_1$. From (5.2), we can simplify as

$$e^{-\hat{z}\psi}\begin{bmatrix}\ddot{p}_{xm}\\ \ddot{p}_{ym}\\ \ddot{p}_{zm}-g\end{bmatrix}\frac{-m}{T_M} = \begin{bmatrix}\sin\theta\cos\phi\\ -\sin\phi\\ \cos\theta\cos\phi\end{bmatrix} \tag{5.4}$$

Thus, one can easily verify that

$$\phi = \sin^{-1}\left(\frac{-\ddot{p}_{xm}\sin\psi+\ddot{p}_{ym}\cos\psi}{T_M/m}\right) \tag{5.5}$$

and hence

$$\theta = \operatorname{atan2}\left(\frac{\ddot{p}_{xm}\cos\psi+\ddot{p}_{ym}\sin\psi}{-\cos\phi\ T_M/m}, \frac{\ddot{p}_{zm}-g}{-\cos\phi\ T_M/m}\right) \tag{5.6}$$

By applying Ψ^{-1} to the derivatives of the Euler angles, we obtain vector ω. Then, one can solve for a_{1s}, b_{1s} and T_T from the Euler equations. Hence, taking the derivatives, we have $w_2 = \dot{T}_T$, $w_3 = \dot{a}_{1s}$, $w_4 = \dot{b}_{1s}$. Hence, we have set up that

$$\begin{aligned}&(P_m, \dot{P}_m, \phi, \theta, \psi, \omega^b, T_M, T_T, a_{1s}, b_{1s}, w_1, \dot{w}_1, \ddot{w}_1, w_2, w_3, w_4)\\ &\mapsto (P_m, \dot{P}_m, \ddot{P}_m, P_m^{(3)}, P_m^{(4)}, P_m^{(5)}, \psi, \dot{\psi}, \ddot{\psi}, \psi^{(3)})\end{aligned}$$

is a diffeomorphism on sets where $\phi \neq \pm\pi/2$ and $\theta \neq \pm\pi/2$ showing that the model system is indeed differentially flat. □

Similarly, one can show that in coordinated flight mode, $\{p_{xm}, p_{ym}, p_{zm}, \beta\}|_{\beta\equiv 0}$ are flat outputs of the model system.

6. Nonlinear Control Design Based on Outer Flatness

In general, the full helicopter dynamics are neither feedback linearizable or differentially flat, especially, if full rotary wing dynamics and actuator dynamics are considered. However, based on the natural time scale separation, one can treat a helicopter system as a system formed by cascading two systems, an "outer system"(the position dynamics) and an "inner system" (the rest of the system). If the outer system is differentially flat, a scheme called outer flatness is proposed by [34] for generating an inner trajectory for an inner system to track. This scheme has been successfully used for generating a two stage control synthesis for many systems which are not completely flat. Such a scheme which utilizes the flatness of the outer system is roughly illustrated in Figure 6.

In the figure, P_O is the outer system which is flat, and P_I is the inner system which is not necessarily flat. Given a desired output trajectory, say $y_d^O(\cdot)$, the mapping F in Figure 6 utilizes the flatness property of the outer system to generate an desired output trajectory $y_d^I(\cdot)$ for the inner system. The control synthesis for the overall system then reduces to the design of an inner system controller, C, which drives the inner system output $y^I(t) \to y_d^I(t)$ (exponentially) as $t \to \infty$. As the inner system output converges, one can show that the outer system output converges to the desired one, $y^O(t) \to y_d^O(t)$ as $t \to \infty$. That is, the overall system asymptotically tracks the desired trajectory.

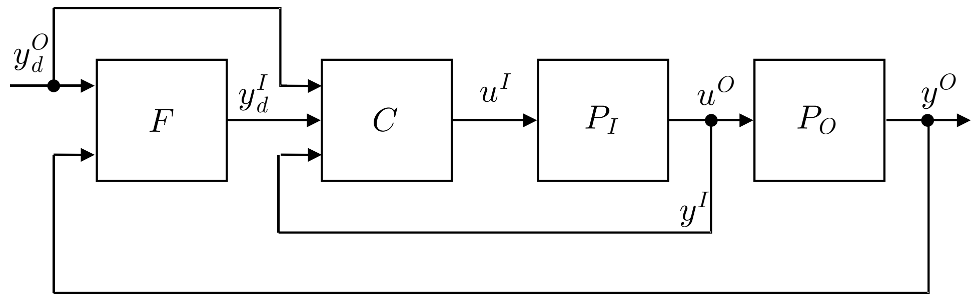

FIGURE 6. Partitioned inner and outer systems.

