Introduction to Mobile Robotics

Bayes Filter – Particle Filter and Monte Carlo Localization

Wolfram Burgard

Motivation

- Estimating the state of a dynamical system is a fundamental problem
- The Recursive Bayes Filter is an effective approach to estimate the belief about the state of a dynamical system
  - How to represent this belief?
  - How to maximize it?
- Particle filters are a way to **efficiently** represent an arbitrary **(non-Gaussian) distribution**
- Basic principle
  - Set of state hypotheses ("particles")
  - Survival-of-the-fittest

Bayes Filters

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1, \ldots, u_t, z_t)$$

Bayes $$= \eta \; P(z_t \mid x_t, u_1, z_1, \ldots, u_t) \; P(x_t \mid u_1, z_1, \ldots, u_t)$$

Markov $$= \eta \; P(z_t \mid x_t) \; P(x_t \mid u_1, z_1, \ldots, u_t)$$

Total prob. $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) \; P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \; dx_{t-1}$$

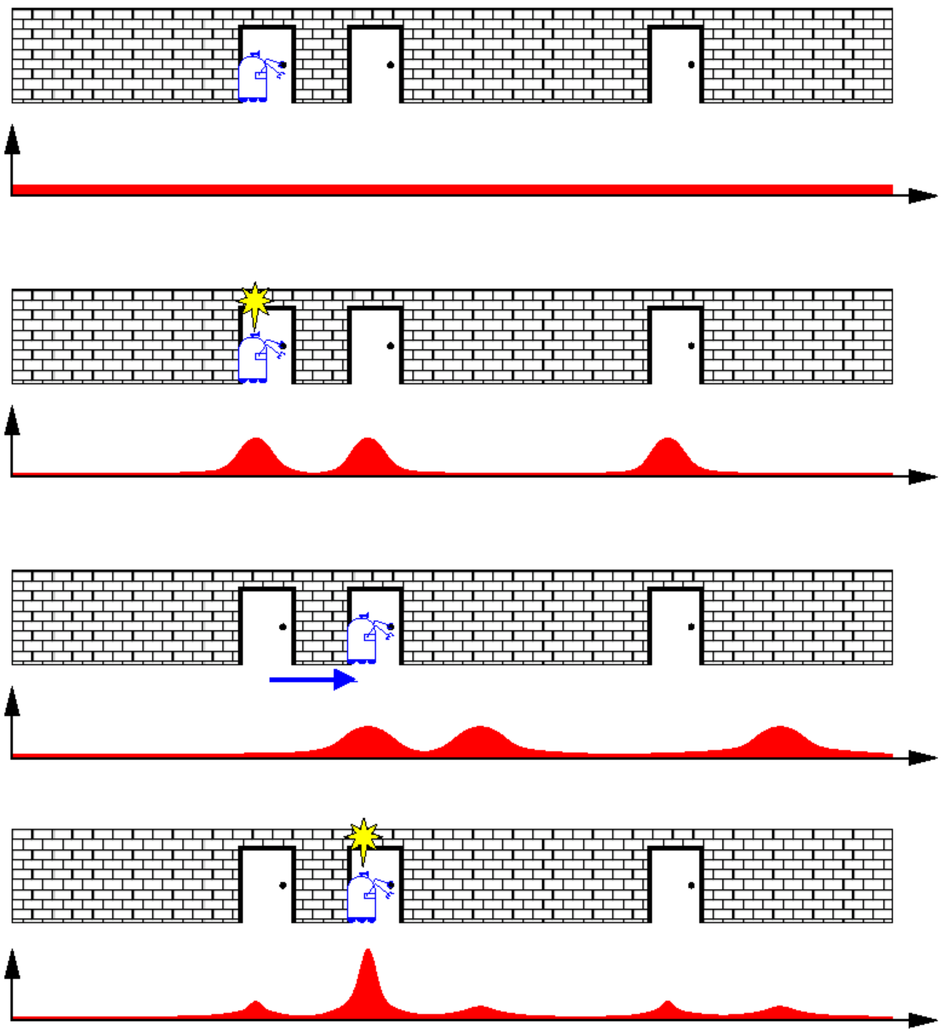
Markov $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \; dx_{t-1}$$

Markov $$= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}) \; dx_{t-1}$$

$$\boxed{= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}}$$

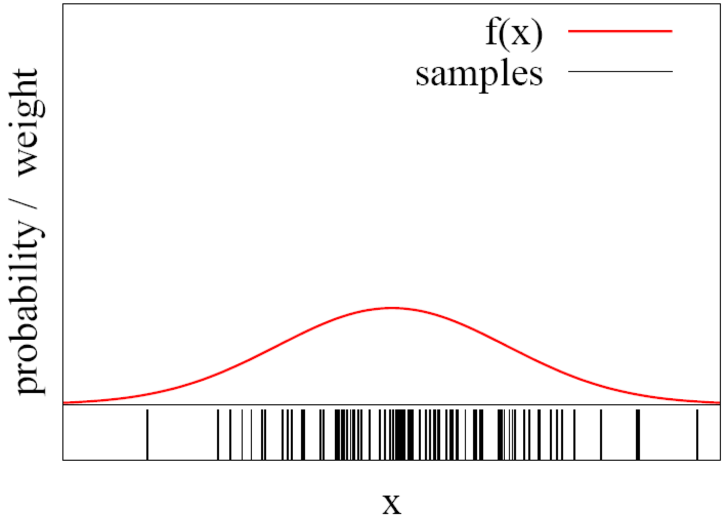
Probabilistic Localization

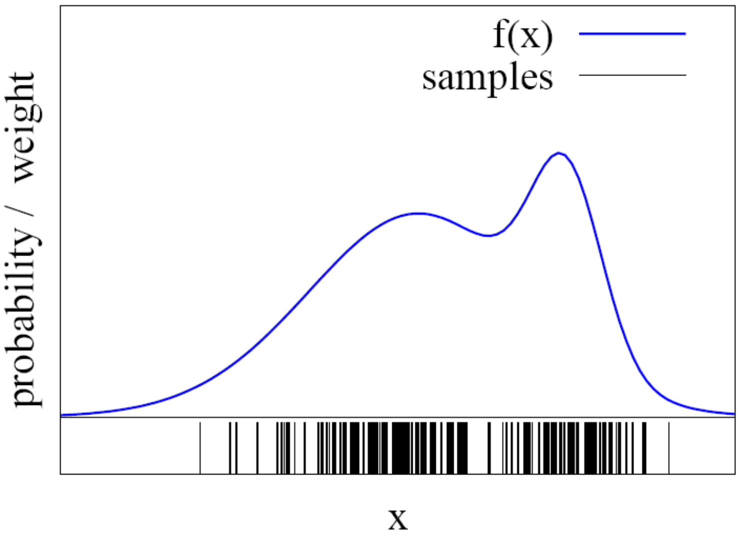
$$Bel(x \mid z, u) \;=\; \alpha\, p(z \mid x) \int_{x'} p(x \mid u, x')\; Bel(x')dx'$$



Function Approximation

- Particle sets can be used to approximate functions



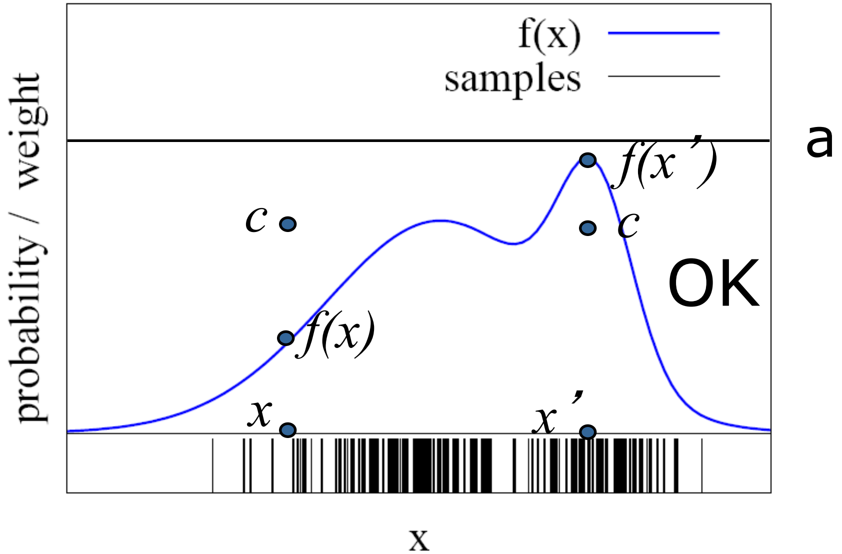


- The more particles fall into an interval, the higher the probability of that interval

- How to draw samples from a function/distribution?

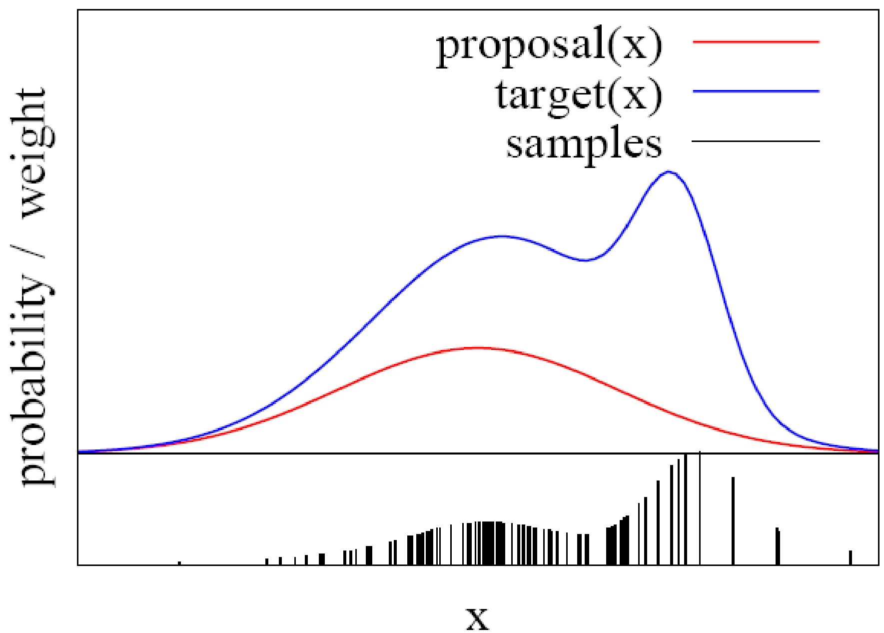
Rejection Sampling

- Let us assume that $f(x) < a$ for all x
- Sample x from a uniform distribution
- Sample c from [0,a]
- if $f(x) > c$ keep the sample
  otherwise reject the sample



Importance Sampling Principle

- We can even use a different distribution g to generate samples from f
- Using an importance weight w, we can account for the "differences between g and f"
- $w = f / g$
- f is called target
- g is called proposal
- Pre-condition: $f(x)>0 \rightarrow g(x)>0$



Particle Filter Representation

- Set of weighted samples

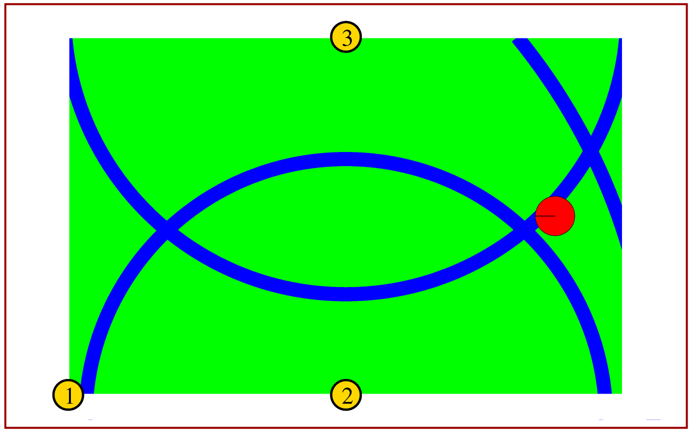
$$S = \left\{ \left\langle \underbrace{s^{[i]}}_{\text{State hypothesis}}, \underbrace{w^{[i]}}_{\text{Importance weight}} \right\rangle \mid i = 1, \ldots, N \right\}$$

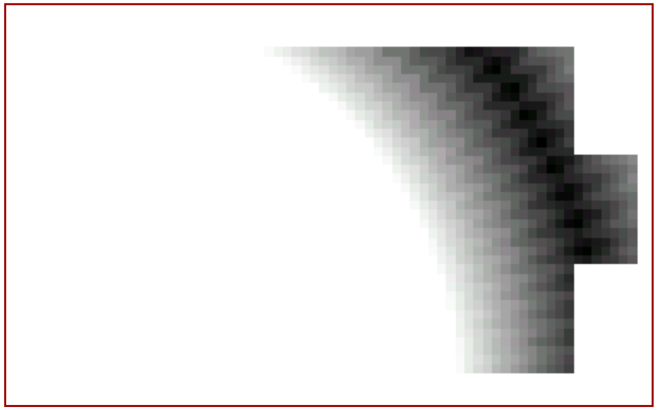
- The samples represent the posterior

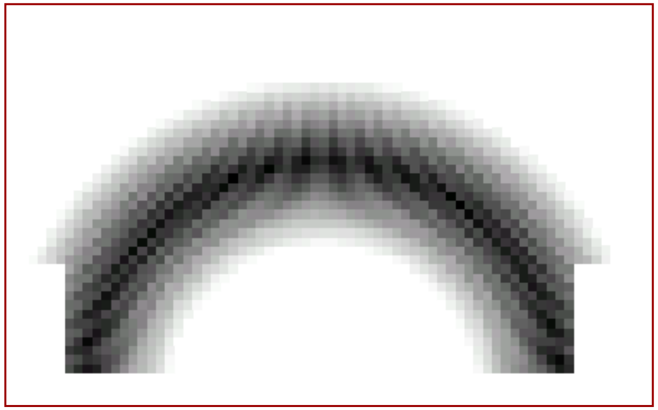
$$p(x) = \sum_{i=1}^{N} w_i \cdot \delta_{s^{[i]}}(x)$$

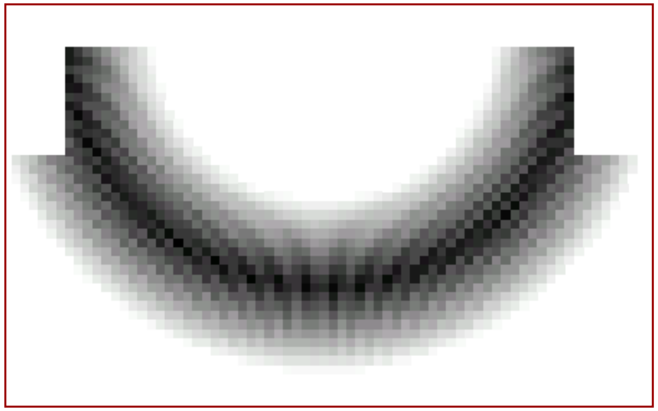
Importance Sampling with Resampling: Landmark Detection Example

