Nonlinear Optimization for Optimal Control

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Many slides and figures adapted from Stephen Boyd

[optional] Boyd and Vandenberghe, Convex Optimization, Chapters 9 – 11
[optional] Betts, Practical Methods for Optimal Control Using Nonlinear Programming

Bellman's curse of dimensionality

- n-dimensional state space
- Number of states grows exponentially in n (assuming some fixed number of discretization levels per coordinate)
- In practice
  - Discretization is considered only computationally feasible up to 5 or 6 dimensional state spaces even when using
    - Variable resolution discretization
    - Highly optimized implementations

This Lecture: Nonlinear Optimization for Optimal Control

- Goal: find a sequence of control inputs (and corresponding sequence of states) that solves:

$$
\begin{array}{ll}
\min_{u,x} & \sum_{t=0}^{H} g(x_t, u_t) \\
\text{subject to} & x_{t+1} = f(x_t, u_t) \ \ \forall t \\
 & u_t \in \mathcal{U}_t \ \ \forall t \\
 & x_t \in \mathcal{X}_t \ \ \forall t
\end{array}
$$

- Generally hard to do. We will cover methods that allow to find a local minimum of this optimization problem.
- Note: iteratively applying LQR is one way to solve this problem if there were no constraints on the control inputs and state

Outline

- **Unconstrained minimization**
  - **Gradient Descent**
  - Newton's Method
- Equality constrained minimization
- Inequality and equality constrained minimization

Unconstrained Minimization

$$\boxed{\min_x f(x) \qquad (1)}$$

(Implicitly assumed x can be chosen from the entire domain of f, often $\mathbb{R}^n$.)

- If x* satisfies:

$$\begin{aligned} \nabla_x f(x^*) &= 0 \quad (2) \\ \nabla_x^2 f(x^*) &\succeq 0 \quad (3) \end{aligned}$$

then x* is a local minimum of f.

- In simple cases we can directly solve the system of n equations given by (2) to find candidate local minima, and then verify (3) for these candidates.
- In general however, solving (2) is a difficult problem. Going forward we will consider this more general setting and cover numerical solution methods for (1).

Steepest Descent

- Idea:
  - Start somewhere
  - Repeat: Take a small step in the steepest descent direction

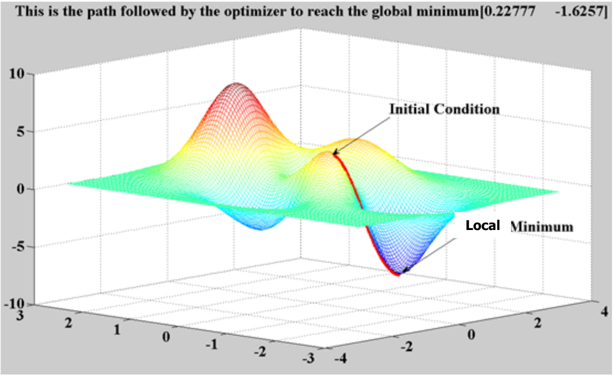


Figure source: Mathworks

Steep Descent

- Another example, visualized with contours:

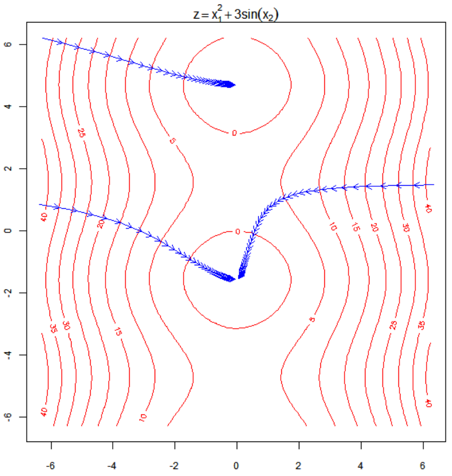


Figure source: yihui.name

Steepest Descent Algorithm

```
1. Initialize x
2. Repeat
    1. Determine the steepest descent direction $\Delta x$
    2. Line search.  Choose a step size $t > 0$.
    3. Update.  $x := x + t \Delta x$.
3. Until stopping criterion is satisfied
```

What is the Steepest Descent Direction?

Assuming a smooth function, we have that

$$f(x_0 + \Delta x) \approx f(x_0) + \nabla_x f(x_0)^\top \Delta x$$

The (locally at x_0) direction of steepest descent is given by:

$$\begin{aligned}\Delta x^* &= \arg \min_{\Delta x : \|\Delta x\|_2 = 1} f(x_0) + \nabla_x f(x_0)^\top \Delta x \\ &= \arg \min_{\Delta x : \|\Delta x\|_2 = 1} \nabla_x f(x_0)^\top \Delta x\end{aligned}$$

As we have all $a, b \in \mathbb{R}^n$ that $\min_{b:\|b\|_2=1} a^\top b$ is achieved for $b = -\frac{a}{\|a\|_2}$, we have that the steepest descent direction

$$\Delta x^* = -\nabla_x f(x_0)$$

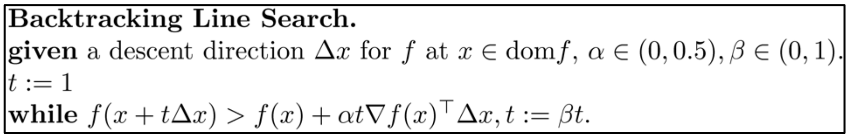
Stepsize Selection: Exact Line Search

$$t = \arg \min_{s \geq 0} f(x + s\Delta x)$$

- Used when the cost of solving the minimization problem with one variable is low compared to the cost of computing the search direction itself.