It has been shown that the helicopter dynamics are *approximately* differentially flat with the position and heading $\{P, \psi\}$ as the flat outputs. The approximation is based on the assumption that the coupling terms a_{1s}, b_{1s}, T_T are small and can be neglected in the model. The outer system, P_0, is defined as

$$P_O : \begin{cases} \ddot{P} = \dfrac{1}{m} R(\Theta) \begin{bmatrix} 0 \\ 0 \\ -T_M \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} + h(x) \\ \psi = \psi \end{cases}$$

with

$$h(x) \quad = \quad \frac{1}{m} R(\Theta) \begin{bmatrix} -T_M \sin a_{1s} \\ T_M \sin b_{1s} - T_T \\ -T_M(\cos a_{1s} \cos b_{1s} - 1) \end{bmatrix}$$

where the inputs are $u^O = y^I = [\phi, \theta, \psi, T_M]^T$, and the outputs are $y^O = [p_x, p_y, p_z, \dot{p}_x, \dot{p}_y, \dot{p}_z, \psi]^T$. The inner system, P_I, are described by (4.2), (4.3) and in addition the *rotary wing dynamics* which can be approximated by the following equations(for details see [24])

$$T_M = c_{M1}\theta_M + c_{M3}\theta_M^3, \quad T_T = c_{T1}\theta_T + c_{T3}\theta_T^3, \quad a_{1s} = -B, \quad b_{1s} = A$$

for near hovering operation, where θ_M, θ_T are the main and tail rotor collective pitch, and B, A are the longitudinal and lateral cyclic pitch. Here, we have $y^I = [\phi\ \theta\ \psi\ T_M]^T$ and we define $u^I = [\theta_M\ \theta_T\ B\ A]^T$. One must notice that this approximation introduces a small non-vanishing modeling error $h(x)$ which depends on $\Theta, T_M, a_{1s}, b_{1s}, T_T$. We will soon show its effect on the stability of the closed-loop system.

The control design for the overall system is be based on an assumption that there exists a controller C such that $e^I = 0$ is an exponentially stable equilibrium point for the inner error system

$$\dot{e}^I = f^I(e^I, e^O, t)|_{e^O=0}, \ f^I(0, 0, t) = 0$$

where $e^O = [(P - P_d)^T, (\dot{P} - \dot{P}_d)^T]^T$ and $e^I = y^I - y_d^I$. There have been various design methodologies proposed for the controller of the inner system, *e.g.* [16]. The details of the design of inner controller deployed in here, please refer to [12]. In this section, we are only interested in the performance of the overall system assuming such a controller C is already available and satisfies the specified property.

The design of the outer controller, F, is based on the flatness property of the approximated outer system. As shown in previous section, for the model system (4.4), there exists a diffeomorphism for showing that the system is differentially flat. By using only part of the results, on could obtain a smooth mapping from the outer system output to the inner system output

$$\begin{array}{rcl} \Phi : \mathbb{R}^3 \times S & \rightarrow & S^3 \times \mathbb{R} \\ (\xi_1, \xi_2, \xi_3, \xi_4) & \mapsto & (\phi_1, \phi_2, \phi_3, \phi_4) \end{array}$$

where

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} \xi_{02} \\ \text{atan2}(\frac{\xi_1 \cos\xi_4 + \xi_2 \sin\xi_4}{-(\xi_{01}/m)\cos\xi_{02}}, \frac{\xi_3 - 1}{-(\xi_{01}/m)\cos\xi_{02}}) \\ \xi_4 \\ \xi_{01} \end{bmatrix} \tag{6.1}$$

with $\xi_{01} = m\sqrt{\xi_1^2 + \xi_2^2 + (\xi_3 - g)^2}$, $\xi_{02} = \sin^{-1}(\frac{-\xi_1 \sin\xi_4 + \xi_2 \cos\xi_4}{\xi_{01}/m})$, in which Φ is defined on sets with $\phi_1, \phi_2 \neq \pm\pi/2$. In particular, $\Phi(\ddot{p}_{xm}, \ddot{p}_{ym}, \ddot{p}_{zm}, \psi) = (\phi, \theta, \psi, T_M)$. Suppose that the desired output trajectory of the outer system, y_d^O, and its derivative are given. To obtain the desired trajectory of the inner system, we define a pseudo-input vector

$$v_p = \ddot{P}_d + K_v(\dot{P} - \dot{P}_d) + K_p(P - P_d) \tag{6.2}$$

where $v_p \in \mathbb{R}^3$, $K_p,\ K_v \in \mathbb{R}^{3\times 3}$ are control parameters. With the above pseudo-input, the desired values of ϕ, θ, T_M are given by

$$(\phi_d, \theta_d, \psi, T_{Md}) = \Phi(v_{xp}, v_{yp}, v_{zp}, \psi). \tag{6.3}$$

Thus, we obtain the desired output of the inner system $y_d^I = [\phi_d, \theta_d, \psi_d, T_{Md}]^T$.

Clearly, if the inner system *exactly* tracks the desired trajectory (Θ_d, T_{Md}), that is, $y_d^I = y^I$ in Figure 6, then the behavior of the overall closed-loop system is specified by the outer system only, which, due to chosen the control law (6.2), is *approximately* a linear system with poles assigned by the parameters K_v, K_p.