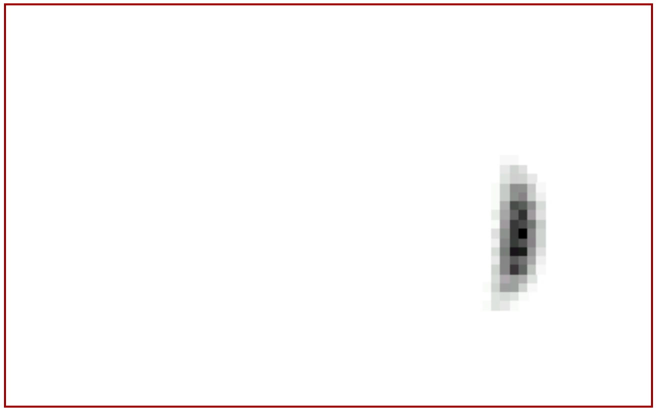
Distributions



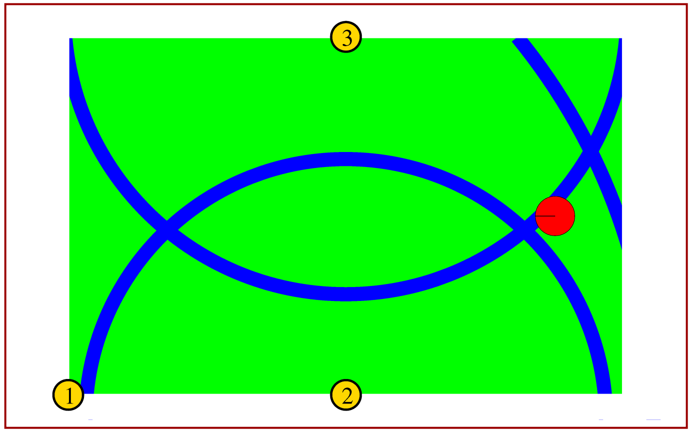


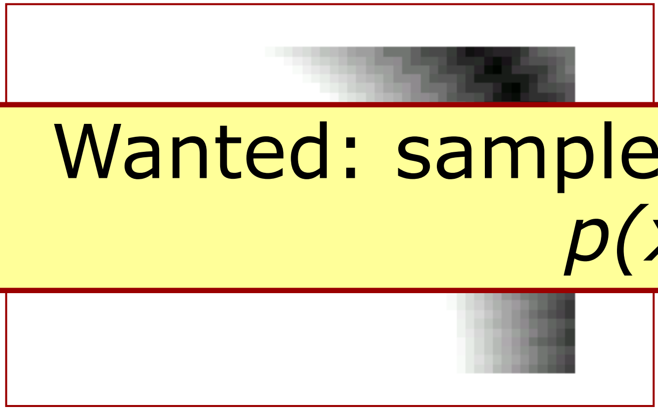


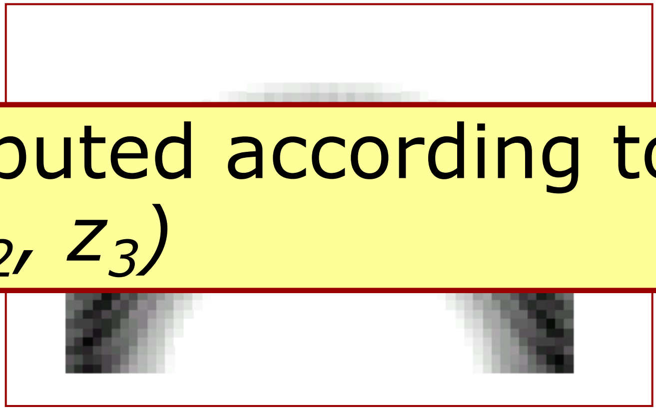




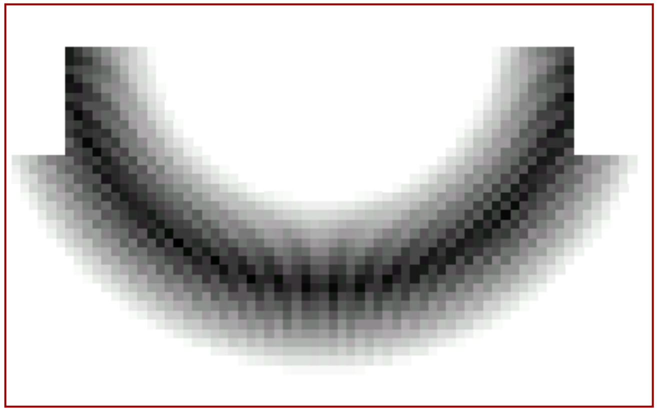
Distributions

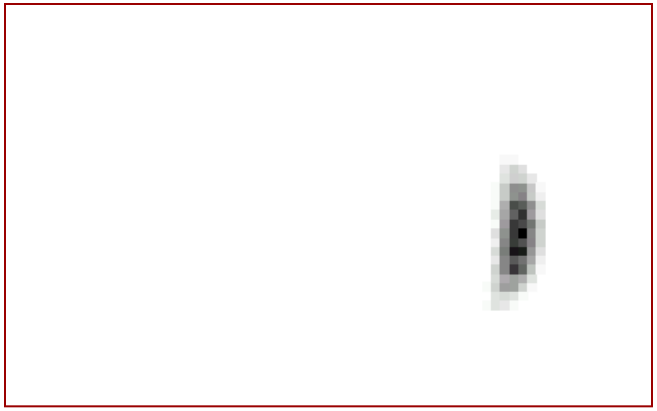






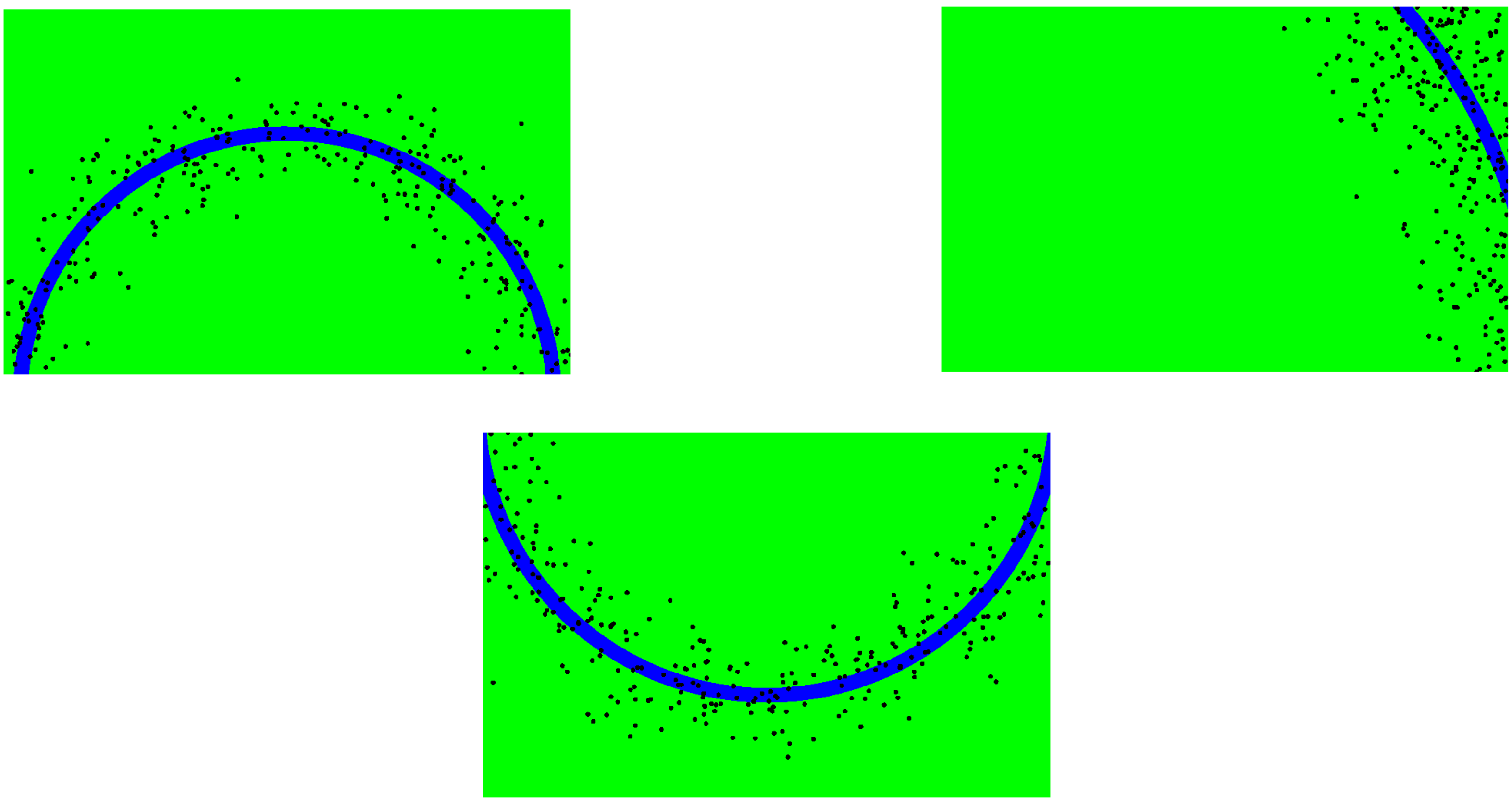
Wanted: samples distributed according to
$p(x|\ z_1, z_2, z_3)$





This is Easy!

We can draw samples from $p(x|z_l)$ by adding noise to the detection parameters.



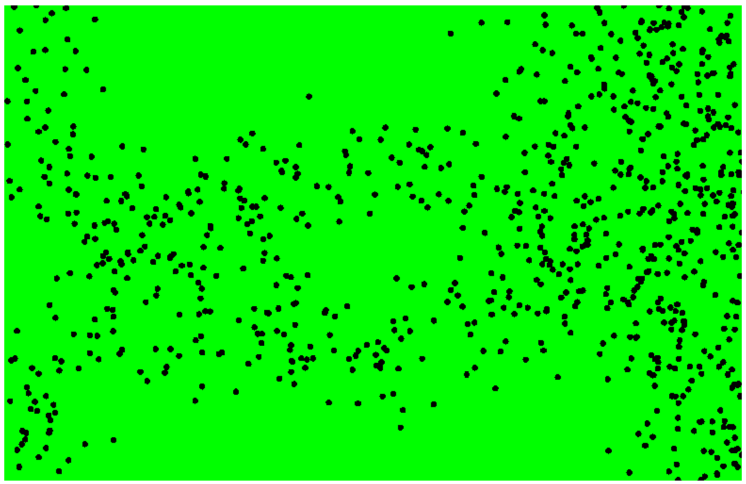
Importance Sampling

$$\text{Target distribution f}: p(x \mid z_1, z_2, \ldots, z_n) = \frac{\prod_k p(z_k \mid x) \quad p(x)}{p(z_1, z_2, \ldots, z_n)}$$

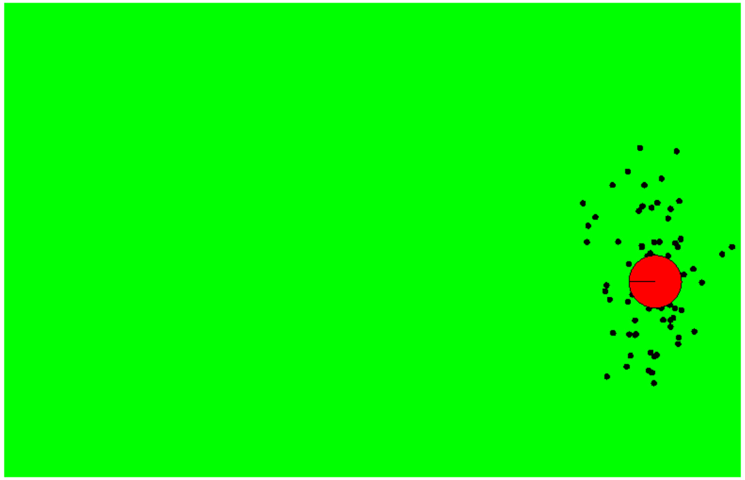
$$\text{Sampling distribution g}: p(x \mid z_l) = \frac{p(z_l \mid x) p(x)}{p(z_l)}$$

$$\text{Importance weights w}: \frac{f}{g} = \frac{p(x \mid z_1, z_2, \ldots, z_n)}{p(x \mid z_l)} = \frac{p(z_l) \prod_{k \neq l} p(z_k \mid x)}{p(z_1, z_2, \ldots, z_n)}$$

Importance Sampling with Resampling



Weighted samples



After resampling

Particle Filter Localization

$$Bel(x \mid z, u) = \boxed{\alpha\, p(z \mid x)} \int_{x'} \boxed{p(x \mid u, x')} \boxed{Bel(x')dx'}$$

1. Draw x' from $Bel(x)$

2. Draw x from $p(x \mid u, x')$

3. Importance factor for x : $w = \alpha\, p(z \mid x)$

4. Re-sample

Rejection Sampling

2. Draw x from $p(x \mid u, x')$

- Let us assume that $f(x) < \mathrm{a}$ for all x
- Sample x from a uniform distribution
- Sample c from [0,a]
- if $f(x) > c$ keep the sample
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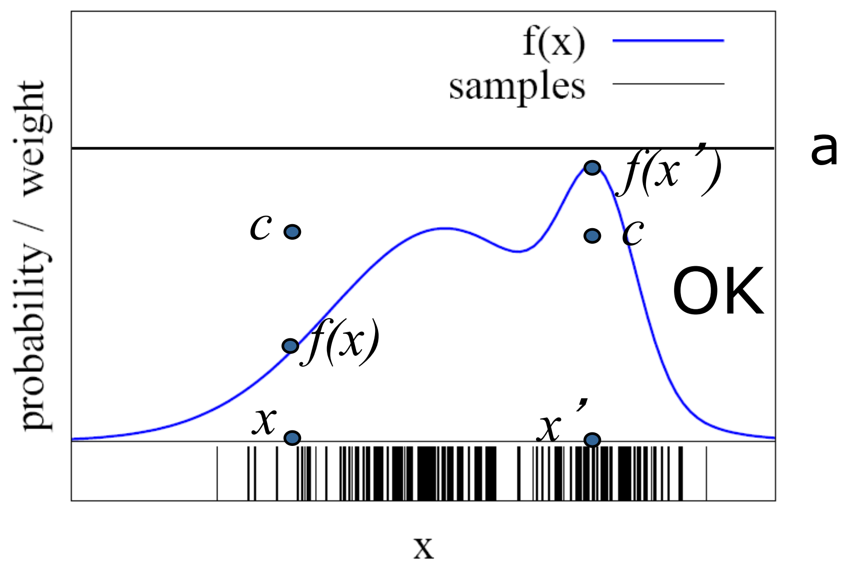

Importance Sampling Principle

3. Importance factor for $x : w = \alpha\, p(z \mid x)$

- We can even use a different distribution g to generate samples from f
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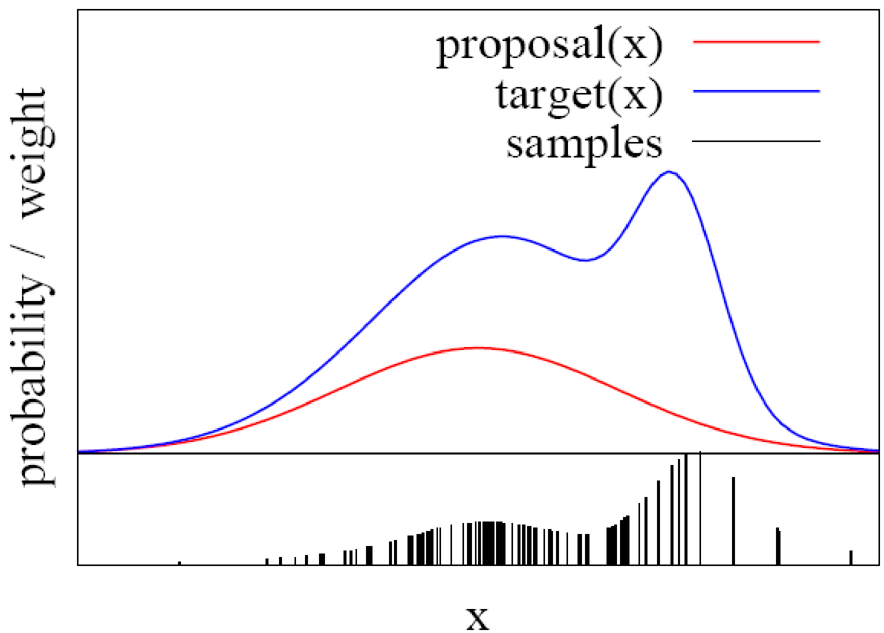
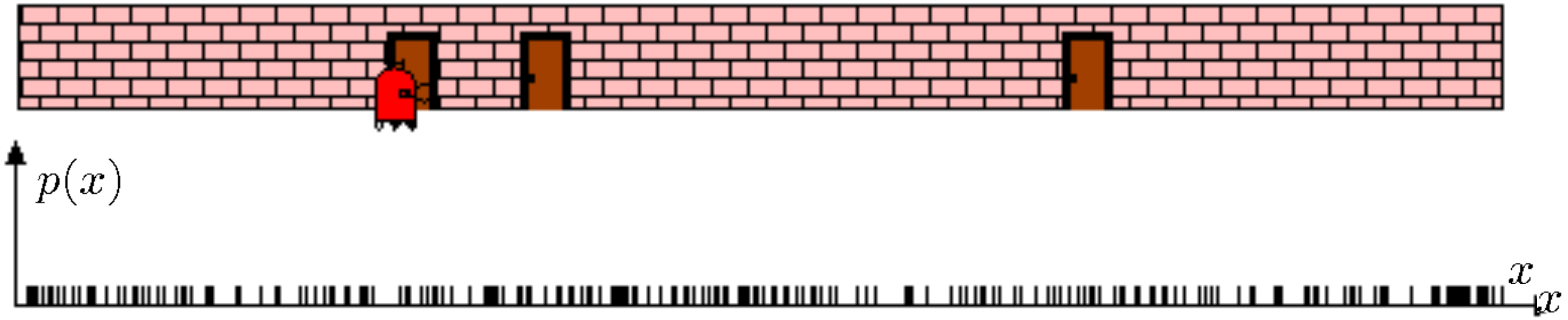