Stepsize Selection: Backtracking Line Search

- Inexact: step length is chose to approximately minimize f along the ray {x + t Δx | t $\geq$ 0}



```
**Backtracking Line Search.**
**given** a descent direction $\Delta x$ for $f$ at $x \in \mathbf{dom} f$, $\alpha \in (0, 0.5)$, $\beta \in (0, 1)$.
$t := 1$
**while** $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^\top \Delta x$, $t := \beta t$.
```

Stepsize Selection: Backtracking Line Search

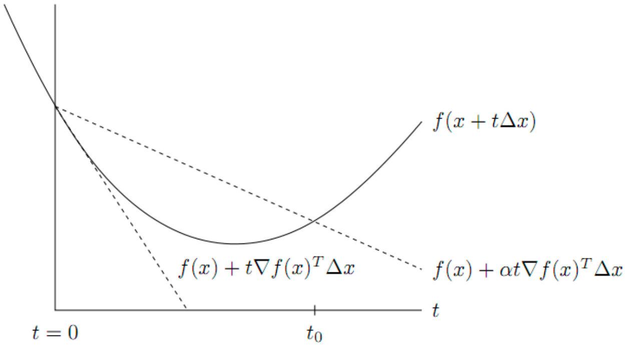
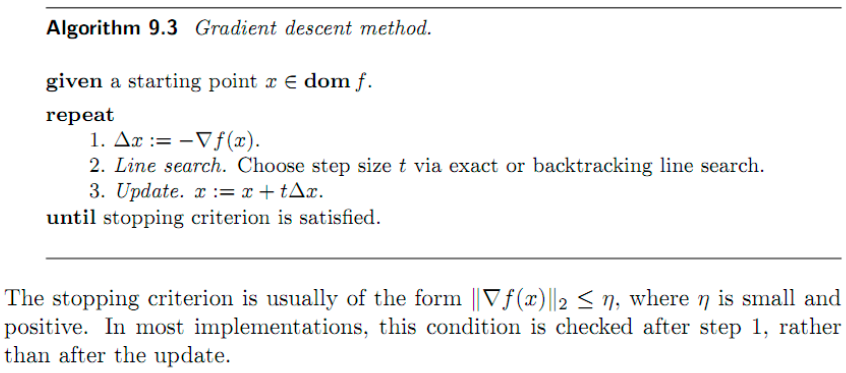

$f(x + t\Delta x)$
$f(x) + t\nabla f(x)^T \Delta x$
$f(x) + \alpha t\nabla f(x)^T \Delta x$
t
$t = 0$
t_0

Figure 9.1 *Backtracking line search.* The curve shows f, restricted to the line over which we search. The lower dashed line shows the linear extrapolation of f, and the upper dashed line has a slope a factor of α smaller. The backtracking condition is that f lies below the upper dashed line, *i.e.*, $0 \leq t \leq t_0$.

Figure source: Boyd and Vandenberghe

Gradient Descent Method



Algorithm 9.3 *Gradient descent method.*

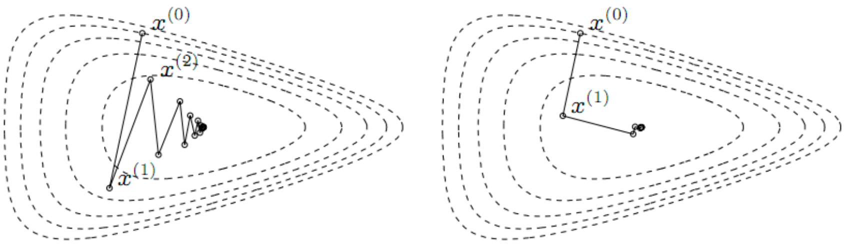
```
given a starting point $x \in \mathbf{dom}\, f$.
repeat
    1. $\Delta x := -\nabla f(x)$.
    2. Line search. Choose step size $t$ via exact or backtracking line search.
    3. Update. $x := x + t\Delta x$.
until stopping criterion is satisfied.
```

The stopping criterion is usually of the form $\|\nabla f(x)\|_2 \leq \eta$, where η is small and positive. In most implementations, this condition is checked after step 1, rather than after the update.

Figure source: Boyd and Vandenberghe

Gradient Descent: Example 1

$$f(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$$



backtracking line search

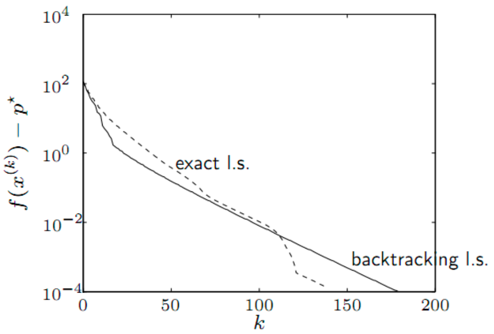
exact line search

Figure source: Boyd and Vandenberghe

Gradient Descent: Example 2

a problem in $\mathbf{R}^{100}$

$$f(x) = c^T x - \sum_{i=1}^{500} \log(b_i - a_i^T x)$$



'linear' convergence, *i.e.*, a straight line on a semilog plot

Figure source: Boyd and Vandenberghe

Gradient Descent: Example 3

$$f(x) = (1/2)(x_1^2 + \gamma x_2^2) \qquad (\gamma > 0)$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k$$

- very slow if $\gamma \gg 1$ or $\gamma \ll 1$
- example for $\gamma = 10$:

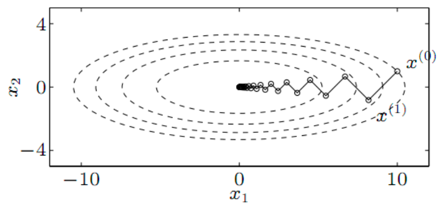
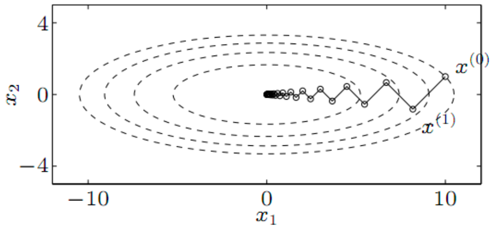
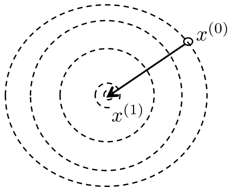


Figure source: Boyd and Vandenberghe

Gradient Descent Convergence



Condition number = 10



Condition number = 1

- For quadratic function, convergence speed depends on ratio of highest second derivative over lowest second derivative ("condition number")
- In high dimensions, almost guaranteed to have a high (=bad) condition number
- Rescaling coordinates (as could happen by simply expressing quantities in different measurement units) results in a different condition number

Outline

- **Unconstrained minimization**
  - Gradient Descent
  - **Newton's Method**
- Equality constrained minimization
- Inequality and equality constrained minimization

Newton's Method

- 2nd order Taylor Approximation rather than 1st order:

$$f(x+\Delta x) \approx f(x) + \nabla f(x)^\top \Delta x + \frac{1}{2}\Delta x^\top \nabla^2 f(x) \Delta x$$

assuming $\nabla^2 f(x) \succeq 0$, the minimum of the 2nd order approximation is achieved at: $\Delta x_{\rm nt} = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$

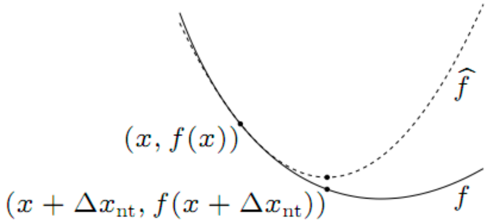
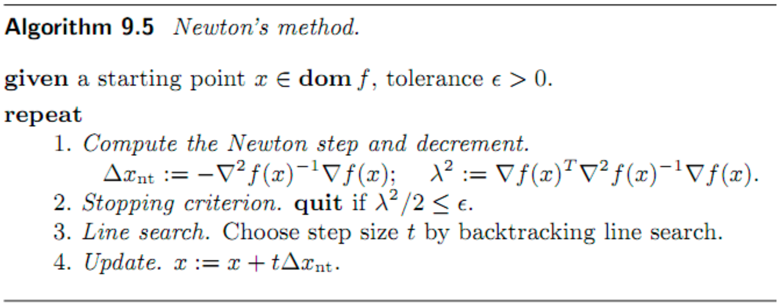


Figure source: Boyd and Vandenberghe

Newton's Method



```
Algorithm 9.5  Newton's method.

given a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.
repeat
    1. Compute the Newton step and decrement.
        $\Delta x_{\rm nt} := -\nabla^2 f(x)^{-1}\nabla f(x)$;   $\lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$.
    2. Stopping criterion. quit if $\lambda^2/2 \le \epsilon$.
    3. Line search. Choose step size $t$ by backtracking line search.
    4. Update. $x := x + t\Delta x_{\rm nt}$.
```

Figure source: Boyd and Vandenberghe

Affine Invariance

- Consider the coordinate transformation y = A x

- If running Newton's method starting from $x^{(0)}$ on f(x) results in

  $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$

- Then running Newton's method starting from $y^{(0)} = A\, x^{(0)}$ on g(y) = f(A^{-1} y), will result in the sequence

  $y^{(0)} = A\, x^{(0)}, y^{(1)} = A\, x^{(1)}, y^{(2)} = A\, x^{(2)}, \ldots$

- Exercise: try to prove this.

Newton's method when we don't have $\nabla^2 f(x) \succeq 0$

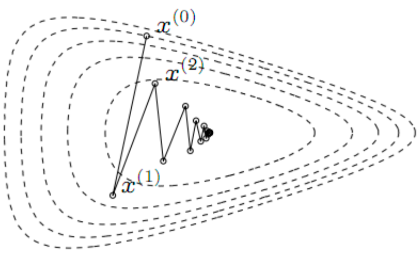
- Issue: now Δx_{nt} does not lead to the local minimum of the quadratic approximation --- it simply leads to the point where the gradient of the quadratic approximation is zero, this could be a maximum or a saddle point

- Three possible fixes, let $X \Lambda X^\top = \nabla^2 f(x)$ be the eigenvalue decomposition.
  - Fix 1: Replace $\nabla^2 f(x)$ with $X \bar{\Lambda} X^\top$, with $\bar{\Lambda}$ a diagonal matrix with $\bar{\Lambda}_{i,i} = \max(0, \Lambda_{i,i})$.
  - Fix 2: Replace $\nabla^2 f(x)$ with $X \bar{\Lambda} X^\top$, with $\bar{\Lambda}$ a diagonal matrix with $\bar{\Lambda}_{i,i} = \Lambda_{i,i} + (-1) * \min_j \Lambda_{j,j}$
  - Fix 3: Use a gradient descent step, rather than a Newton step, in the current iteration.