Now if we summarize all conditions so far and rewrite the dynamics of the overall closed-loop system in terms of the tracking errors e^I and e^O of the inner and outer systems respectively, they have the form

$$\begin{cases} \dot{e}^I &= f^I(e^I, e^O, t) \\ \dot{e}^O &= Ae^O + g^O(e^I, t) + h^O(e^I, e^O, t) \end{cases} \tag{6.4}$$

where

$$g^O(e^I, t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ g(e^I, t) \end{bmatrix}, \quad h^O(e^I, e^O, t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ h(e^I, e^O, t) \end{bmatrix}$$

with

$$g(e^I, t) = \frac{1}{m} R(\Theta) \begin{bmatrix} 0 \\ 0 \\ -T_M \end{bmatrix} - \frac{1}{m} R(\Theta_d) \begin{bmatrix} 0 \\ 0 \\ -T_{Md} \end{bmatrix}.$$

In the above equations, $f^I(e^I, e^O, t)$ is in general a function of both e^I and e^O since the input of the inner system is a function of e^O.

The function $h^O(e^I, e^O, t)$ from (4.4) is a small non-vanishing approximation error, and $g^O(e^I, t)$ vanishes when the inner system exactly tracks the desired trajectory, *i.e.*, $g^O(0, t) = 0$. Since the helicopter model is smooth and many of the parameters are physically bounded, $g^O(e^I, t)$ is in fact (globally) bounded as $\|g^O(e^I, t)\| \leq L_1 \|e^I\|$ for some constant $L_1 > 0$[1] and $f^I(e^I, e^O, t)$ is Lipschitz, *i.e.*,$\|f^I(e_1^I, e_1^O, t) - f^I(e_2^I, e_2^O, t)\| \leq L_2(\|e_1^I - e_2^I\| + \|e_1^O - e_2^O\|)$.

[1]Such a L can be estimated from the system equation (4.1).

Stability Analysis. We now analyze the performance of the overall closed-loop system. As we have argued before, the function f in (6.4) is in general a function of both e^I and e^O. However, in practice, the inner system is usually designed to have a much faster convergence rate than the outer system. To simplify the analysis, for now we assume that the inputs $T_{Md}(\cdot)$ and $\Theta_d(\cdot)$ of the inner system are approximately constant, and thus f is only a function of e^I (the more general case will be presented afterwards).

Recall that given an general system $\dot{x} = f(x,t)$, by the Lyapunov theorem and its converse [25], the system is exponentially stable if and only if there exists a *Lyapunov function* $V(x,t)$ satisfying

$$\alpha_1\|x\|^2 \leq V(x,t) \leq \alpha_2\|x\|^2 \tag{6.5}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -\alpha_3\|x\|^2 \tag{6.6}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_4\|x\| \tag{6.7}$$

for some positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$.

However, we may write

$$f^I(e^I, e^O, t) = f^I(e^I, 0, t) + d^I(e^I, e^O, t)$$

where $d^I(e^I, e^O, t) = f^I(e^I, e^O, t) - f^I(e^I, 0, t)$. The nominal system $\dot{e}^I = f^I(e^I, 0, t)$ is exponentially stable as designed. We can apply this theorem to both the nominal outer system $\dot{e}^O = Ae^O$ and the nominal inner system $\dot{e}^I = f^I(e^I, 0, t)$ and denote the corresponding Lyapunov functions as V^O and V^I and the Lyapunov constants as $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ and $\beta_1, \beta_2, \beta_3, \beta_4 > 0$ respectively. Then for the overall system, we have the result

Theorem 6.1. *Consider the following perturbed system*

$$\left\{ \begin{array}{lclcl} \dot{e}^I & = & f^I(e^I, e^O, t) & = & f^I(e^I, 0, t) + d^I(e^I, e^O, t) \\ \dot{e}^O & = & Ae^O + g^O(e^I, t) & & \end{array} \right. \tag{6.8}$$

where $g^O(e^I, t)$ is a perturbation term that satisfies $\|g^O(e^I, t)\| \leq L_1\|e^I\|$ for some $L_1 > 0$. If, for $f^I(e^I, e^O, t)$, there exists $L_2 > 0$ such that $\|f^I(e_1^I, e_1^O, t) - f^I(e_2^I, e_2^O, t)\| \leq L_2(\|e_1^I - e_2^I\| - \|e_1^O - e_2^O\|)$, then the overall system is exponentially stable if the product of the two Lipschitz constants satisfies the inequality

$$L_1 \cdot L_2 < \frac{\alpha_3}{\alpha_4} \cdot \frac{\beta_3}{\beta_4}. \tag{6.9}$$