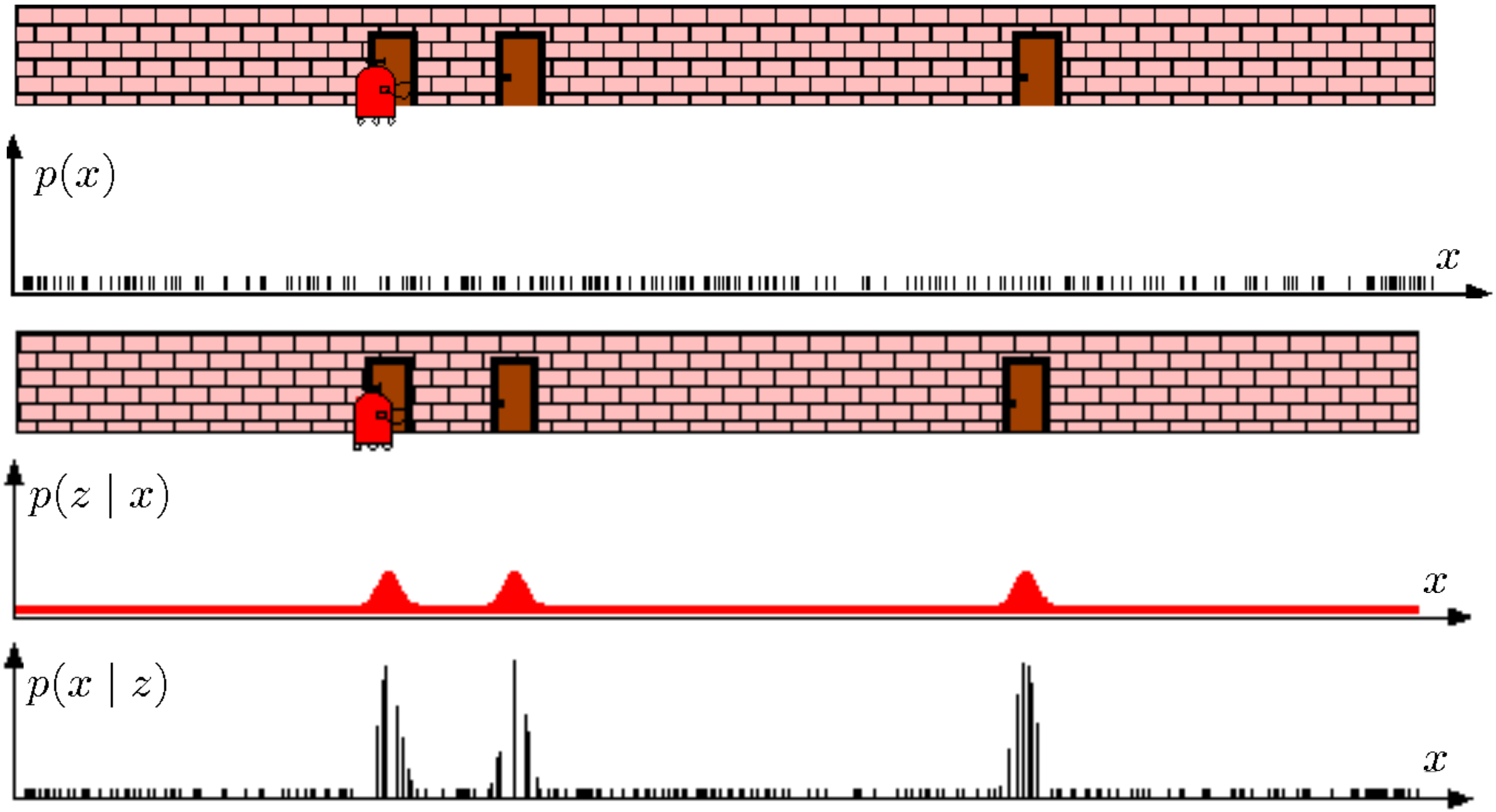
Particle Filters



Sensor Information: Importance Sampling

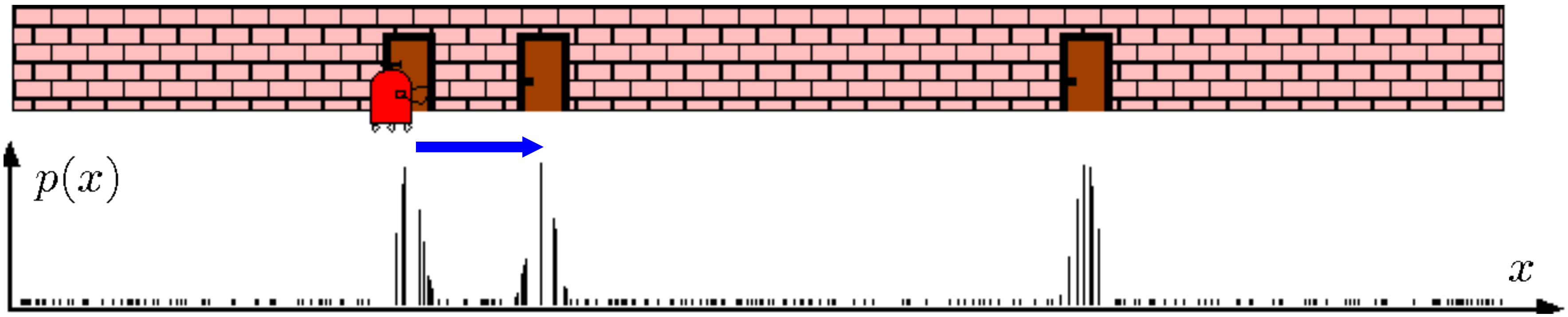
$$Bel(x) \leftarrow \alpha\, p(z \mid x)\, Bel^{-}(x)$$

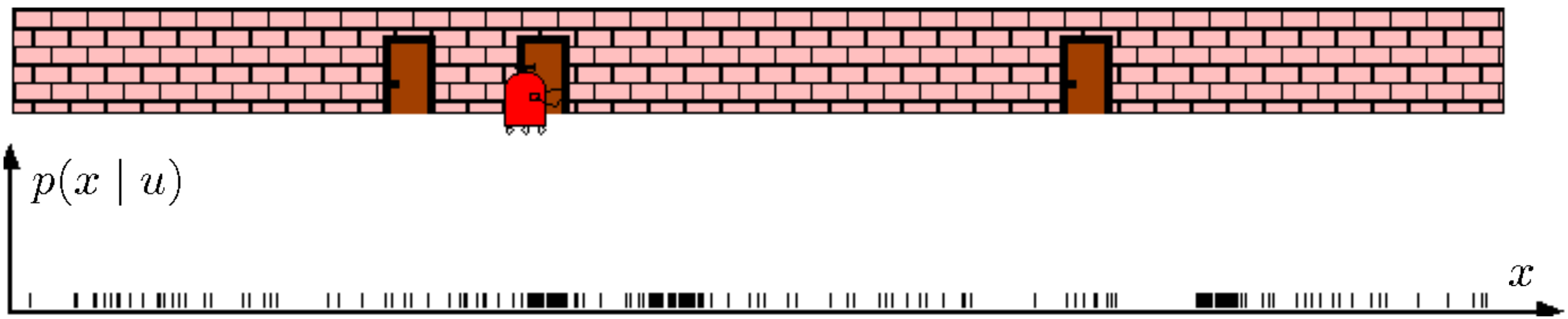
$$w \leftarrow \frac{\alpha\, p(z \mid x)\, Bel^{-}(x)}{Bel^{-}(x)} = \alpha\, p(z \mid x)$$



Robot Motion

$$Bel^-(x) \leftarrow \int p(x \mid u, x')\, Bel(x')\, \mathrm{d}x'$$





Sensor Information: Importance Sampling

$$Bel(x) \leftarrow \alpha\, p(z \mid x)\, Bel^{-}(x)$$

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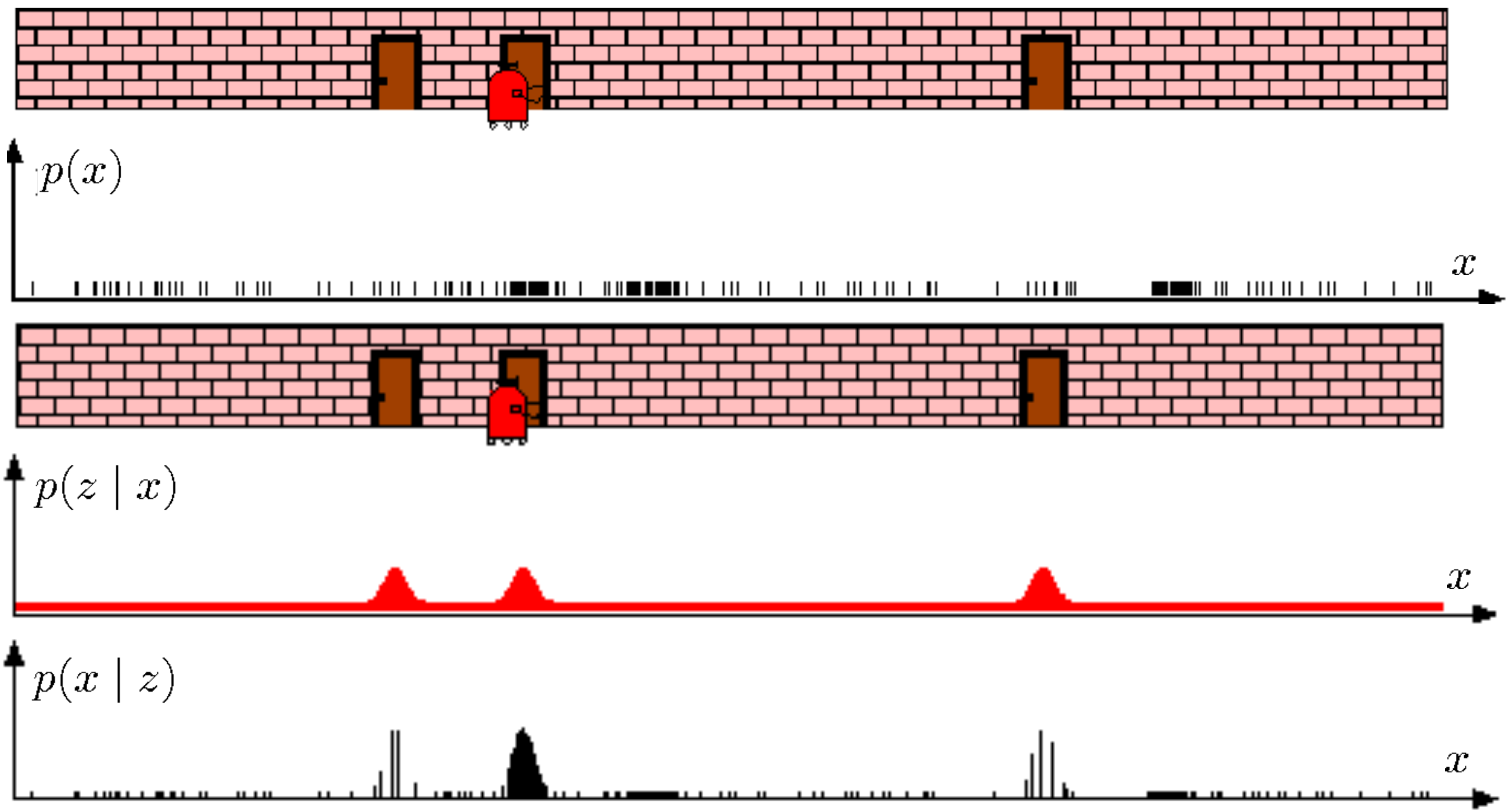
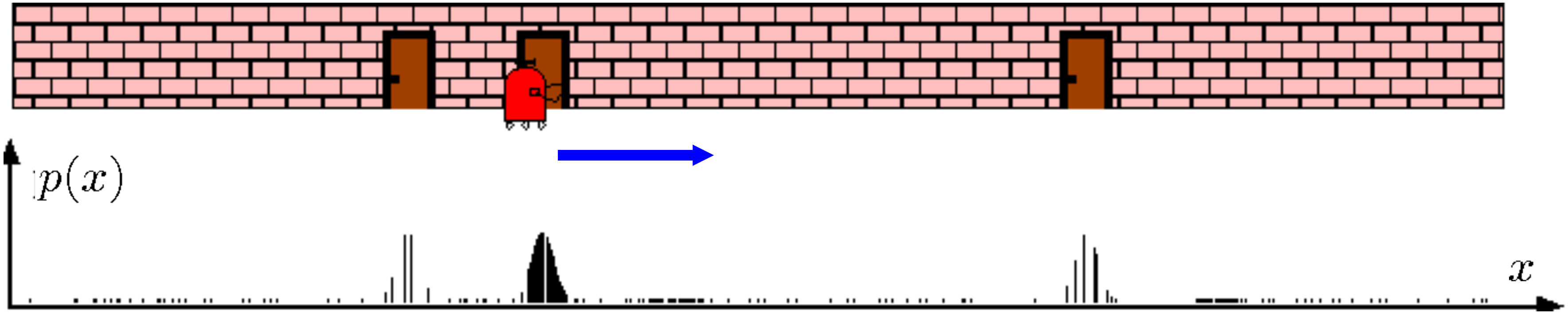
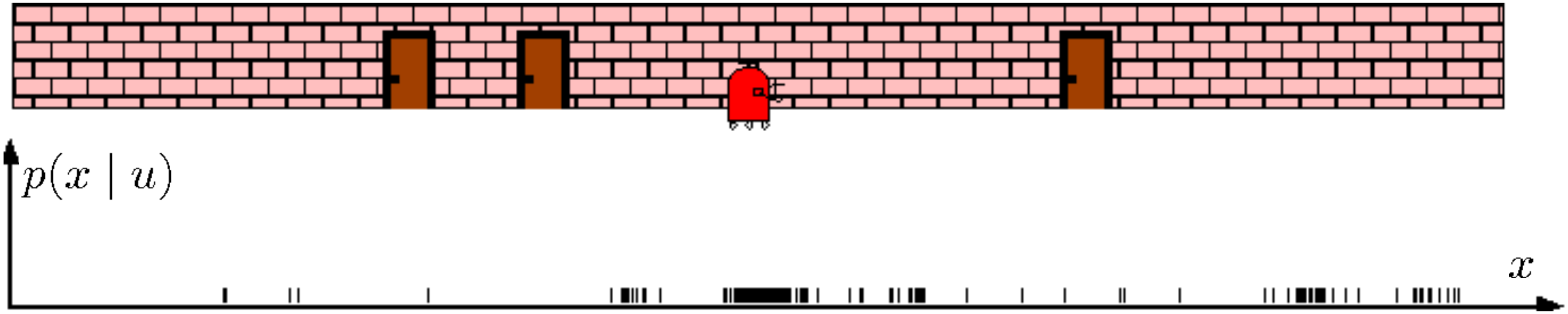


Robot Motion

$$Bel^{-}(x) \leftarrow \int p(x \mid u, x')\, Bel(x')\, \mathrm{d}x'$$





Particle Filter Algorithm

- Sample the next generation for particles using the proposal distribution

- Compute the importance weights :

$$weight = target\ distribution \,/\, proposal\ distribution$$

- Resampling: “Replace unlikely samples by more likely ones”

Particle Filter Algorithm

1. Algorithm **particle_filter**(S_{t-1}, u_t, z_t):
2. $S_t = \emptyset, \quad \eta = 0$
3. **For** $i = 1, \ldots, n$ *Generate new samples*
4. Sample index *j(i)* from the discrete distribution given by w_{t-1}
5. Sample x_t^i from $p(x_t \mid x_{t-1}, u_t)$ using $x_{t-1}^{j(i)}$ and u_t
6. $w_t^i = p(z_t \mid x_t^i)$ *Compute importance weight*
7. $\eta = \eta + w_t^i$ *Update normalization factor*
8. $S_t = S_t \cup \{< x_t^i, w_t^i >\}$ *Add to new particle set*
9. **For** $i = 1, \ldots, n$
10. $w_t^i = w_t^i / \eta$ *Normalize weights*

Particle Filter Algorithm

$$Bel(x_t) = \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, Bel(x_{t-1})\, dx_{t-1}$$

draw x^i_{t-1} from $Bel(x_{t-1})$

draw x^i_t from $p(x_t \mid x^i_{t-1}, u_t)$

Importance factor for x^i_t:

$$w^i_t = \frac{\text{target distribution}}{\text{proposal distribution}}$$

$$= \frac{\eta\, p(z_t \mid x_t)\, p(x_t \mid x_{t-1}, u_t)\, Bel\ (x_{t-1})}{p(x_t \mid x_{t-1}, u_t)\, Bel\ (x_{t-1})}$$

$$\propto p(z_t \mid x_t)$$

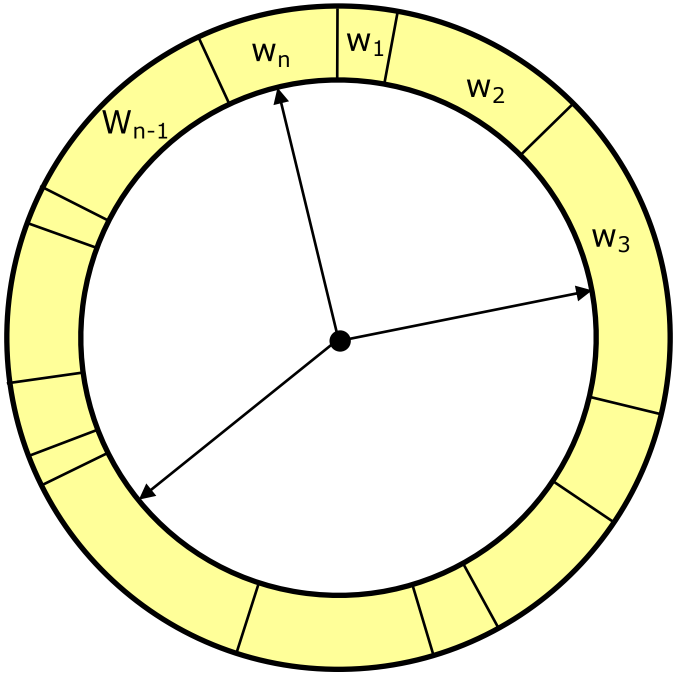
Resampling

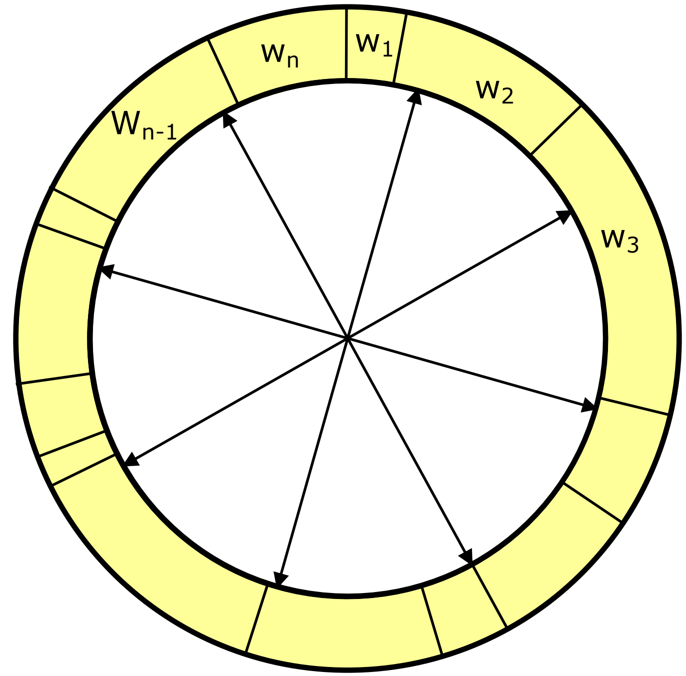
- **Given**: Set S of weighted samples.