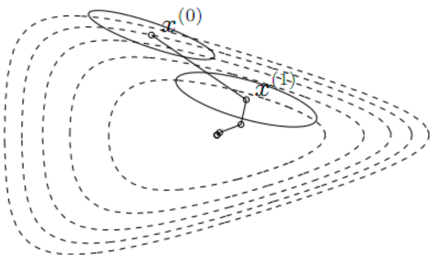
In my experience Fix 2 works best.

Example 1

$$f(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$$



gradient descent with backtracking line search



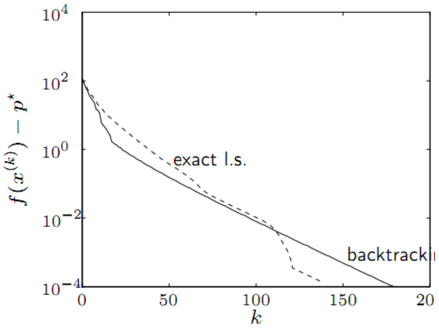
Newton's method with backtracking line search

Figure source: Boyd and Vandenberghe

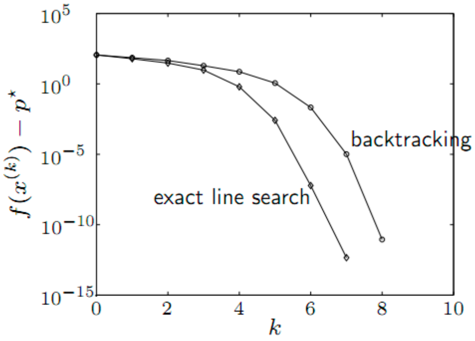
Example 2

a problem in R^{100}

$$f(x) = c^T x - \sum_{i=1}^{500} \log(b_i - a_i^T x)$$



gradient descent



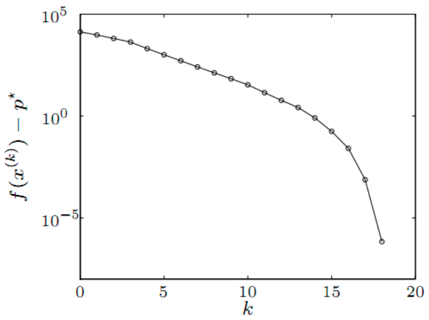
Newton's method

Figure source: Boyd and Vandenberghe

Larger Version of Example 2

example in R^{10000} (with sparse a_i)

$$f(x) = -\sum_{i=1}^{10000} \log(1 - x_i^2) - \sum_{i=1}^{100000} \log(b_i - a_i^T x)$$



- backtracking parameters $\alpha = 0.01$, $\beta = 0.5$.
- performance similar as for small examples

Gradient Descent: Example 3

$$f(x) = (1/2)(x_1^2 + \gamma x_2^2) \qquad (\gamma > 0)$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k$$

- very slow if $\gamma \gg 1$ or $\gamma \ll 1$
- example for $\gamma = 10$:

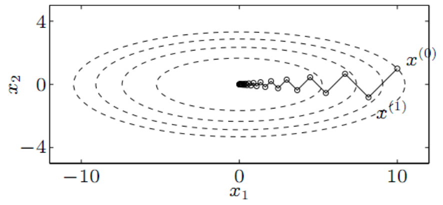
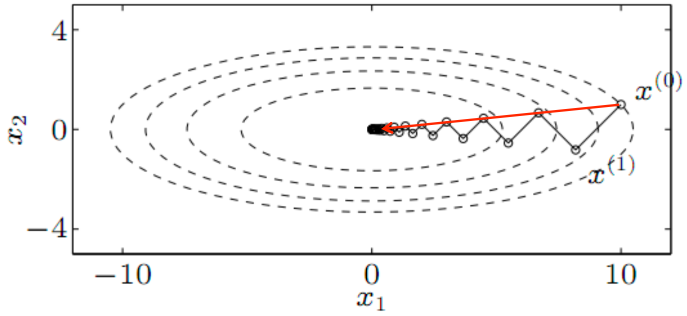


Figure source: Boyd and Vandenberghe

Example 3



- Gradient descent
- Newton's method (converges in one step if f convex quadratic)

Quasi-Newton Methods

- Quasi-Newton methods use an approximation of the Hessian
  - Example 1: Only compute diagonal entries of Hessian, set others equal to zero. Note this also simplfies computations done with the Hessian.
  - Example 2: natural gradient --- see next slide

Natural Gradient

- Consider a standard maximum likelihood problem:

$$\max_\theta f(\theta) = \max_\theta \sum_i \log p(x^{(i)};\theta)$$

- Gradient:

$$\frac{\partial f(\theta)}{\partial \theta_p} = \sum_i \frac{\partial \log p(x^{(i)};\theta)}{\partial \theta_p} = \sum_i \frac{\partial p(x^{(i)};\theta)}{\partial \theta_p} \frac{1}{p(x^{(i)};\theta)}$$

- Hessian:

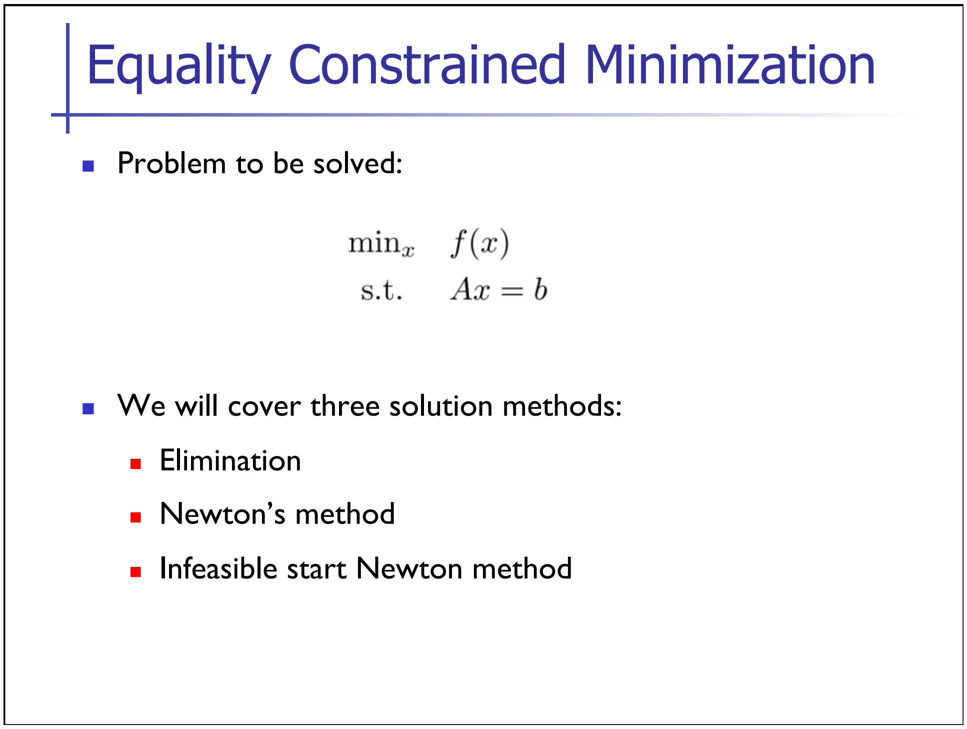
$$\frac{\partial^2 f(\theta)}{\partial \theta_q \partial \theta_p} = \sum_i \frac{\partial^2 p(x^{(i)};\theta)}{\partial \theta_q \partial \theta_p} \frac{1}{p(x^{(i)};\theta)} - \frac{\partial p(x^{(i)};\theta)}{\partial \theta_q} \frac{1}{p(x^{(i)};\theta)} \frac{\partial p(x^{(i)};\theta)}{\partial \theta_p} \frac{1}{p(x^{(i)};\theta)}$$

$$\nabla^2 \log f(\theta) = \sum_i \frac{\nabla^2 p(x^{(i)};\theta)}{p(x^{(i)};\theta)} - \left(\nabla \log p(x^{(i)};\theta)\right) \left(\nabla \log p(x^{(i)};\theta)\right)^\top$$

- Natural gradient only keeps the 2nd term
  1: faster to compute (only gradients needed); 2: guaranteed to be negative definite; 3: found to be superior in some experiments

Outline

- Unconstrained minimization
  - Gradient Descent
  - Newton's Method
- **Equality constrained minimization**
- Inequality and equality constrained minimization

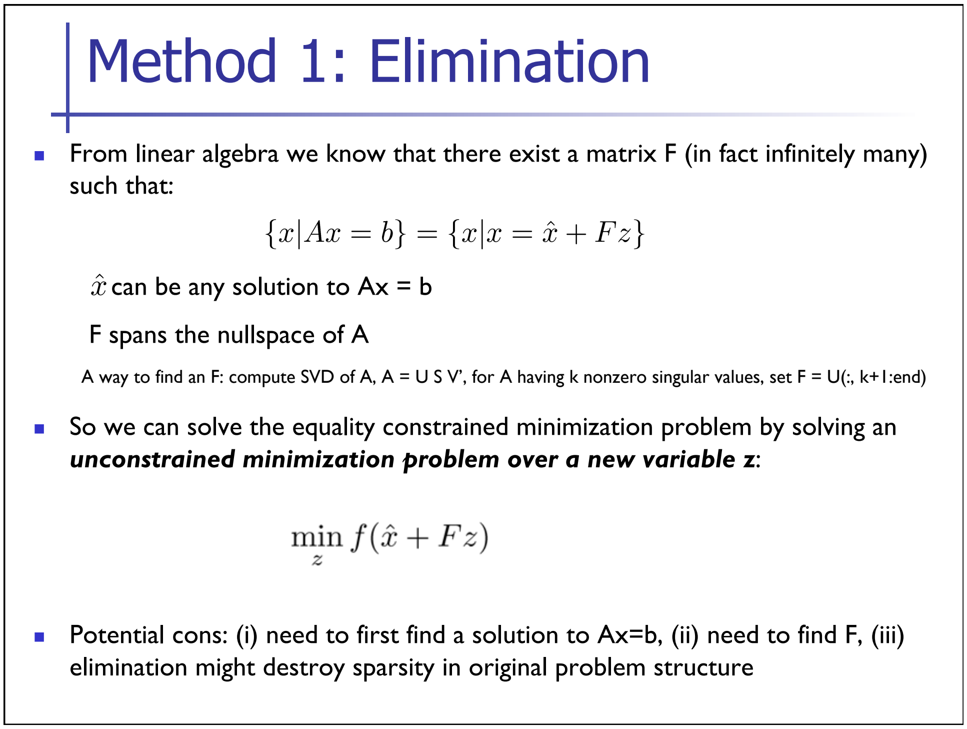


Equality Constrained Minimization

- Problem to be solved:

$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & Ax = b \end{array}$$

- We will cover three solution methods:
  - Elimination
  - Newton's method
  - Infeasible start Newton method