Proof. Since $f^I(e^I, e^O, t)$ is Lipschitz, we have $\|d^I(e^I, e^O, t)\| \leq L_2\|e^O\|$. We consider the candidate Lyapunov function $V = V^I + \mu V^O$ for the overall system. Then we have

$$\begin{aligned} \dot{V} = \dot{V}^I + \mu\dot{V}^O &\leq -\beta_3\|e^I\|^2 + \beta_4 L_2\|e^I\|\|e^O\| - \mu\alpha_3\|e^O\|^2 + \mu\alpha_4 L_1\|e^O\|\|e^I\| \\ &= -(\|e^I\|, \|e^O\|)Q(\|e^I\|, \|e^O\|)^T \end{aligned}$$

where the matrix $Q \in \mathbb{R}^{2\times 2}$ is

$$Q = \begin{bmatrix} \beta_3 & -\frac{1}{2}(\beta_4 L_2 + \mu\alpha_4 L_1) \\ -\frac{1}{2}(\beta_4 L_2 + \mu\alpha_4 L_1) & \mu\alpha_3 \end{bmatrix}.$$

Q is positive definite if and only if $\det(Q) > 0$. That is, there exists $\mu > 0$ such that

$$-\alpha_4^2 L_1^2 \mu^2 + (4\beta_3\alpha_3 - 2\beta_4 L_2 \alpha_4 L_1)\mu - \beta_4^2 L_2^2 > 0.$$

This is true if and only if the discriminant of the quadratic function of μ on the left hand side is positive which yields $L_1 \cdot L_2 < \frac{\alpha_3}{\alpha_4} \cdot \frac{\beta_3}{\beta_4}$. □

This theorem states a very interesting fact about the system (6.8): heuristically, α_3 and β_3 are proportional to the convergence rates of the outer and inner systems respectively,[2] hence the stability of the perturbed

[2] A more precise estimates of the convergences rates are given by $\frac{\alpha_3}{2\alpha_2}$ and $\frac{\beta_3}{2\beta_2}$.

systems requires *only* that the *product* of the Lipschitz constants of the perturbation terms is less than the *product* of the two convergence rates, regardless of the rate of each individual system.

The stability of a similar model of the overall closed-loop system has been studied before in [34], however, no explicit conditions are provided under which a μ exists such that the overall system is stable. Here, Theorem 6.1 give more detailed and useful results in characterizing the properties of the closed-loop system.

Although we have established the conditions for the system (6.8) to be exponentially stable, estimates of its Lyapunov constants indeed depend on L_1, L_2 and all the Lyapunov constants of the inner and outer systems. These constants can be optimized by maximizing the smaller eigenvalue of Q with respect to μ. We here omit the detail and carry on the analysis by assuming that the system (6.8) is exponentially stable and its Lyapunov constants are denoted by $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$. We now want to estimate the effect of the non-vanishing error term h on the performance of the closed-loop system (6.4). In general, we can no longer expect asymptotic stability when a non-vanishing perturbation is introduced. However, according to [9], we can still have good estimates of a bound on the tracking error and the rate of convergence outside this bound.

Proposition 6.2. *Assume that the system (6.8) has the Lyapunov constants $\{\gamma_i\}_{i=1}^4$. Then, for the closed-loop system (6.4), if $\|h(e^I, e^O, t)\| \leq \delta < \frac{\gamma_3}{2\gamma_4}\sqrt{\frac{\gamma_1}{\gamma_2}}$, then the tracking error of the overall system is bounded by $b = \frac{2\gamma_4}{\gamma_3}\sqrt{\frac{\gamma_2}{\gamma_1}}\delta$, and, outside this bound, the error exponentially decreases with a rate larger than $\lambda = \frac{\gamma_3}{4\gamma_2}$.*

The control parameters K_v and K_p can be adjusted so as to minimize the error bound b. For the helicopter model, the error term $h^O(e^I, e^O, t)$ is usually extremely small, as is δ. We can also choose the control parameters such that the inner and outer systems have very fast rates of convergence, hence a large γ_3. Consequently, the error bound b is very small, and usually barely noticeable in simulations and experiments.

7. Simulation Results

We apply our approximate method and compare with the exact method on the true system. The initial conditions are $p_x = 0.1g$, $p_y = 0.05g$, $p_x = 0.2g$, $v_x^p = v_y^p = v_z^p = 0$, $\psi = 0.01$, $g = 9.8$ and other states are in trim conditions. For both control designs, the dominant conjugate poles are $-1.4 \pm 1.4283i$ and other poles are placed at -5. To illustrate the idea, we set y_d and all their derivatives equal to zero, and the controllers are required to hover the helicopter back to origin while turning the heading to zero.

The result of using exact linearization is shown in Figure 7, and that of using approximate linearization is shown in Figure 8. Both controllers are successful in stabilizing the outputs. However, in the exact method the internal dynamics, roll and pitch angles, are excited and continue to oscillate. Furthermore, it exhibits large control effort. While applying the approximate control law, the internal dynamics are stabilized. The control inputs are kept relatively small throughout the simulation, and it validates the assumption made on a_{1s}, b_{1s} and T_T.