- **Wanted** : Random sample, where the probability of drawing x_i is given by w_i.

- Typically done n times with replacement to generate new sample set S'.

Resampling





- Roulette wheel
- Binary search, n log n

- Stochastic universal sampling
- Systematic resampling
- Linear time complexity
- Easy to implement, low variance

Resampling Algorithm

1. Algorithm **systematic_resampling**(*S*,*n*):

2. $S' = \emptyset, c_1 = w^1$
3. **For** $i = 2 \ldots n$ ***Generate cdf***
4. $c_i = c_{i-1} + w^i$
5. $u_1 \sim U]0, n^{-1}], i = 1$ ***Initialize threshold***

6. **For** $j = 1 \ldots n$ ***Draw samples …***
7. **While** ($u_j > c_i$) ***Skip until next threshold reached***
8. $i = i + 1$
9. $S' = S' \cup \left\{ < x^i, n^{-1} > \right\}$ ***Insert***
10. $u_{j+1} = u_j + n^{-1}$ ***Increment threshold***

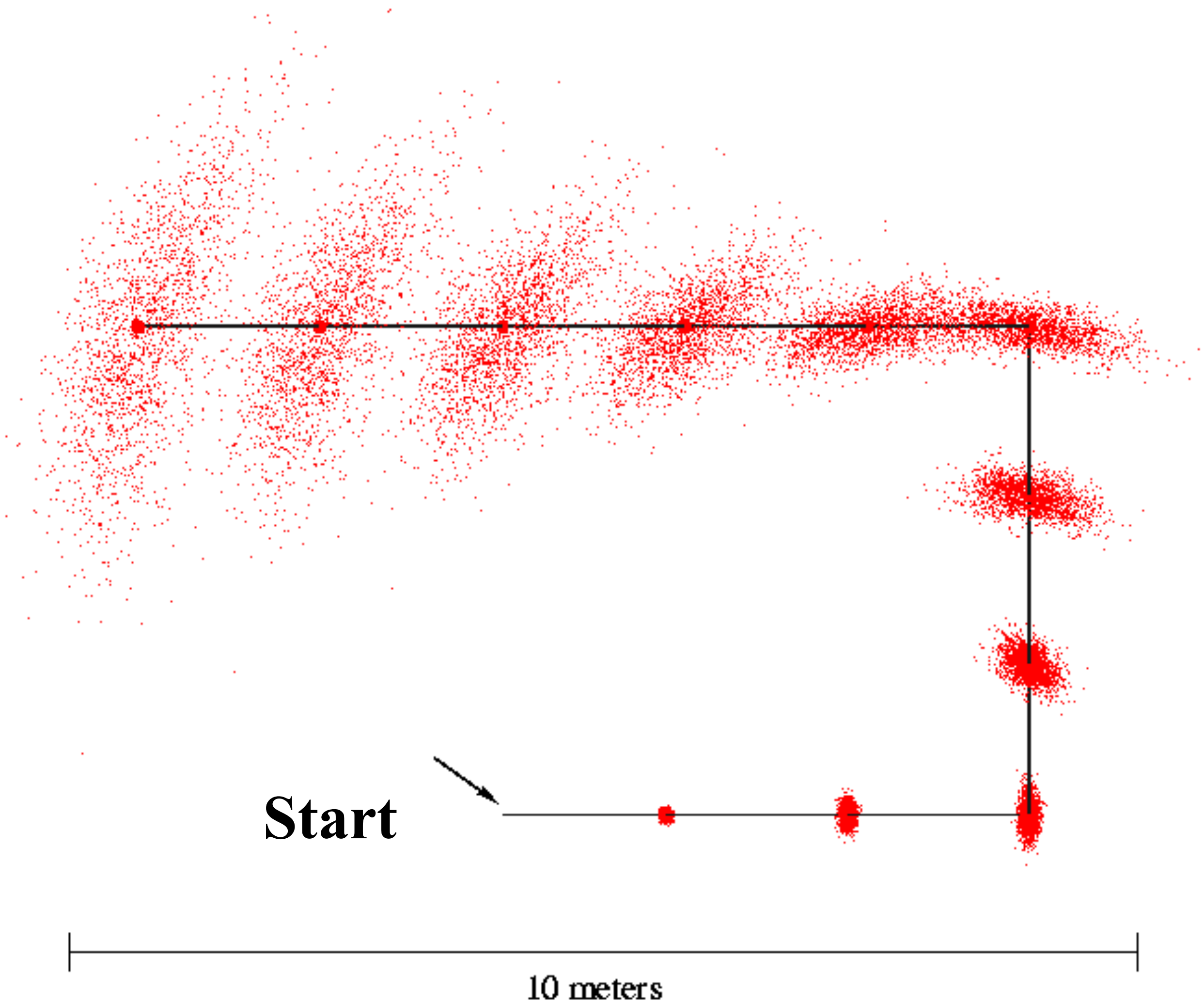
11. **Return** S'

Also called **stochastic universal sampling**

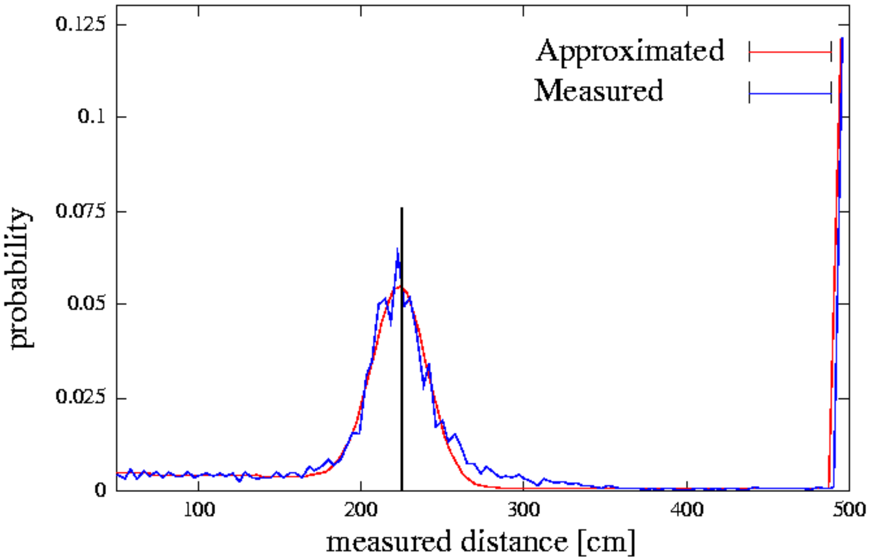
Particle Filters for Mobile Robot Localization

- Each particle is a potential pose of the robot
- Proposal distribution is the motion model of the robot (prediction step)
- The observation model is used to compute the importance weight (correction step)

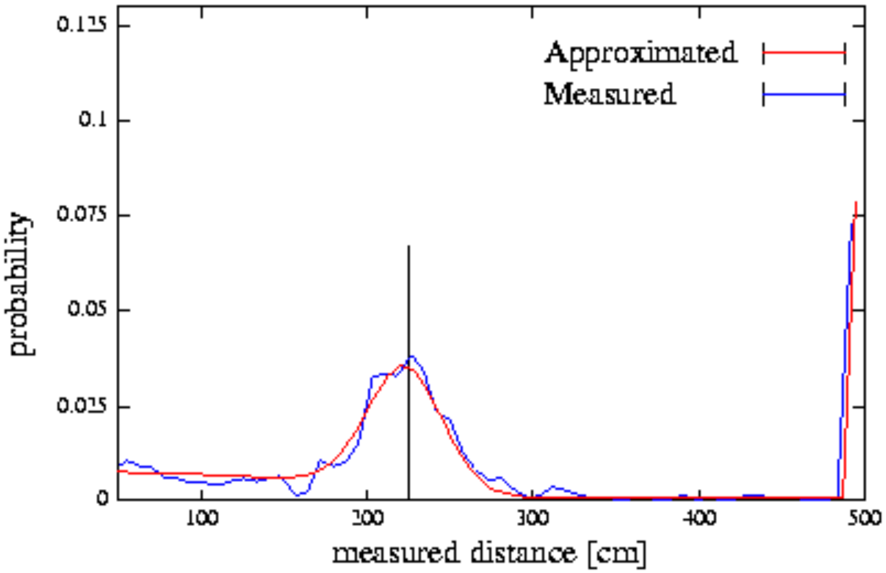
Motion Model



Proximity Sensor Model (Reminder)



Laser sensor



Sonar sensor

Mobile Robot Localization Using Particle Filters (1)

- Each particle is a potential pose of the robot
- The set of weighted particles approximates the posterior belief about the robot's pose (target distribution)

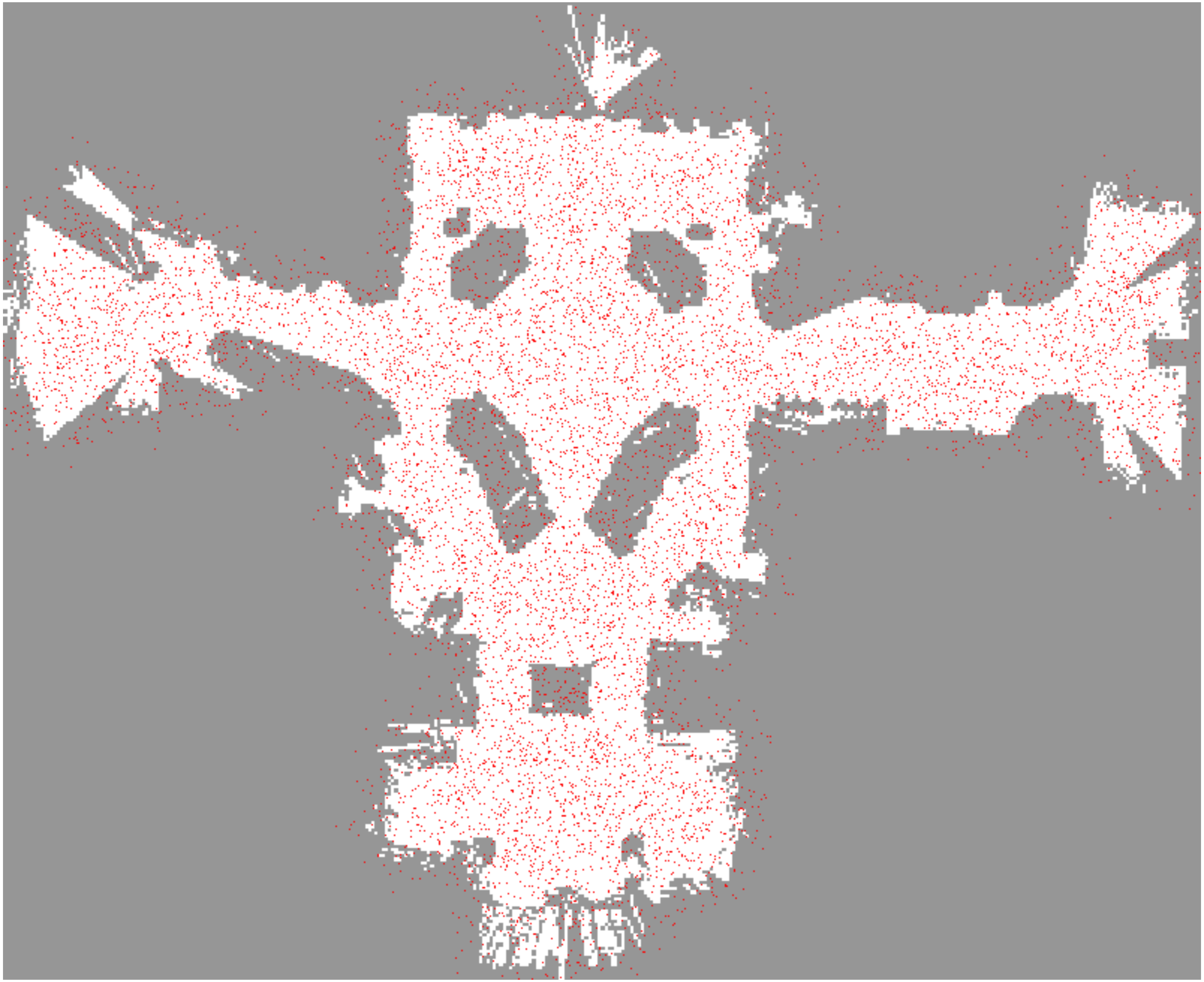
Mobile Robot Localization Using Particle Filters (2)

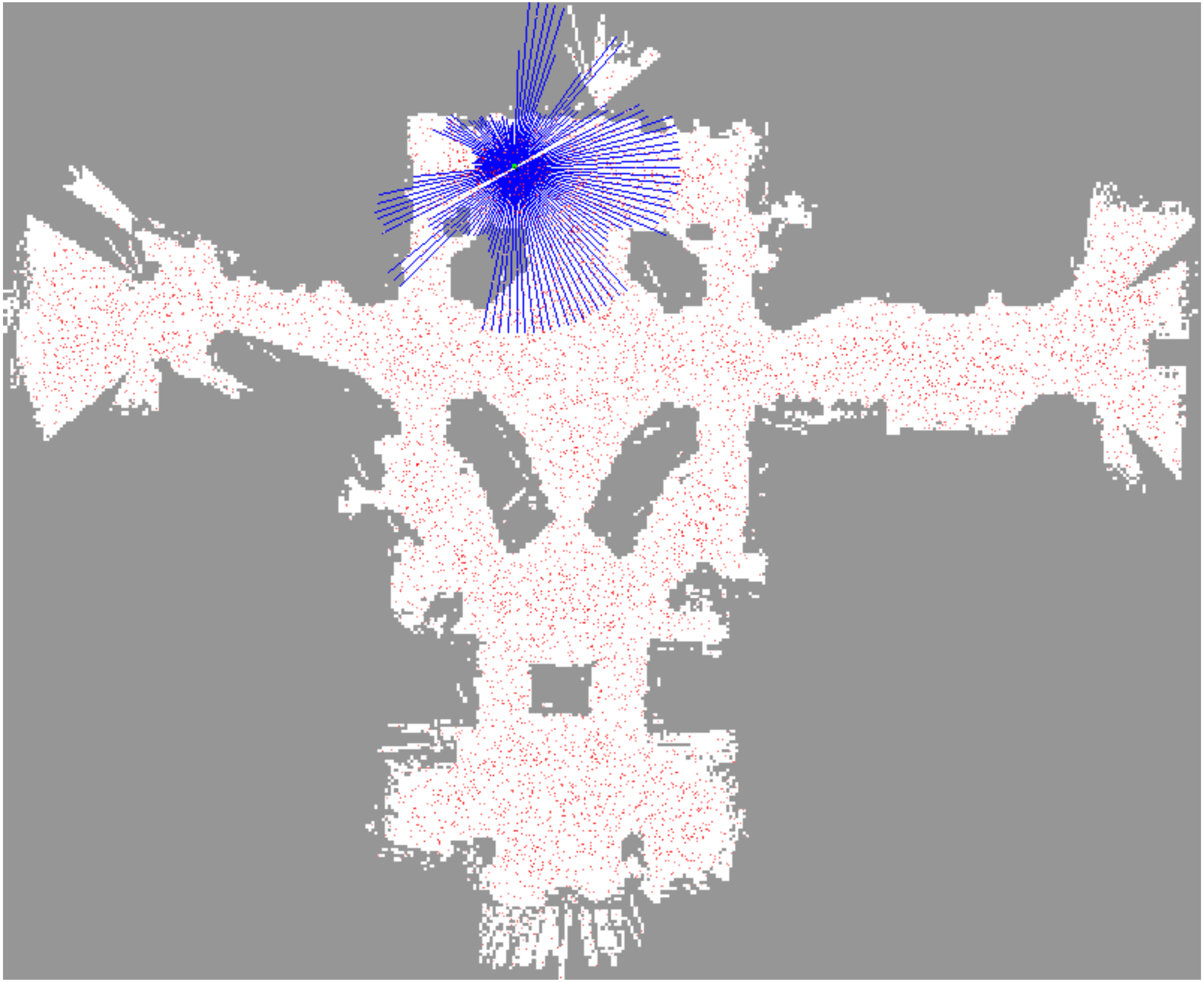
- Particles are drawn from the motion model (proposal distribution)
- Particles are weighted according to the observation model (sensor model)
- Particles are resampled according to the particle weights