Method 1: Elimination

- From linear algebra we know that there exist a matrix F (in fact infinitely many) such that:

$$\{x | Ax = b\} = \{x | x = \hat{x} + Fz\}$$

$\hat{x}$ can be any solution to Ax = b

F spans the nullspace of A

A way to find an F: compute SVD of A, A = U S V', for A having k nonzero singular values, set F = U(:, k+1:end)

- So we can solve the equality constrained minimization problem by solving an ***unconstrained minimization problem over a new variable z***:

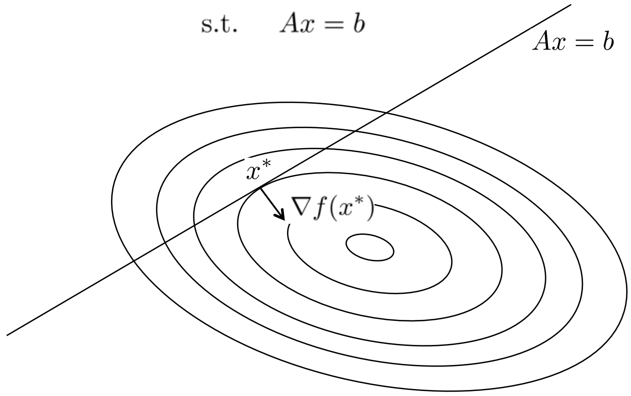
$$\min_z f(\hat{x} + Fz)$$

- Potential cons: (i) need to first find a solution to Ax=b, (ii) need to find F, (iii) elimination might destroy sparsity in original problem structure

Methods 2 and 3 Require Us to First Understand the Optimality Condition

- Recall the problem to be solved:

$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & Ax = b \end{array}$$



Optimality Condition: $Ax^* = b$ and $\nabla f(x^*) + A^\top \nu = 0$

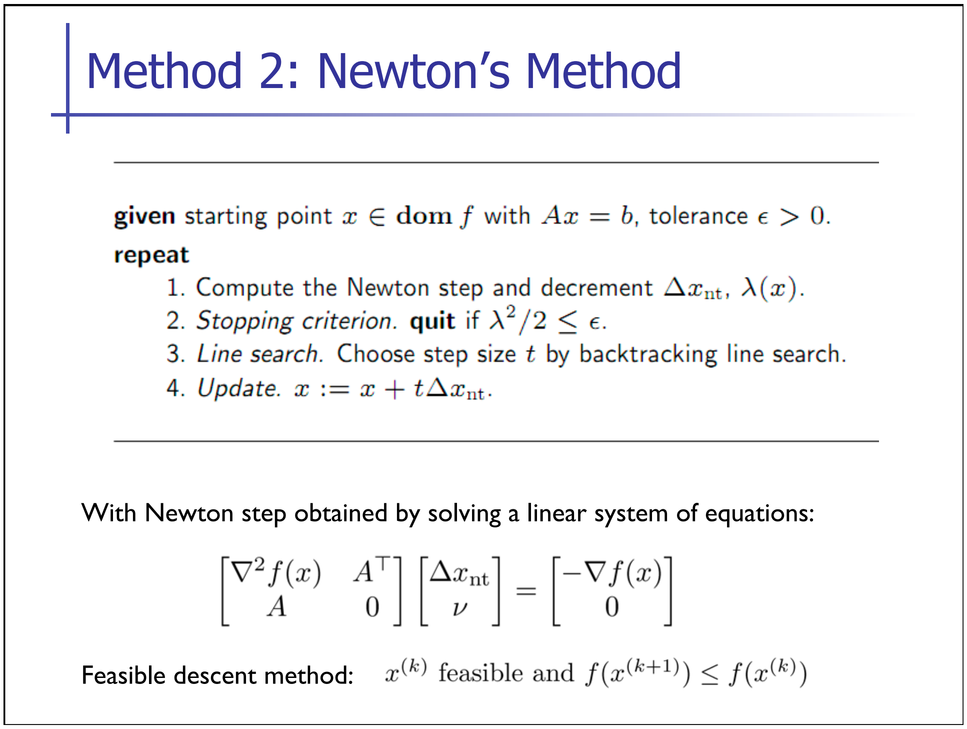
Method 2: Newton's Method

- Problem to be solved: $\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & Ax = b \end{array}$
- Optimality Condition: $Ax^* = b$ and $\nabla f(x^*) + A^\top \nu = 0$
- Assume x is feasible, i.e., satisfies Ax = b, now use 2nd order approximation of f:

$$\begin{array}{ll} \min_{\Delta x} & f(x) + \nabla f(x)^\top \Delta x + \frac{1}{2} \Delta x^\top \nabla^2 f(x) \Delta x \\ \text{s.t.} & A(x + \Delta x) = b \end{array}$$

- → Optimality condition for 2nd order approximation:

$$\begin{bmatrix} \nabla^2 f(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$