We present the simulation result of the propsed nonlinear controller based on outer flatness. The inner controller has poles placed at -5 for controlling the main rotor thrust and at -10 and $-7 \pm 7.1414i$ for controlling the attitude dynamics; the outer controller has both poles placed at -2. With the same initial conditions and desired trajectory, the controller is applied to the exact model and the simulation result is shown in Figure 9. The nonlinear controller stabilizes the both inner and outer system and results in bouned errors. These phenomena have been predicted in Theorem 6.1 and Proposition 6.2.

8. Conclusion

In this paper, the output tracking control design of a helicopter based unmanned aerial vehicle model based on approximate input-output linearization is illustrated. We show that the model cannot be converted into a controllable linear system via exact state space linearization. By neglecting the couplings between rolling/pitching moments and lateral/longitudinal forces, we show that the dynamically extended approximated system with

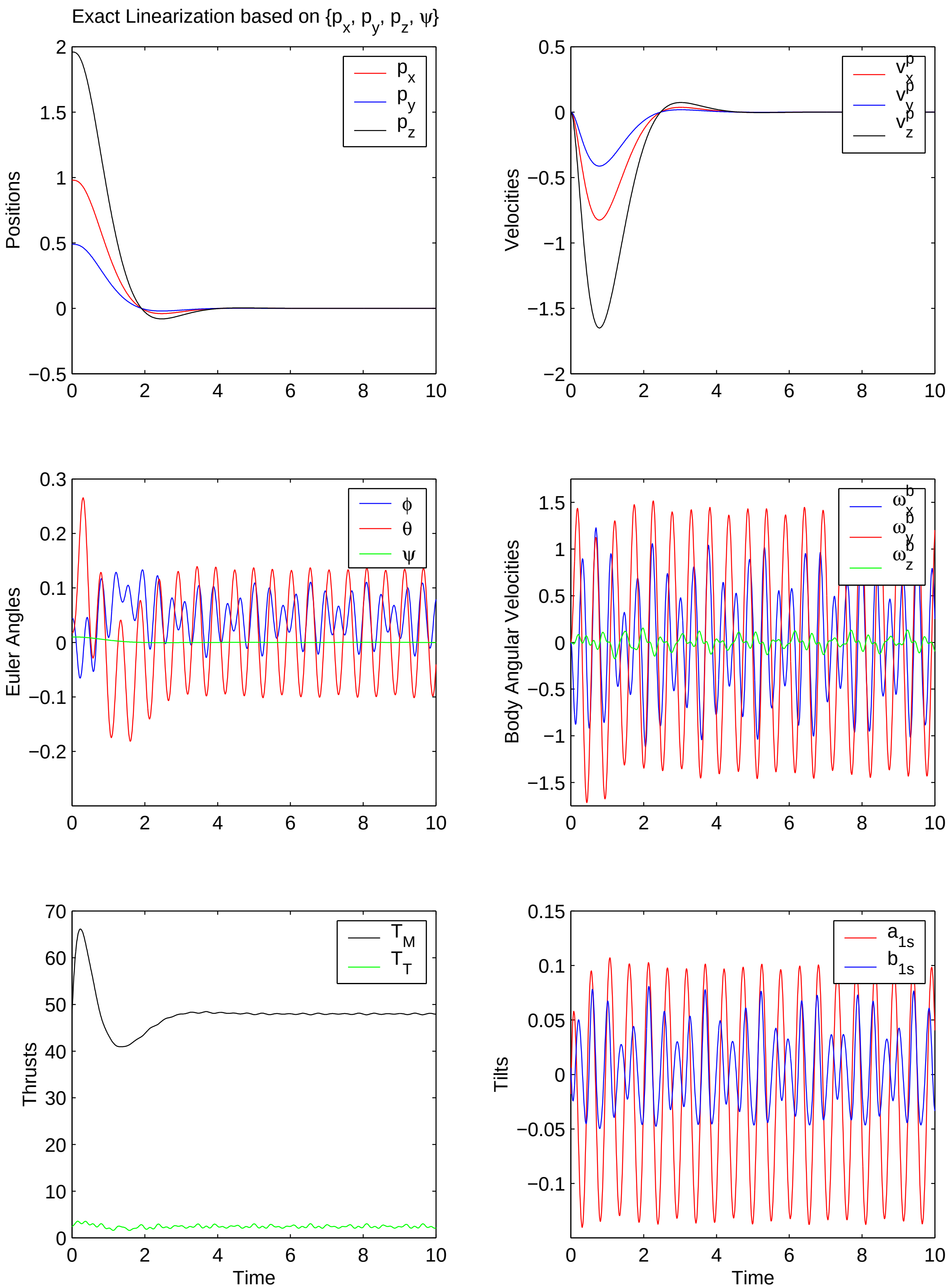

FIGURE 7. Response of the helicopter system by exact input-output linearization with { $p_x,\ p_y,\ p_z,\ \psi$} as outputs