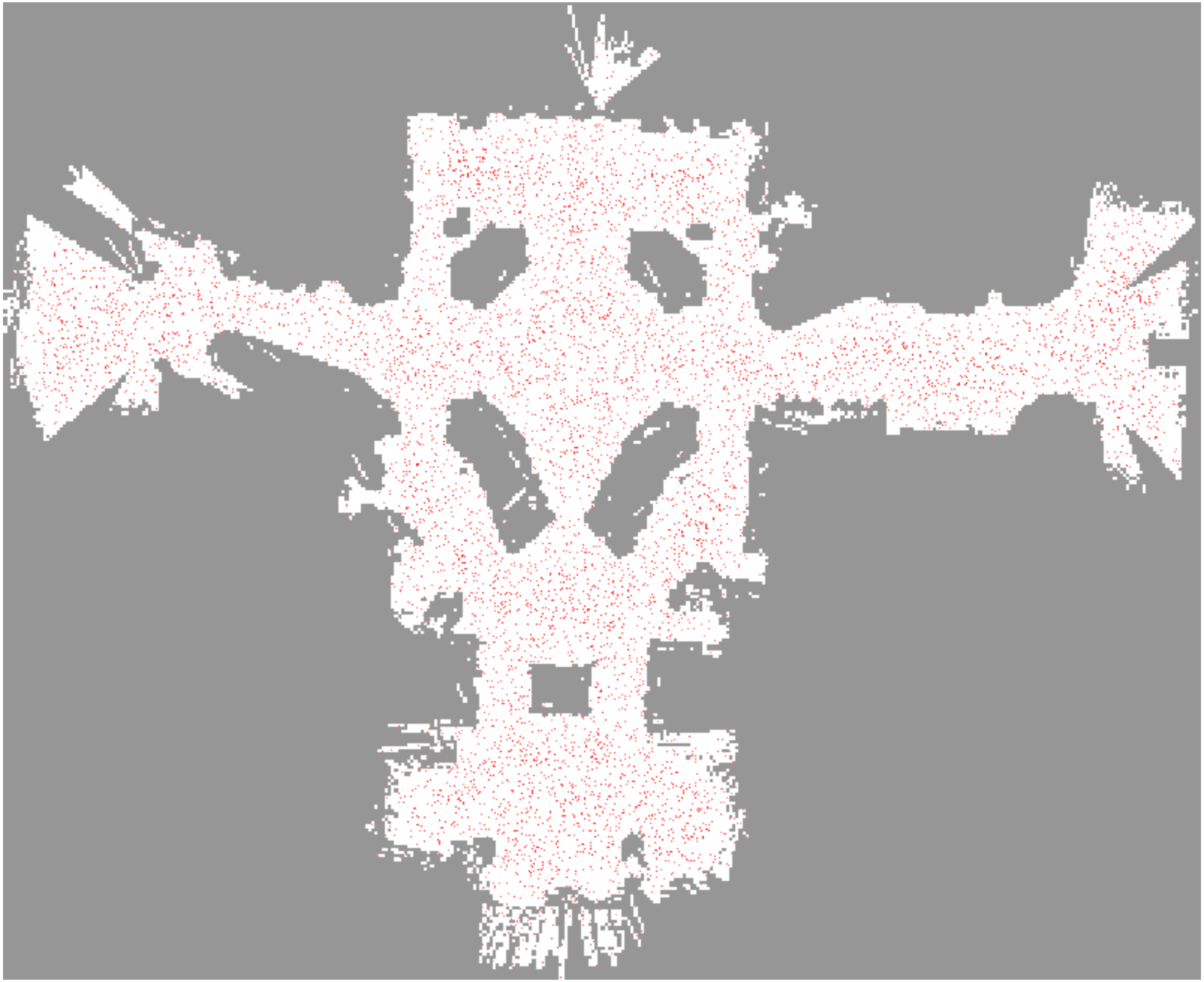
Mobile Robot Localization Using Particle Filters (3)

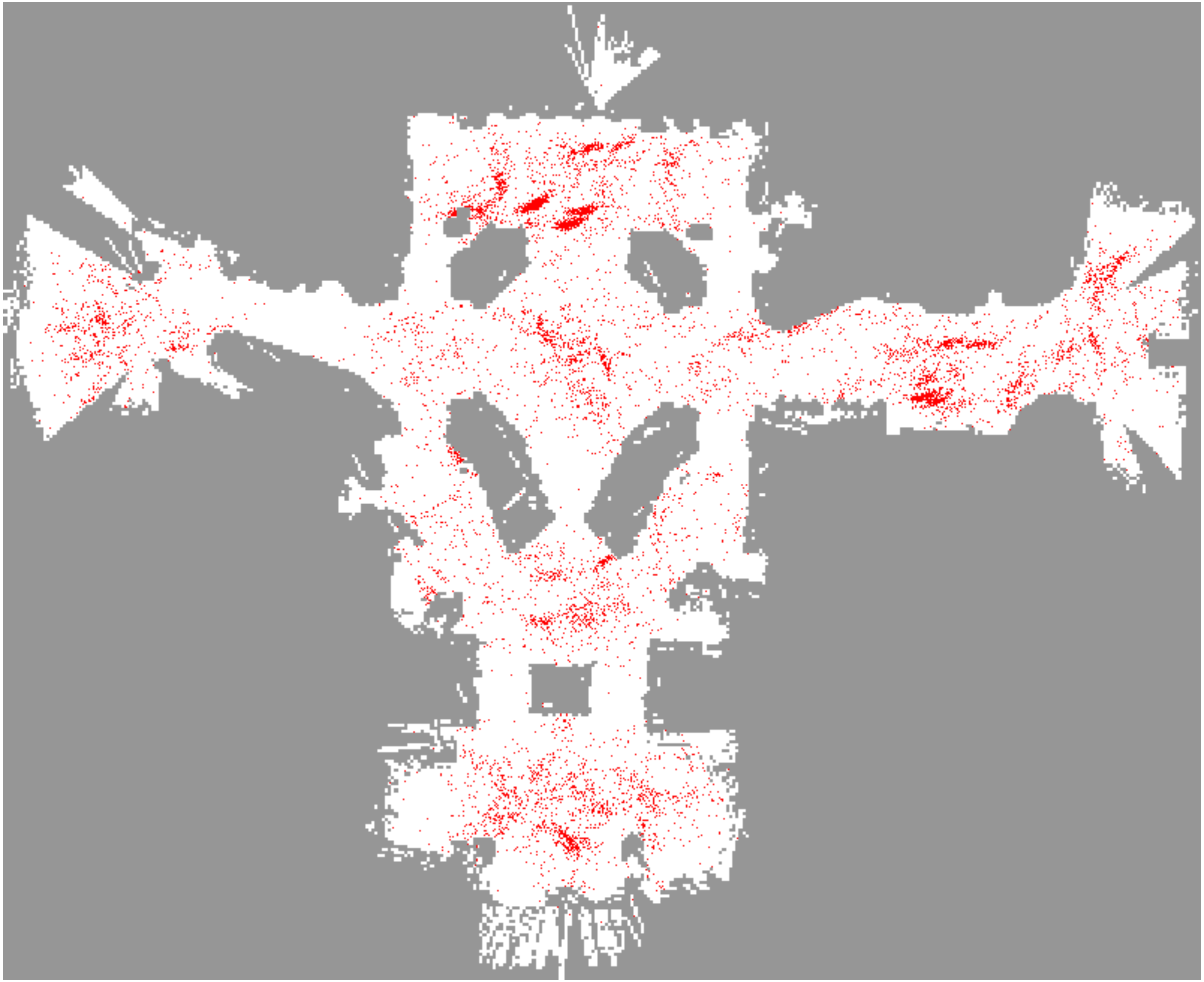
Why is resampling needed?

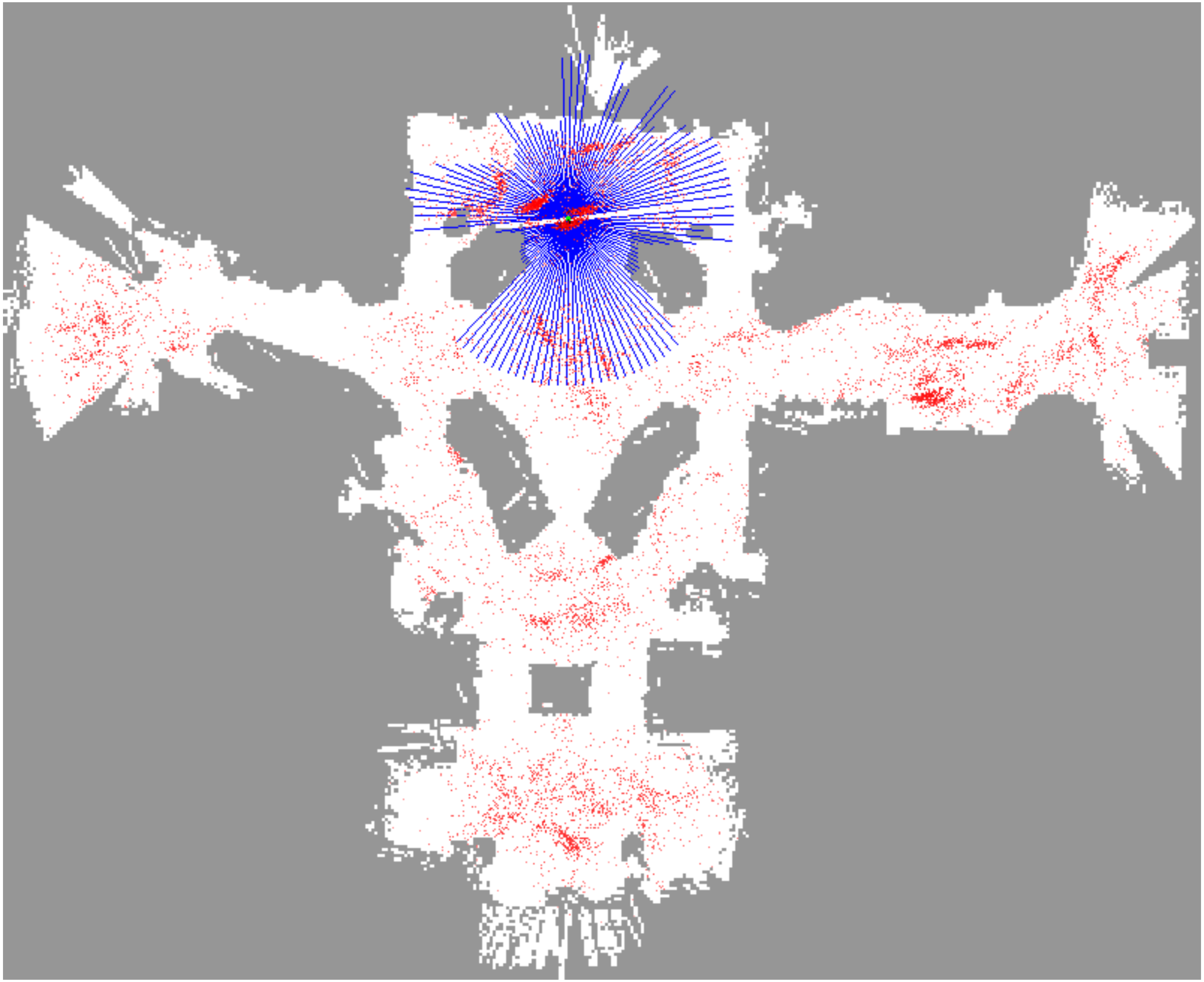
- We only have a finite number of particles
- Without resampling: The filter is likely to loose track of the “good” hypotheses
- Resampling ensures that particles stay in the meaningful area of the state space