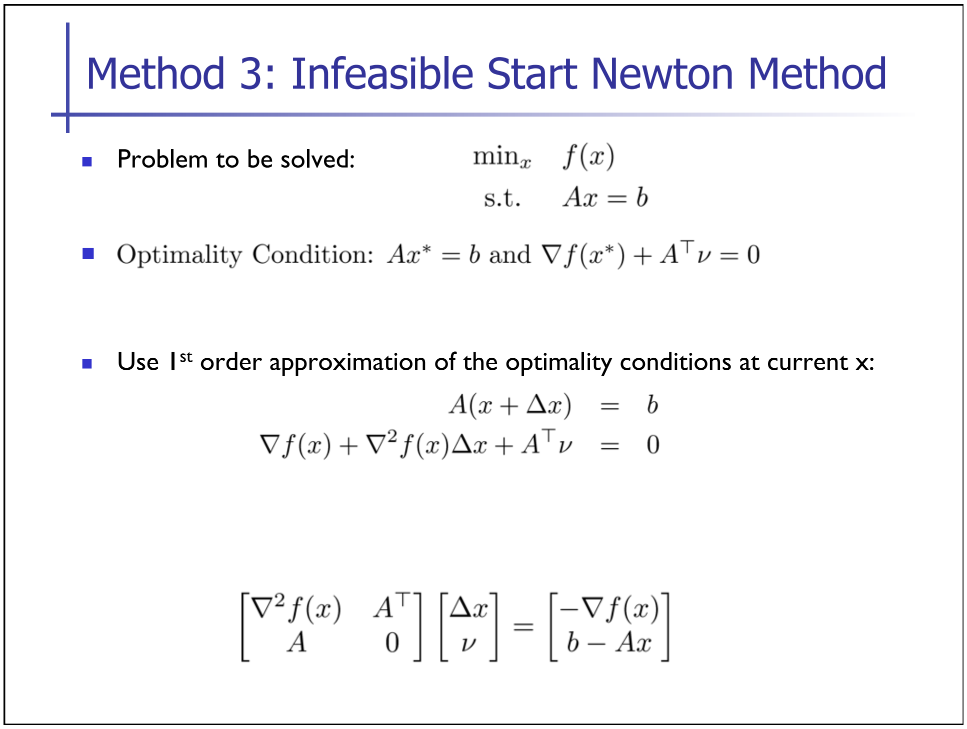
Method 2: Newton's Method

```
given starting point $x \in \mathbf{dom}\, f$ with $Ax = b$, tolerance $\epsilon > 0$.
repeat
    1. Compute the Newton step and decrement $\Delta x_{\mathrm{nt}}$, $\lambda(x)$.
    2. Stopping criterion. quit if $\lambda^2/2 \le \epsilon$.
    3. Line search. Choose step size $t$ by backtracking line search.
    4. Update. $x := x + t\Delta x_{\mathrm{nt}}$.
```

With Newton step obtained by solving a linear system of equations:

$$\begin{bmatrix} \nabla^2 f(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\mathrm{nt}} \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

Feasible descent method: $x^{(k)}$ feasible and $f(x^{(k+1)}) \le f(x^{(k)})$



Method 3: Infeasible Start Newton Method

- Problem to be solved: $\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & Ax = b \end{array}$
- Optimality Condition: $Ax^* = b$ and $\nabla f(x^*) + A^\top \nu = 0$
- Use 1st order approximation of the optimality conditions at current x:

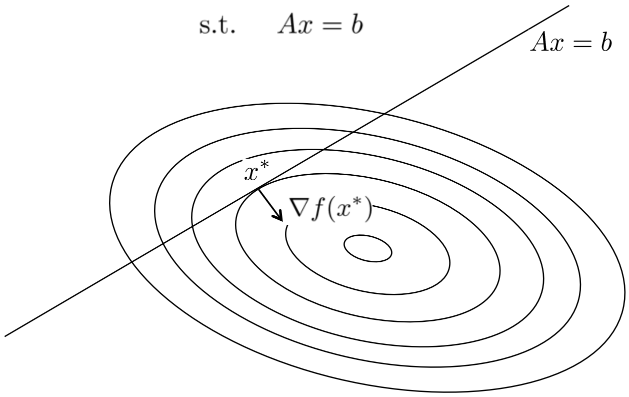
$$\begin{aligned} A(x + \Delta x) &= b \\ \nabla f(x) + \nabla^2 f(x)\Delta x + A^\top \nu &= 0 \end{aligned}$$

$$\begin{bmatrix} \nabla^2 f(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ b - Ax \end{bmatrix}$$

Methods 2 and 3 Require Us to First Understand the Optimality Condition

- Recall the problem to be solved:

$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & Ax = b \end{array}$$



$$\boxed{\textbf{Optimality Condition: } Ax^* = b \text{ and } \nabla f(x^*) + A^\top \nu = 0}$$

Optimal Control

- We can now solve:

$$\begin{array}{ll} \min_{x,u} & \sum_{t=0}^{T} g_t(x_t, u_t) \\ \text{s.t.} & x_{t+1} = A_t x_t + B_t u_t \quad \forall t \end{array}$$

- And often one can efficiently solve

$$\begin{array}{ll} \min_{x,u} & \sum_{t=0}^{T} g_t(x_t, u_t) \\ \text{s.t.} & x_{t+1} = f_t(x_t, u_t) \quad \forall t \end{array}$$

by iterating over (i) linearizing the constraints, and (ii) solving the resulting problem.

Optimal Control: A Complete Algorithm

- Given: $\bar{x}_0$
- For k=0, 1, 2, …, T
  - Solve

$$
\begin{aligned}
\min_{x,u} \quad & \sum_{t=k}^{T} g_t(x_t, u_t) \\
\text{s.t.} \quad & x_{t+1} = f_t(x_t, u_t) \qquad \forall t \in \{k, k+1, \ldots, T-1\} \\
& x_k = \bar{x}_k
\end{aligned}
$$