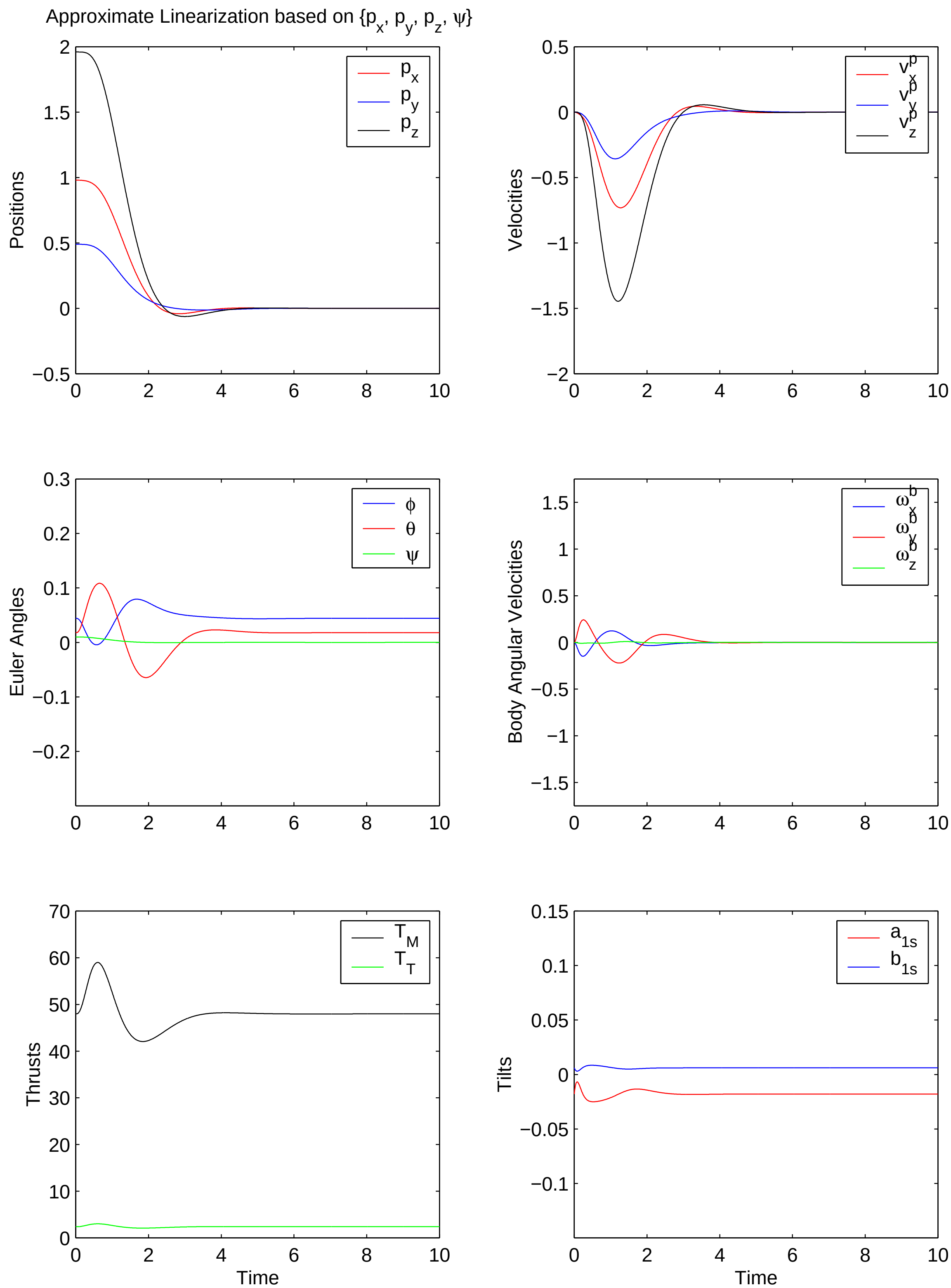

FIGURE 8. Response of the helicopter system by approximate input-output linearization with $\{\, p_x,\ p_y,\ p_z,\ \psi\}$ as outputs