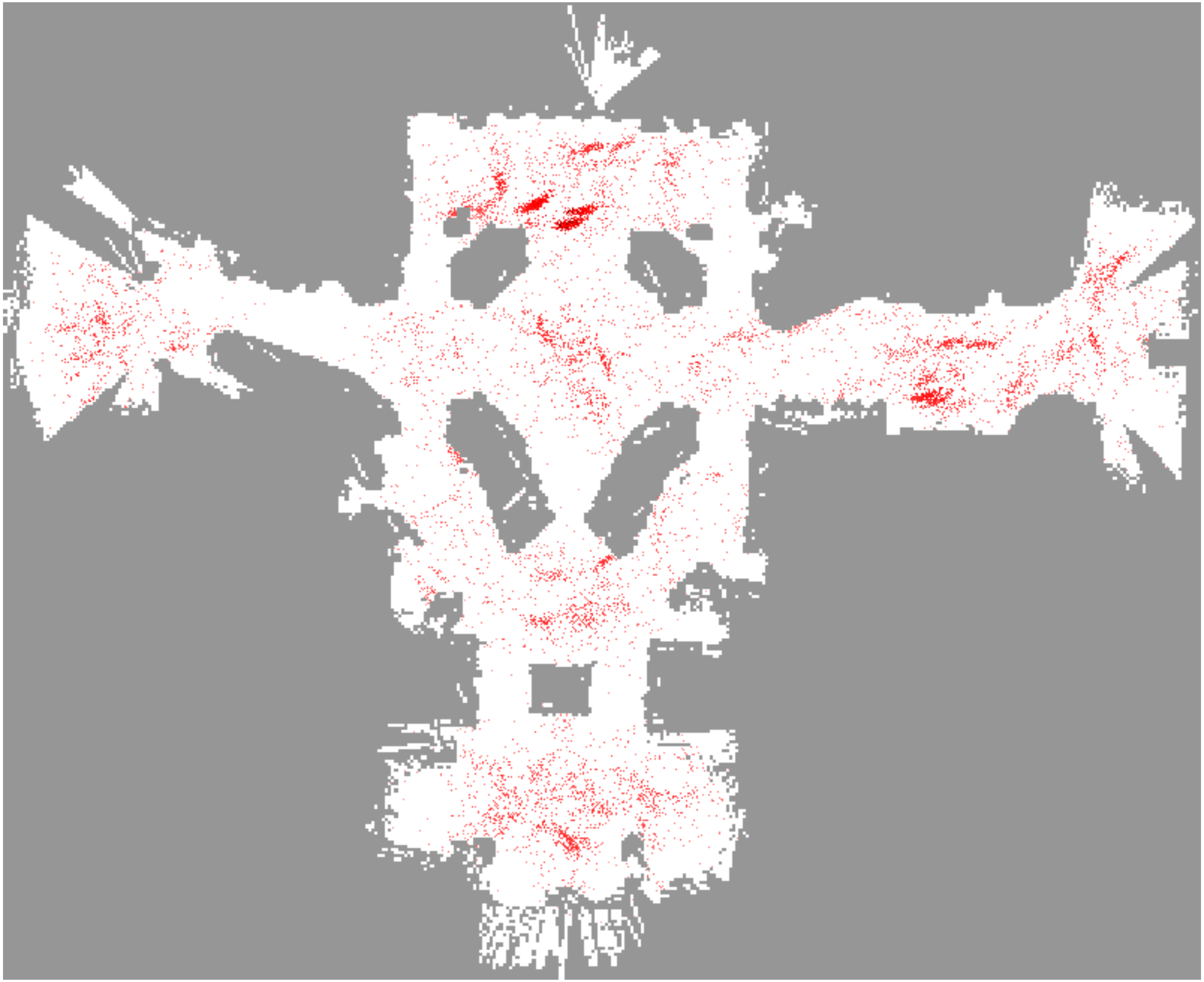


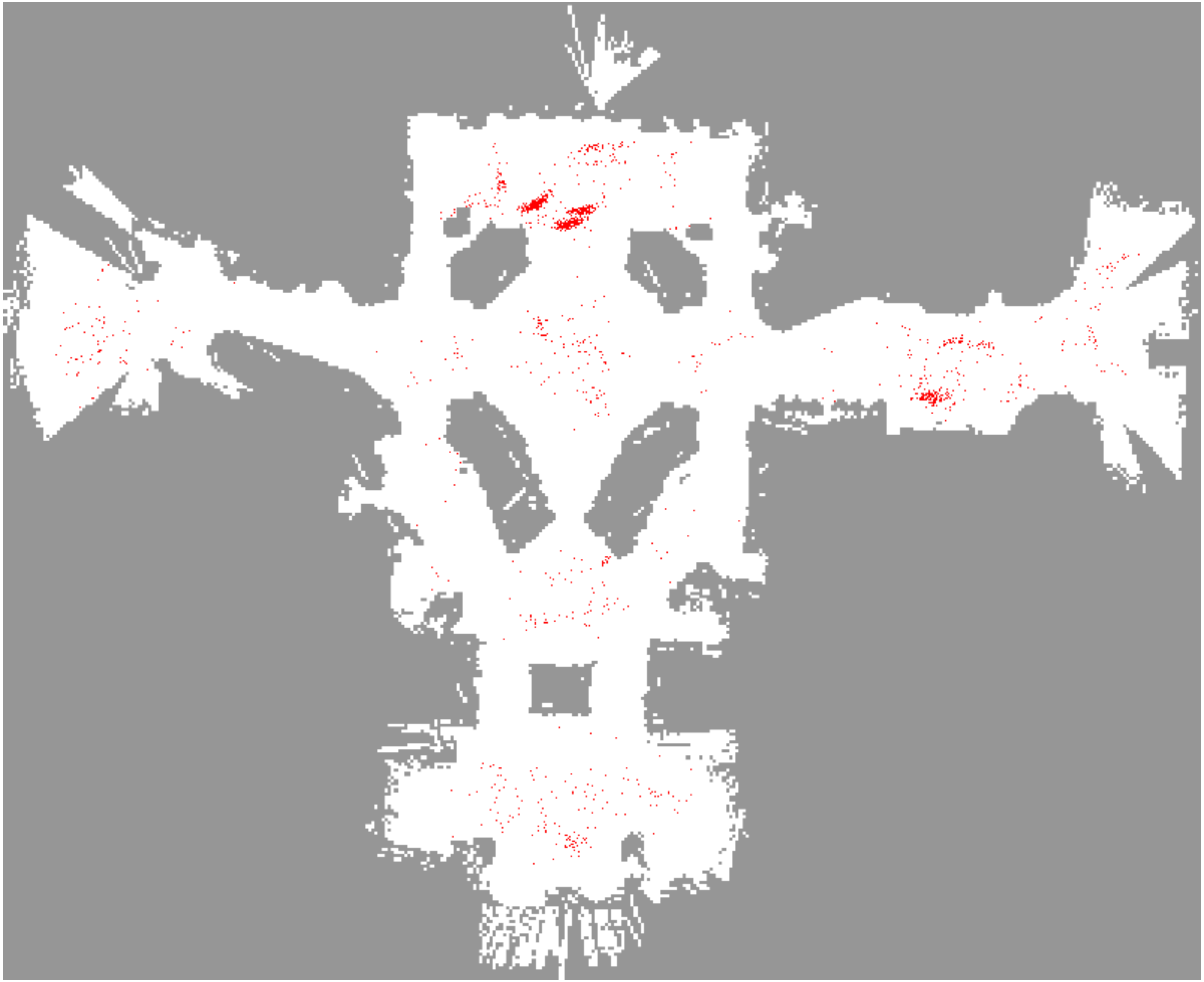


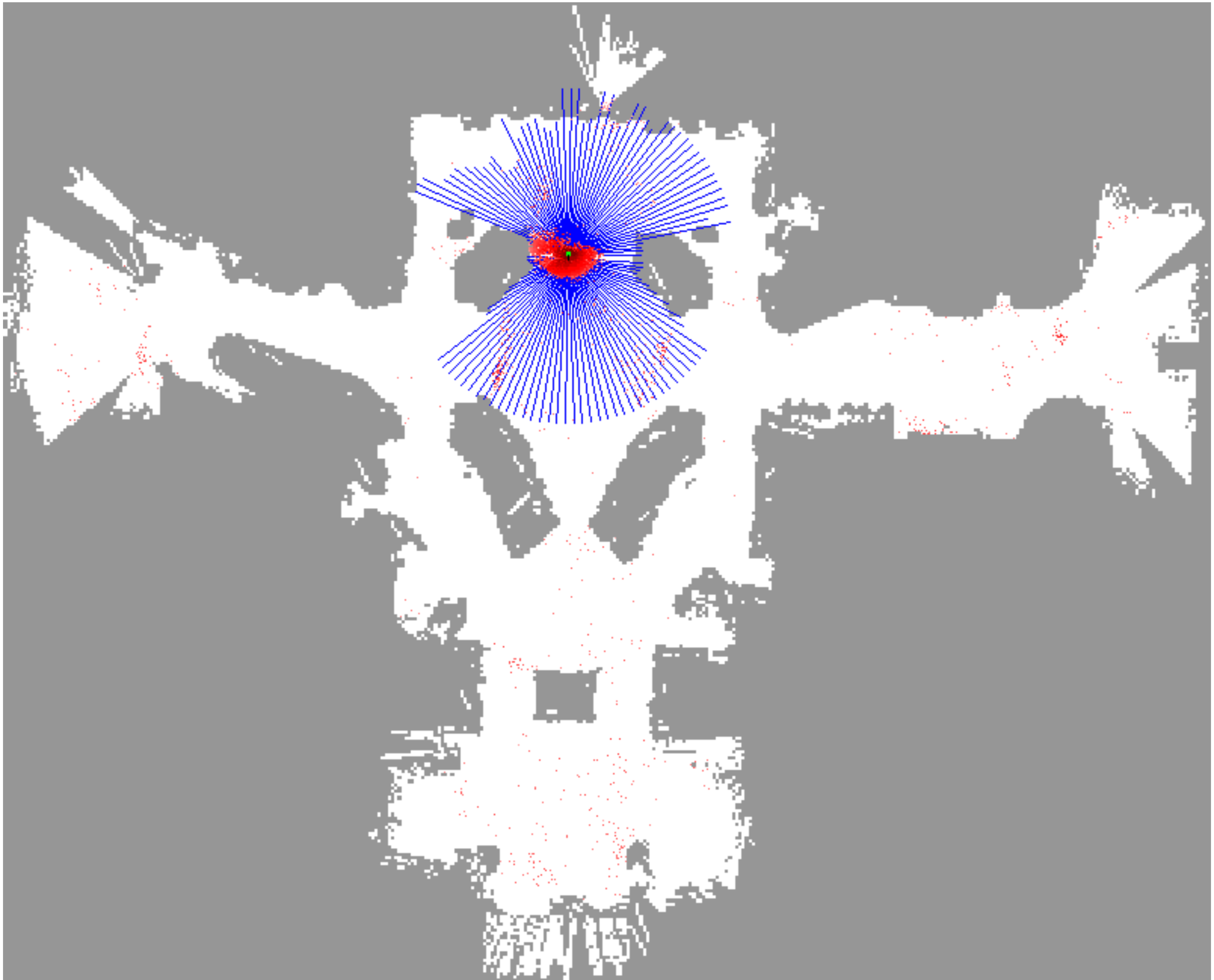


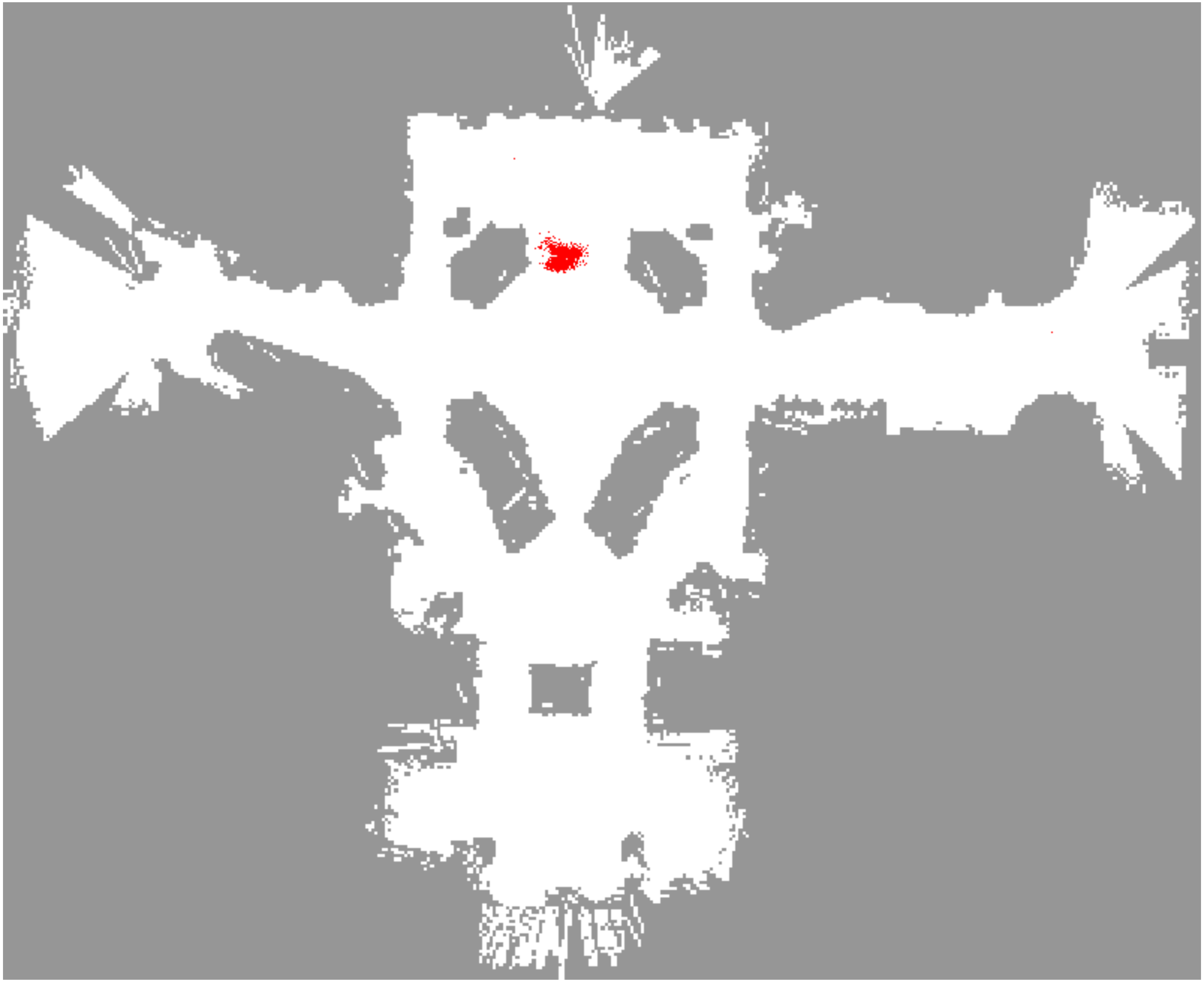


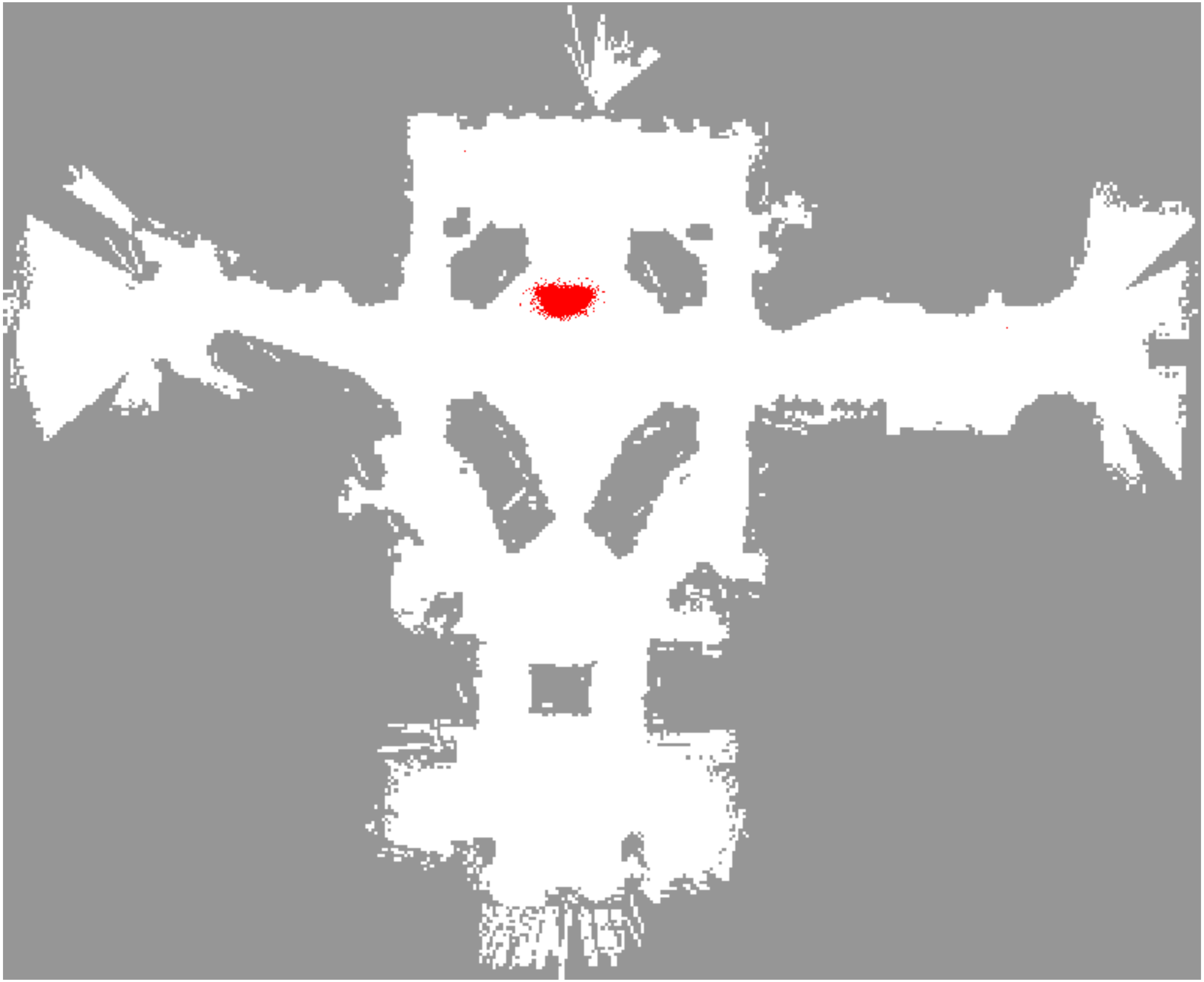


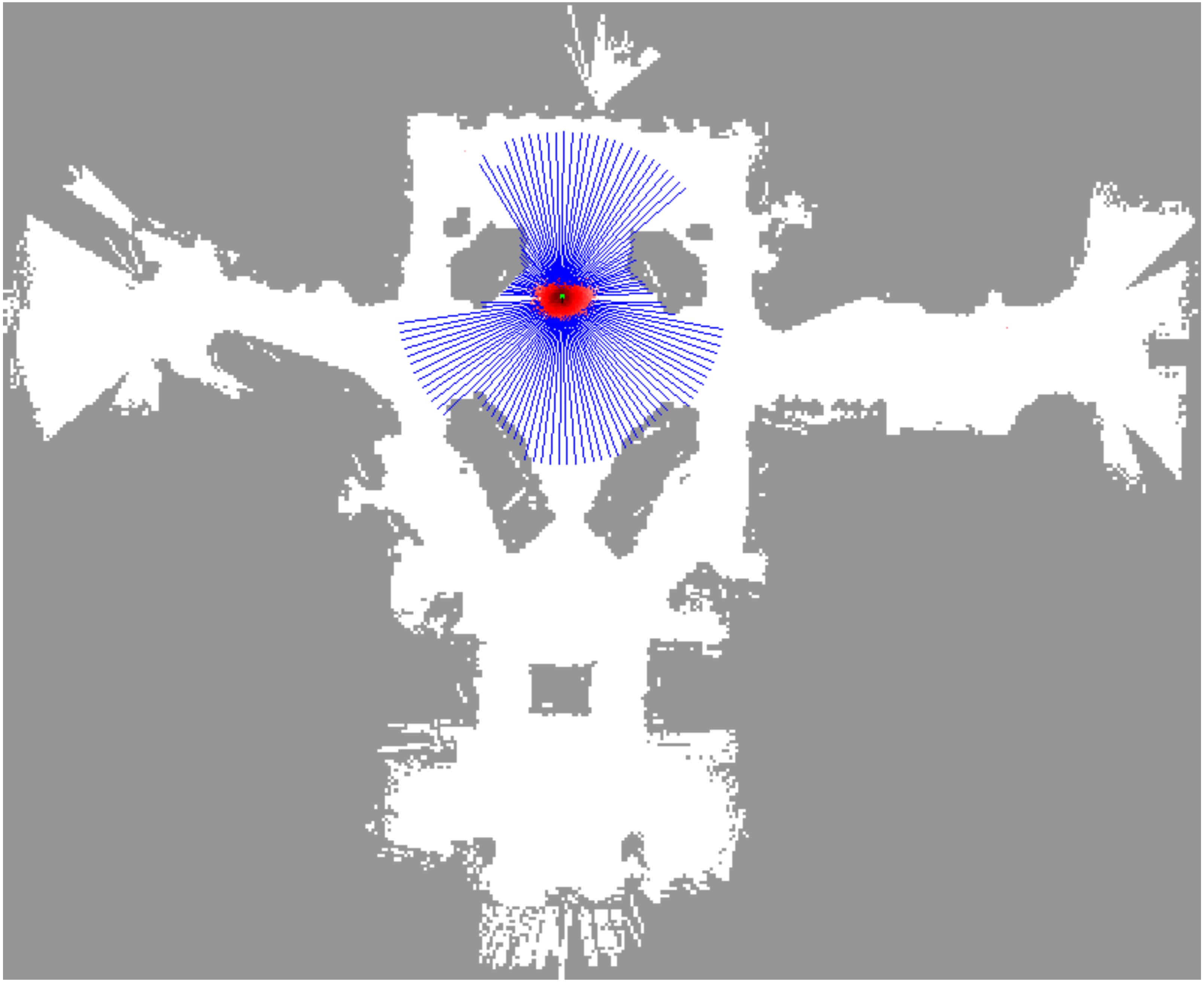


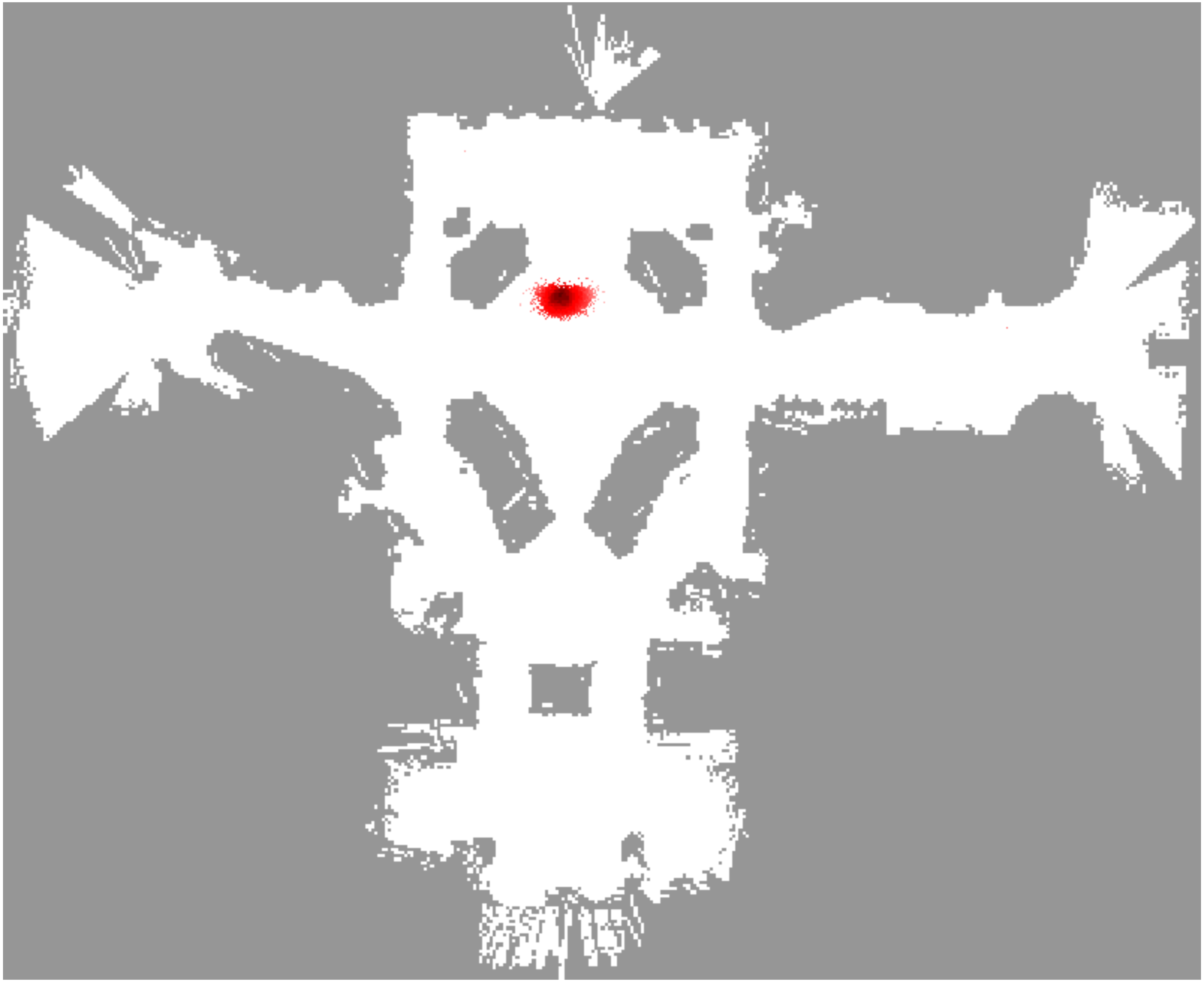


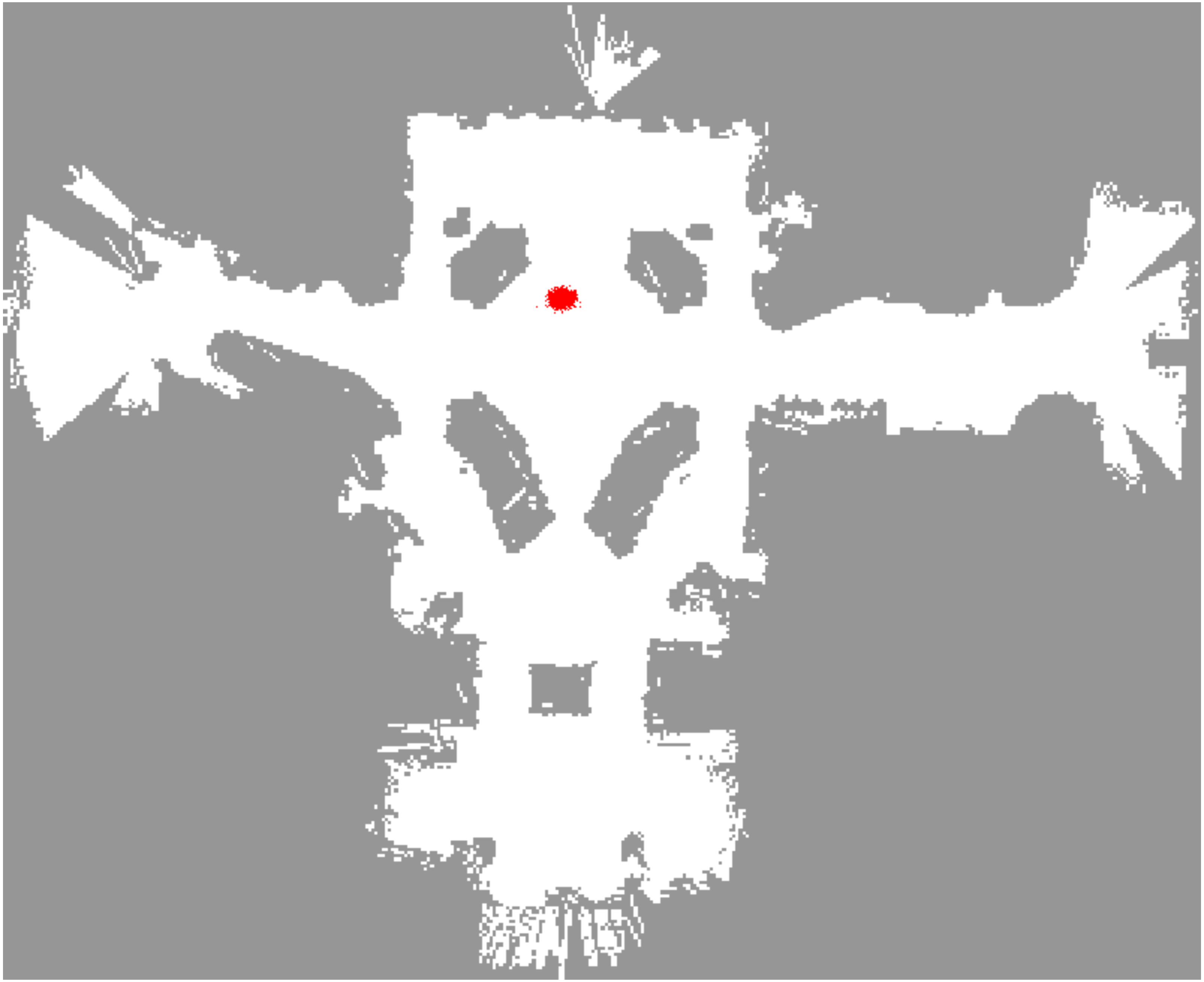


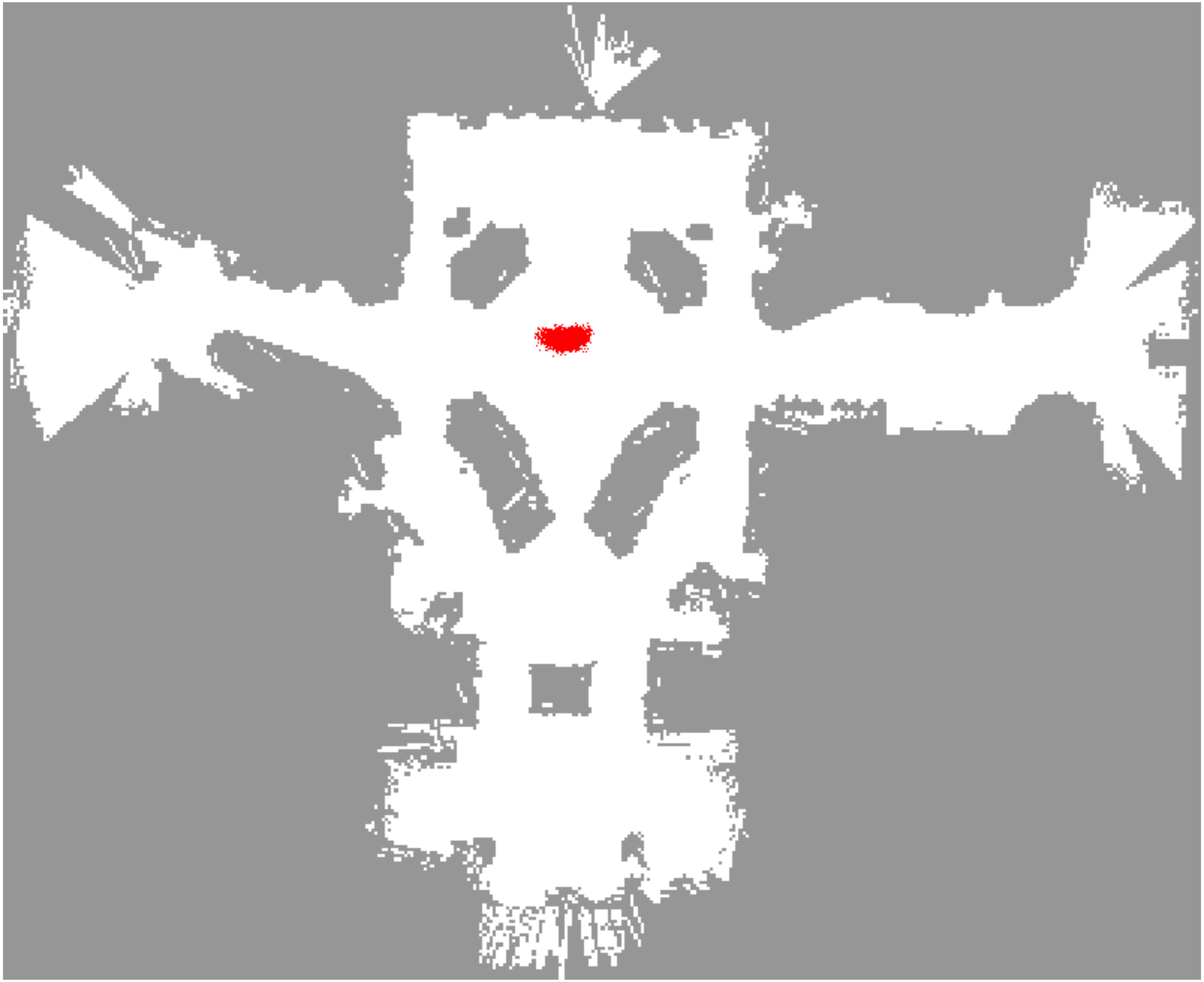


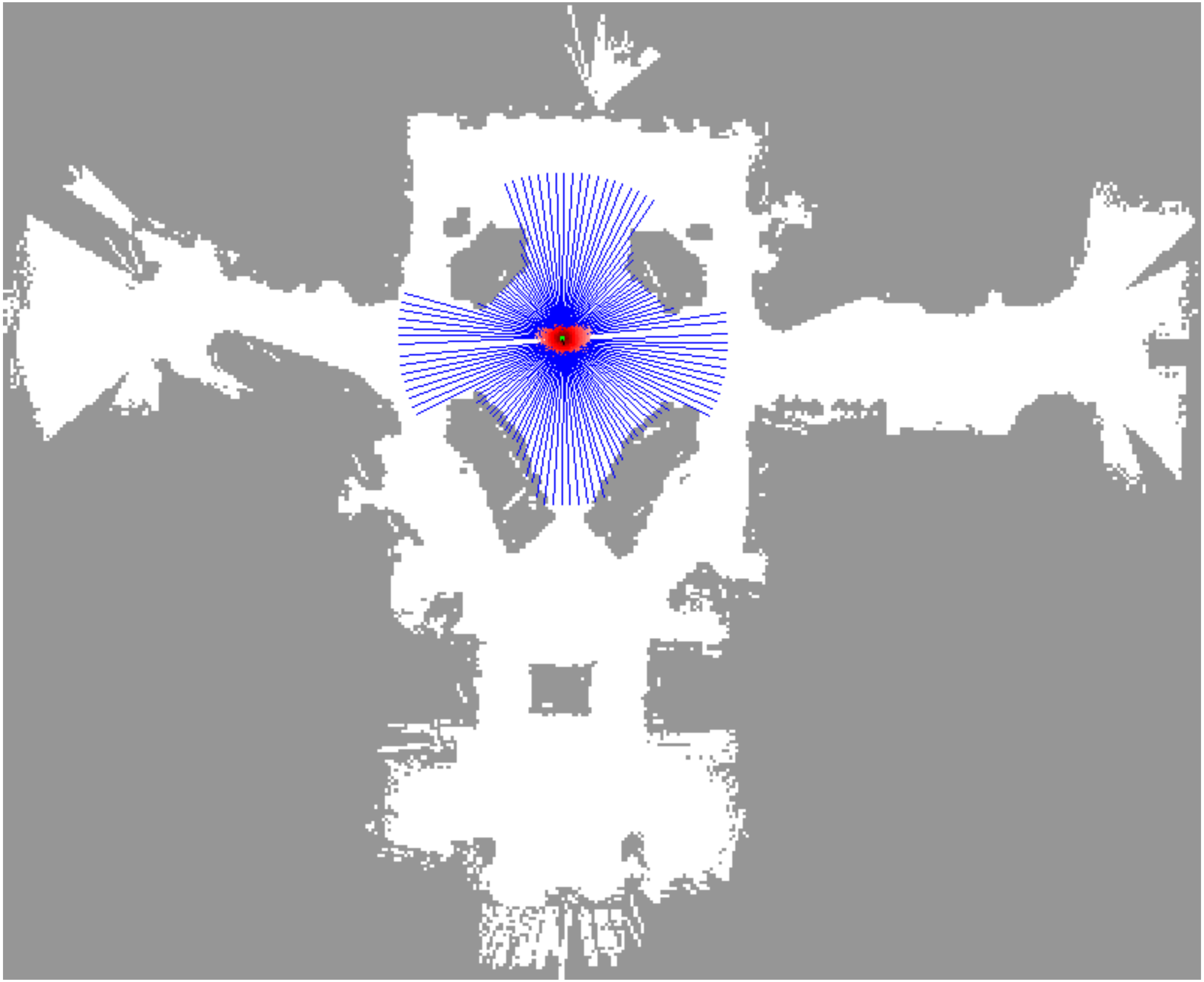




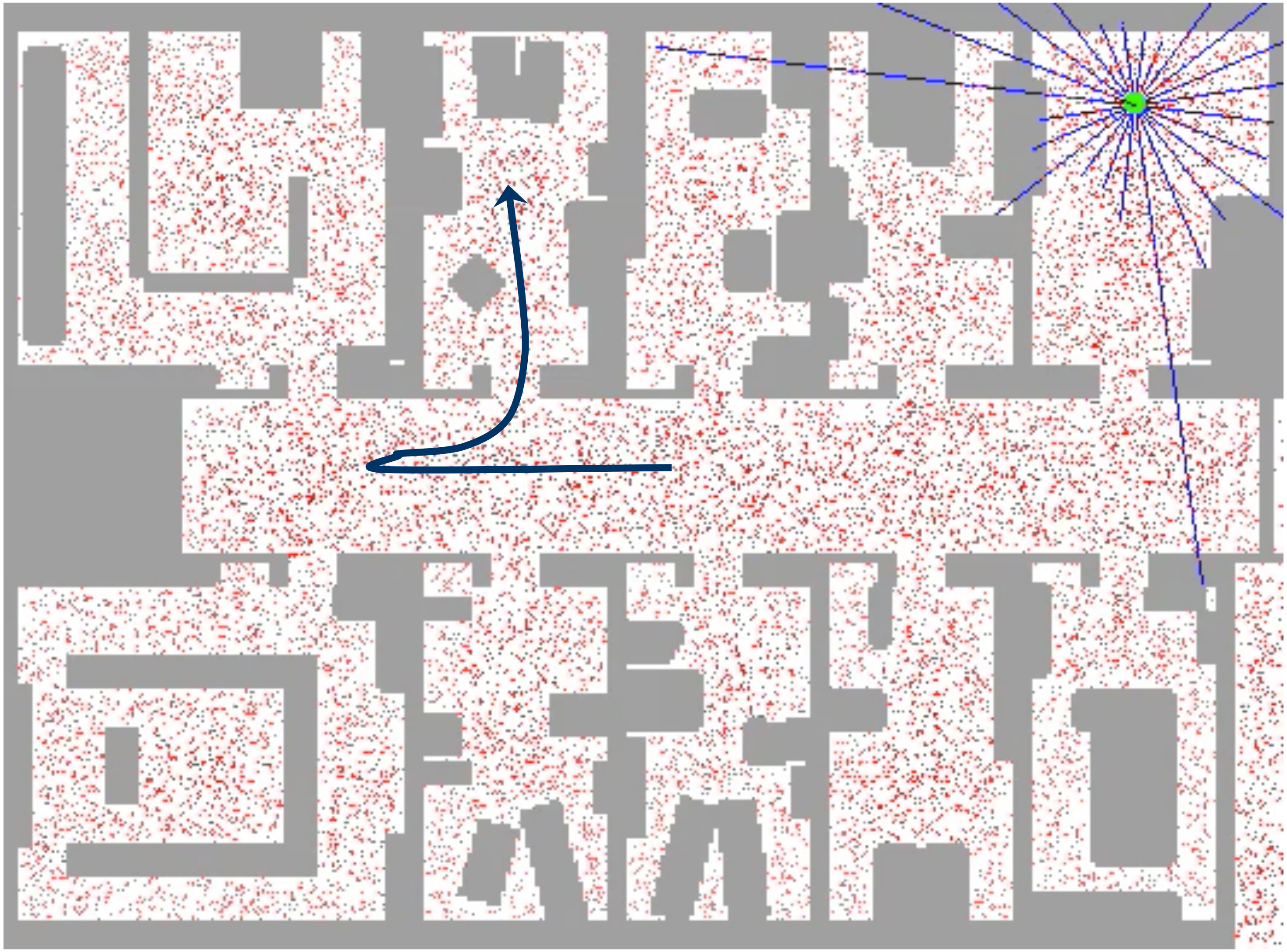




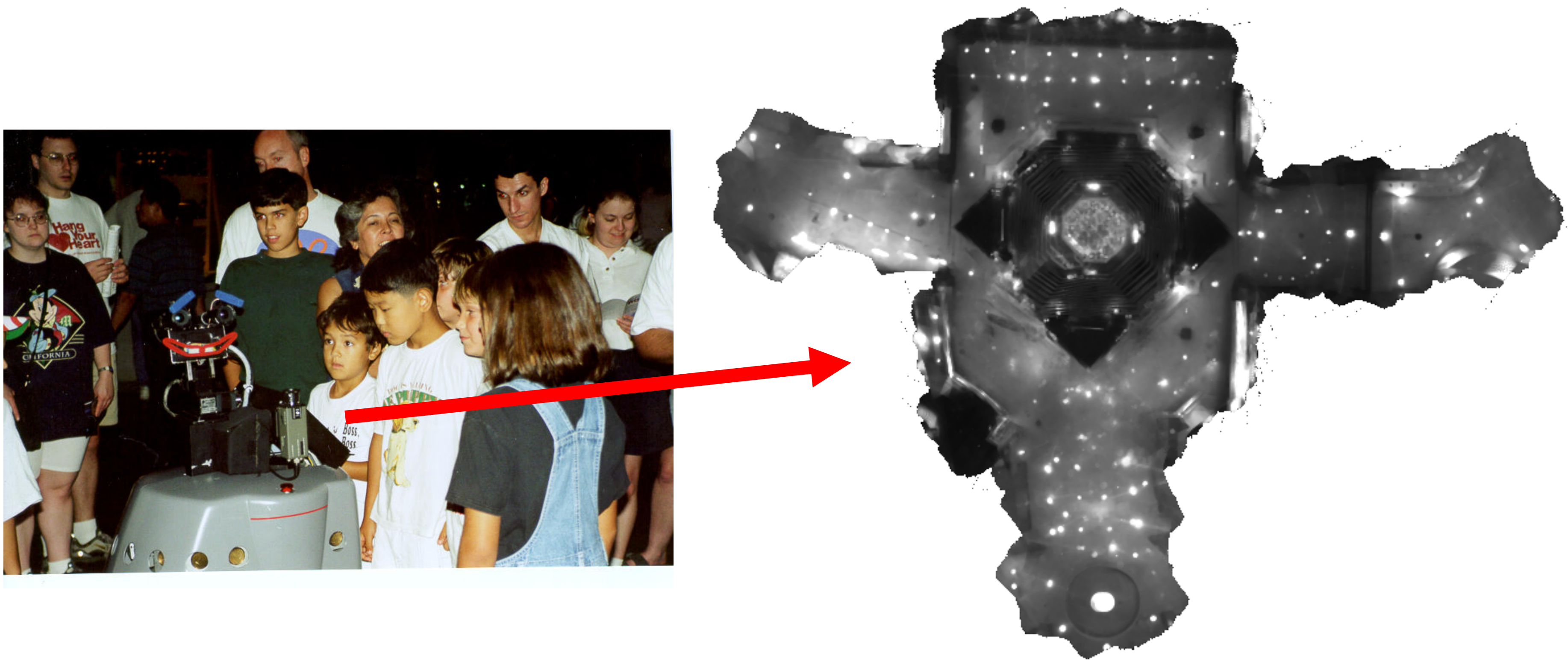




Sample-based Localization (Sonar)