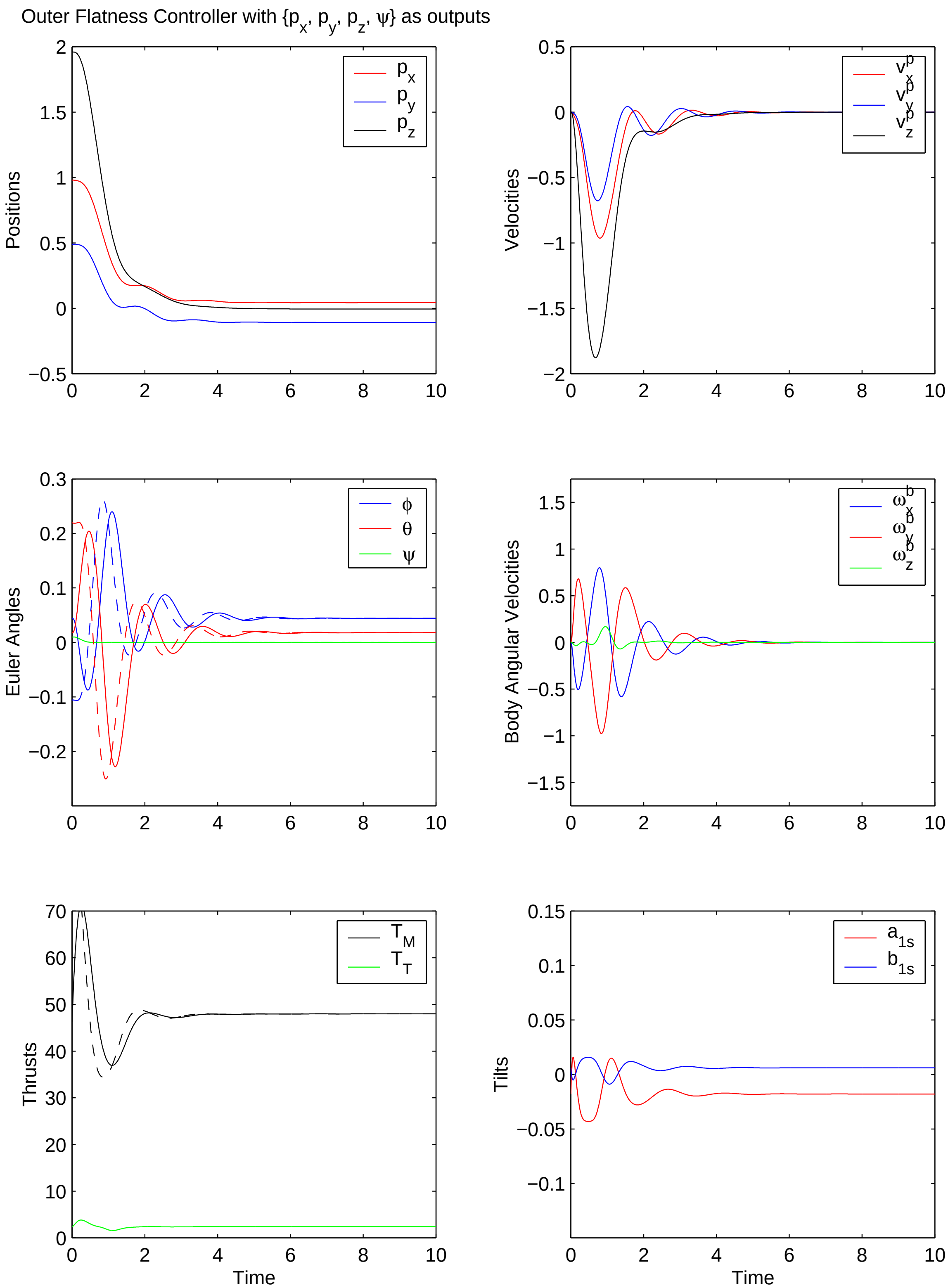

FIGURE 9. Response of helicopter system by nonlinear control design based on outer flatness linearization with $\{\ p_x,\ p_y,\ p_z,\ \psi\}$ as outputs. Notice that the dashed lines represent the desired trajectories of ϕ, θ, T_M.

positions and heading as outputs is linearizable without zero dynamics. We have proved that bounded tracking is achievable by applying the approximate control with a given bounded output trajectory. Next, we derive a diffeomorphism showing that the approximate system with the same outputs is differentially flat. By that, state and input can be expressed as functions of the outputs and their derivatives. Hence, output trajectory generation can take state and input constraints into consideration. Application of path generation for helicopter based on differential flatness can be found in [3].

Based on geometric control theory, we decompose the dynamics into two subsystems: inner and outer systems. A nonlinear controller is proposed based on differential flatness of the outer system. This control design only assumes that the outer system is differentially flat and the inner system is exponentially stable. Also, the assumptions made on the outer system can be applied to many different helicopter models. The nonlinear controller based on outer flatness has been successfully applied in landing an UAV based on computer vision and formation flight of UAV cluster with mesh stability as shown in [27, 28] and [26] respectively. Hence, it is possible to design a nonlinear controller based on the propsed scheme and implement it for controlling actual helicopters.

Simulation results show that the approximate control law produces desired performance without excite the internal states into oscillation. The performance of the control design based on outer flatness is simulated. The nonlinear controller stabilizes both the inner and outer systems and it results in bounded errors.

To reduce the bounded error, one can simply design robust controllers to augment the presented controllers for compensating the effects due to the existence of non-vanishing terms. For disturbance satisfying matching condition, sliding mode controller can be applied. For mismatched disturbance, controllers can be derived using the Backstepping method [15] and Dynamic Surface Control [31]. However, linear techniques such as μ-synthesis [22] can also be used since the systems are linearized by state feedback. Computer Aided Control System Design(CACSD) has enabled the analysis and design of control system. The proofs of Theorem 3.4 and 3.3 are performed symbolically. The description of the system (4.1) and controller derivation using Lie derivatives are computed symbolically.

Figure 10. An unmanned aerial vehicle of Berkeley Aerobot fleet: Ursa Minor.

In future, we will extend our control design to include the fuselage drag force, rotary wing dynamics and actuator dynamics. Robustness issue will be addressed to accommodate the presence of external disturbance and uncertainty of the dynamical model. The final tracking controller will be implemented and tested on a UAV called Ursa Minor as shown in Figure 10, on which we have mounted embedded controller, GPS, INS and wireless Ethernet.

Appendix A. Appendix

A.1. **System parameters.** All variables except for the state variables and inputs are numeric constants, which can be obtained by measurements and experiments. The followings are the values of the constants.

$$
\begin{array}{lllllllll}
I_x & = & 0.142413 & I_y & = & 0.271256 & I_z & = & 0.271492 \\
l_M & = & -0.015 & y_M & = & 0 & h_M & = & 0.2943 \\
h_T & = & 0.1154 & l_T & = & 0.8715 & m & = & 4.9 \\
C_M^Q & = & 0.004452 & D_M^Q & = & 0.6304 & \frac{\partial R_M}{\partial b_{1s}} & = & 25.23 \\
C_T^Q & = & 0.005066 & D_T^Q & = & 0.008488 & \frac{\partial M_M}{\partial a_{1s}} & = & 25.23 \\
c_{M1} & = & 6.4578 & c_{M3} & = & 100.3752 & c_{T1} & = & 0.1837 \\
c_{T3} & = & 0.1545 & & & & & &
\end{array}
$$

The operation regions in radian for a_{1s}, b_{1s} and newton for T_M, T_T are

$$
\begin{array}{ll}
|a_{1s}| \leq 0.4363 & -20.86 \leq T_M \leq 69.48 \\
|b_{1s}| \leq 0.3491 & -5.26 \leq T_T \leq 5.26
\end{array}
$$

A.2. **Proof of Proposition 3.4.** Define a 6×4 matrix

$$
L(x) = \begin{bmatrix} L_{g_1} L_f^2 h_1 & \cdots & L_{g_4} L_f^2 h_1 \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^2 h_6 & \cdots & L_{g_4} L_f^2 h_6 \end{bmatrix}. \tag{A.1}
$$

It can be easily shown that $L(x)$ can be decomposed as the product of two matrices, that is,

$$
L(x) = \begin{bmatrix} \frac{1}{m} R & 0 \\ 0 & \Psi \mathcal{I}^{-1} \end{bmatrix}_{6\times 6} \begin{bmatrix} \dfrac{\partial f^b}{\partial w} \\ \dfrac{\partial \tau^b}{\partial w} \end{bmatrix}_{6 \times 4}. \tag{A.2}
$$

One can observe the left matrix are nonsingular for all $x \in X$ The right matrix has full column rank for all $x \in X$, i.e. the column vectors are linearly independent, and it can be shown by applying Gaussian elimination on columns.

Since $A_{k_1k_2k_3k_4}(x)$ is constructed by extracting the rows of the matrix $L(x)$ according to the indexes $k_1, \cdots, k_4$, we can define $A_{k_1k_2k_3k_4}(x) = E_{k_1k_2k_3k_4} L(x)$, where the extraction matrix $E_{k_1k_2k_3k_4}$ is defined as, in each row, it has a 1 in the position specified by the index and zeros elsewhere. By substitution, we have

$$
A_{k_1k_2k_3k_4}(x) = \underbrace{E_{k_1k_2k_3k_4} \begin{bmatrix} \frac{1}{m} R & 0 \\ 0 & \Psi \mathcal{I}^{-1} \end{bmatrix}}_{A_L} \underbrace{\begin{bmatrix} \dfrac{\partial f^b}{\partial w} \\ \dfrac{\partial \tau^b}{\partial w} \end{bmatrix}}_{A_R}. \tag{A.3}
$$

It is clear that

$$
\mathrm{rank}(A_{k_1k_2k_3k_4}(x)) \leq \min\{\mathrm{rank}(A_L), \mathrm{rank}(A_R)\} = 4.
$$

In order to prove that $A_{k_1k_2k_3k_4}(x)$ has rank equal to 4, we have to show $\mathrm{rank}(A_{k_1k_2k_3k_4}(x)) \geq 4$. The lower bound of Sylvester's inequality theorem is achieved if the rows of A_L lie in the left null space of A_R. By checking the rank of the matrix constructed by appending the basis of left null space of A_R with the transpose of the row vectors of A_L, one can verify symbolically that the vectors of row space of A_L are linearly independent with the vectors in the left null space of A_R. Hence, we have $4 \leq rank(A_{k_1k_2k_3k_4}(x))$ and the result is applied to all 15 possible combinations. Since the upper and lower bound are equal, we prove that $rank(A_{k_1k_2k_3k_4}(x)) = 4$ for all $x \in X$. $\square$

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