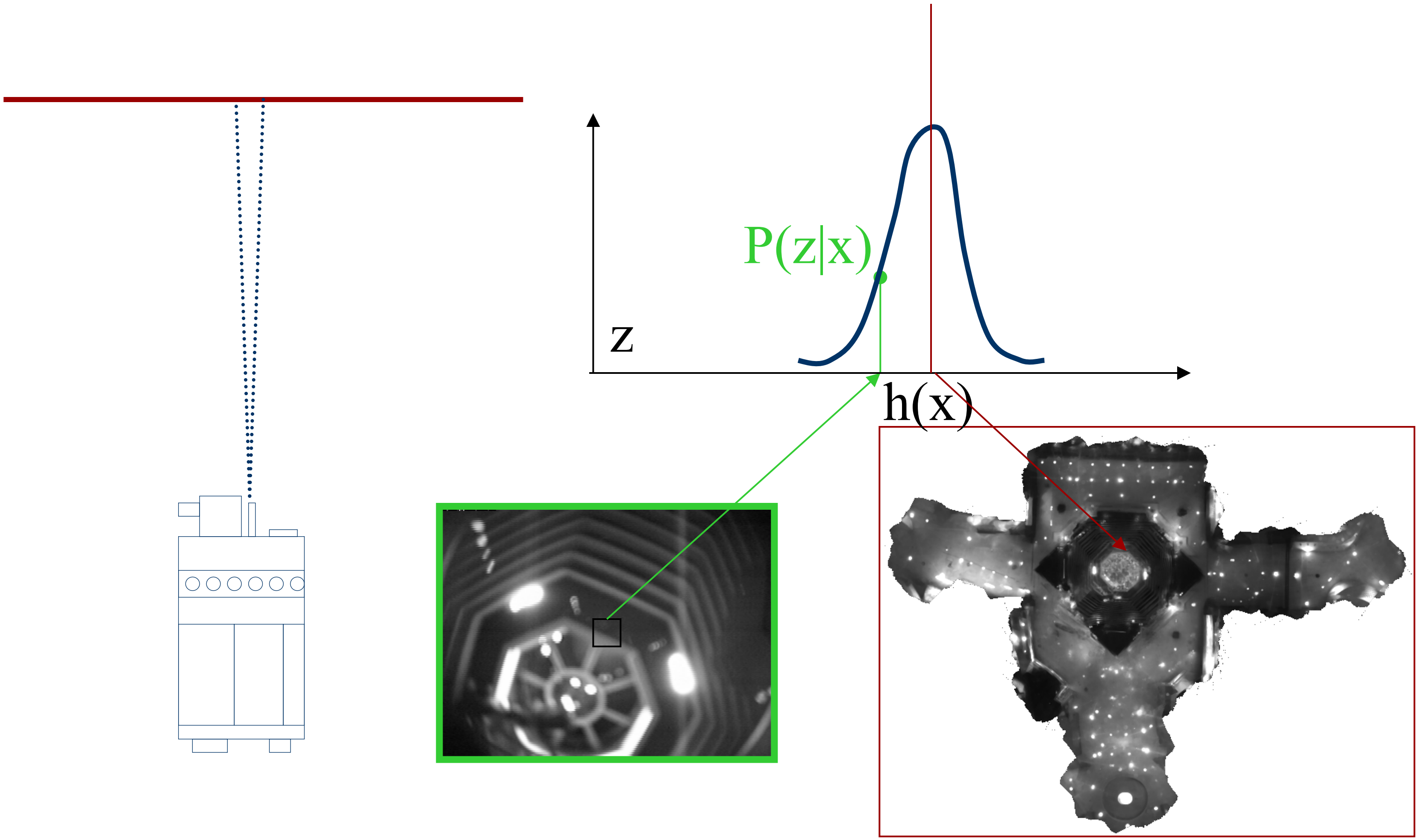


Using Ceiling Maps for Localization



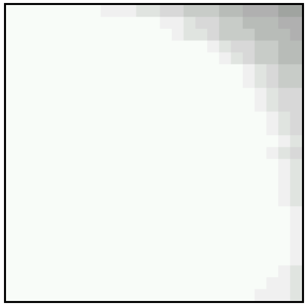
[Dellaert et al. 99]

Vision-based Localization



Under a Light

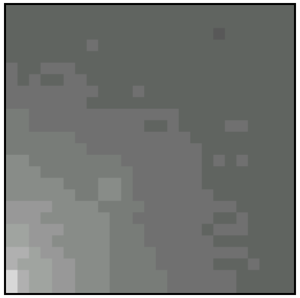
Measurement z:



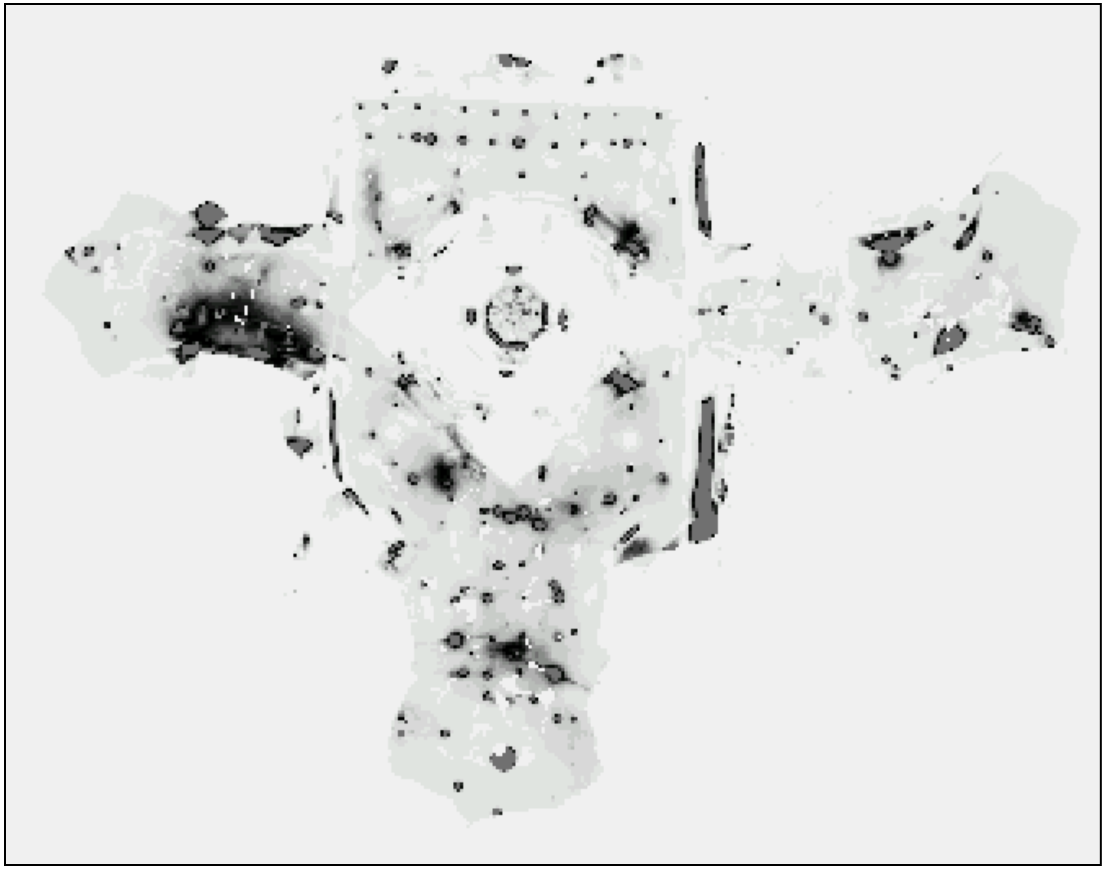
$P(z|x)$:

Next to a Light

Measurement z:



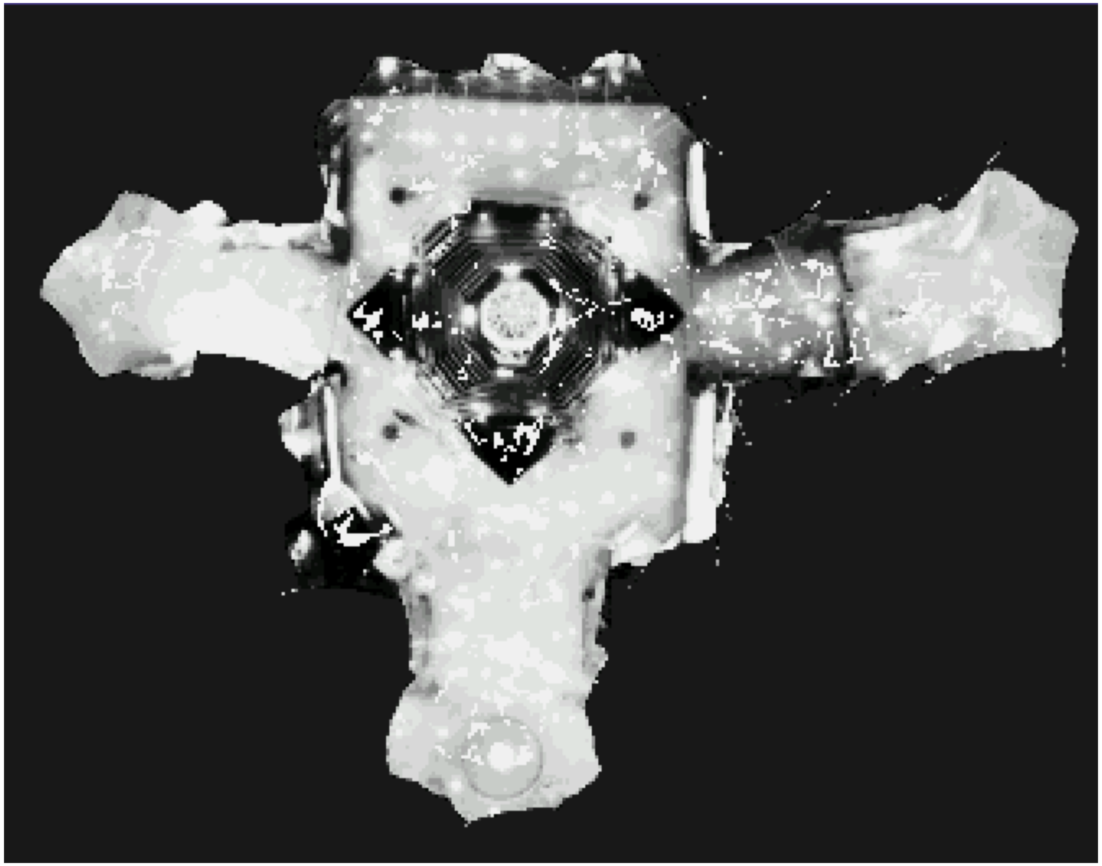
***P(z|x)*:**



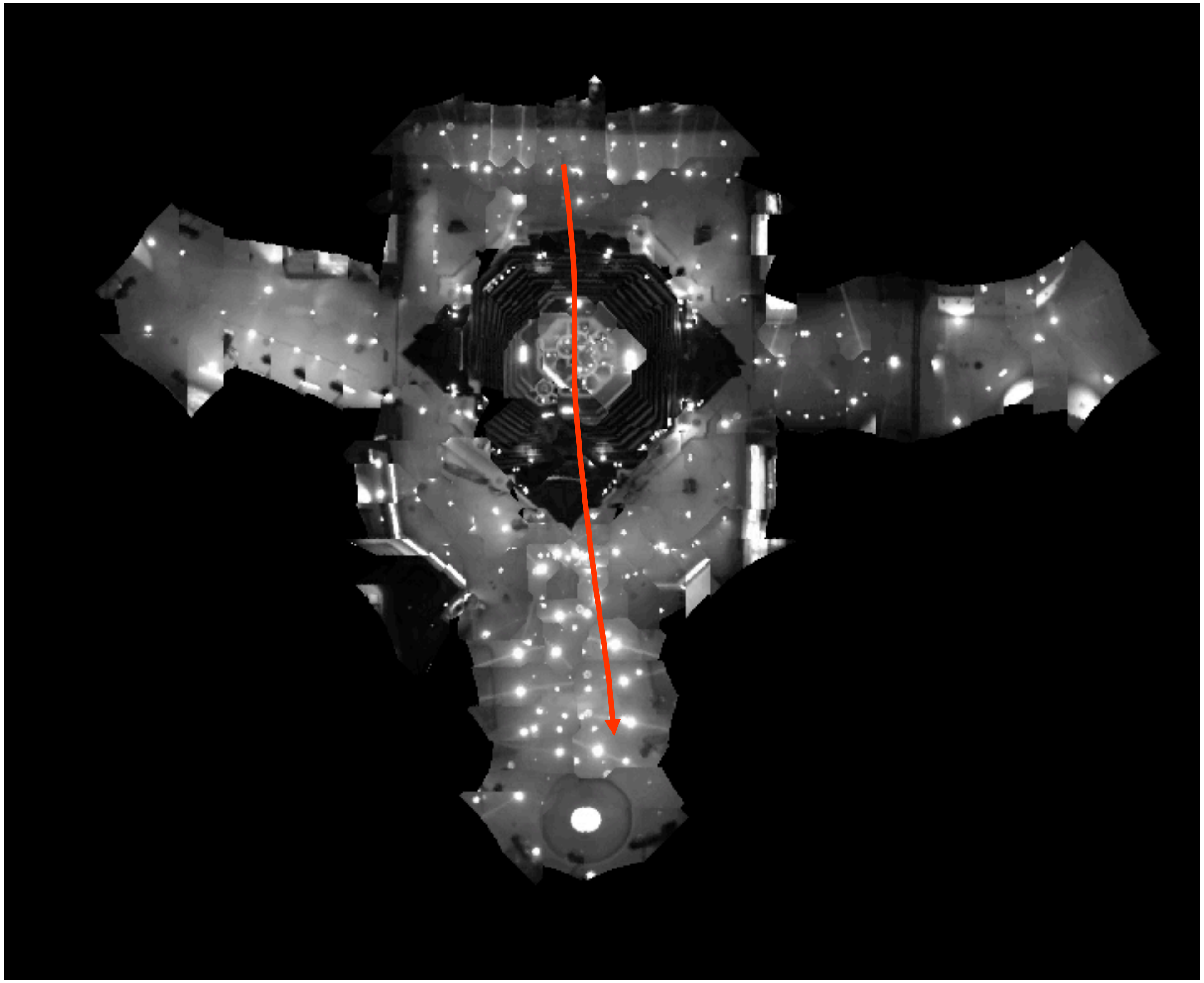
Elsewhere

Measurement z:

***P(z|x)*:**



Global Localization Using Vision



Limitations

- The approach described so far is able
  - to track the pose of a mobile robot and
  - to globally localize the robot

- How can we deal with localization errors (i.e., the kidnapped robot problem)?

Approaches

- Randomly insert a fixed number of samples with randomly chosen poses
- This corresponds to the assumption that the robot can be teleported at any point in time to an arbitrary location
- Alternatively, insert such samples inversely proportional to the average likelihood of the observations (the lower this likelihood the higher the probability that the current estimate is wrong).

Summary – Particle Filters

- Particle filters are an implementation of recursive Bayesian filtering
- They represent the posterior by a set of weighted samples
- They can model arbitrary and thus also non-Gaussian distributions
- Proposal to draw new samples
- Weights are computed to account for the difference between the proposal and the target
- Monte Carlo filter, Survival of the fittest, Condensation, Bootstrap filter

Summary – PF Localization

- In the context of localization, the particles are propagated according to the motion model.
- They are then weighted according to the likelihood model (likelihood of the observations).
- In a re-sampling step, new particles are drawn with a probability proportional to the likelihood of the observation.
- This leads to one of the most popular approaches to mobile robot localization