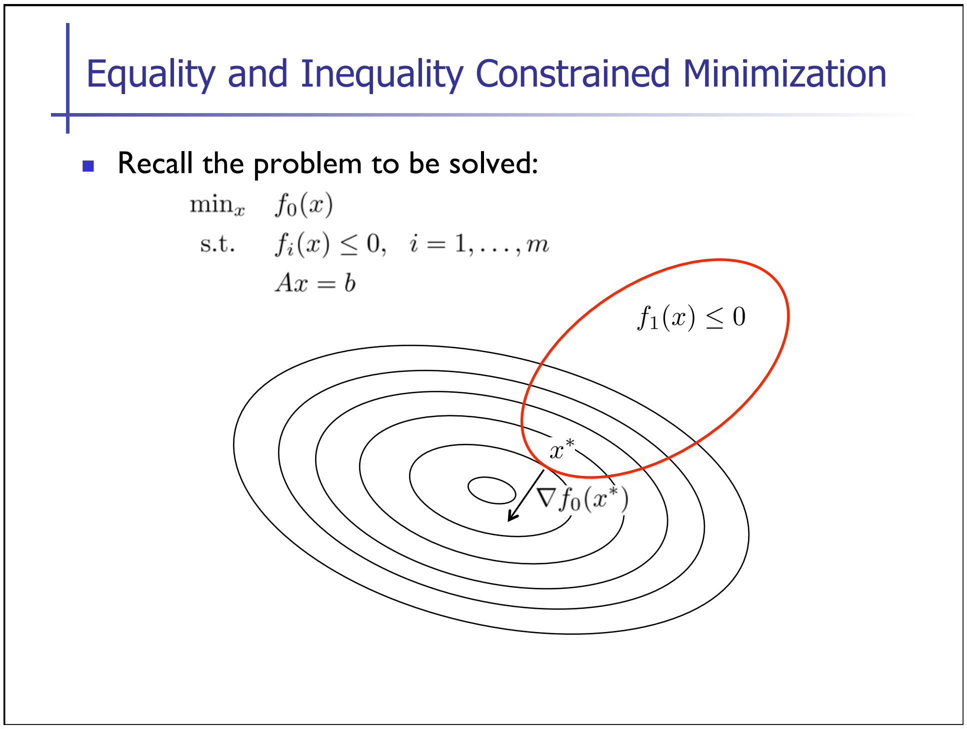
  - Execute u_k
  - Observe resulting state, $\bar{x}_{k+1}$

→ = an instantiation of Model Predictive Control.

→ Initialization with solution from iteration k-1 can make solver very fast (and would be done most conveniently with infeasible start Newton method)

Outline

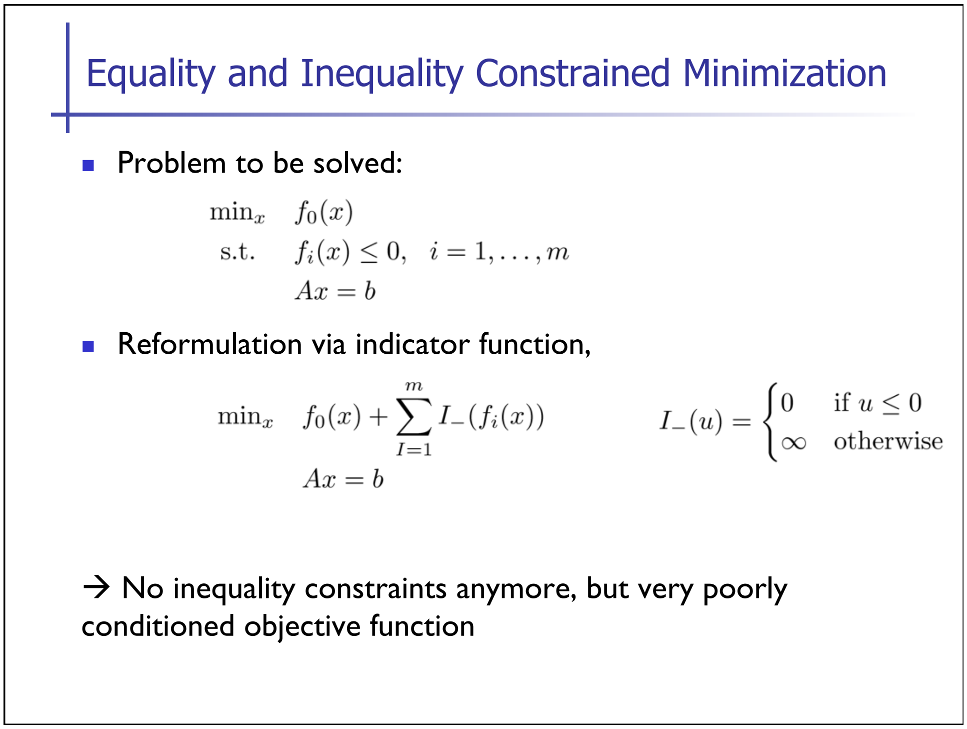
- Unconstrained minimization
- Equality constrained minimization
- **Inequality and equality constrained minimization**



Equality and Inequality Constrained Minimization

- Recall the problem to be solved:

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$



Equality and Inequality Constrained Minimization

- Problem to be solved:

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

- Reformulation via indicator function,

$$\begin{aligned} \min_x \quad & f_0(x) + \sum_{I=1}^{m} I_-(f_i(x)) \\ & Ax = b \end{aligned} \qquad\qquad I_-(u) = \begin{cases} 0 & \text{if } u \le 0 \\ \infty & \text{otherwise} \end{cases}$$

→ No inequality constraints anymore, but very poorly conditioned objective function

Equality and Inequality Constrained Minimization

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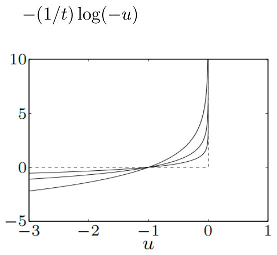
- Approximation via logarithmic barrier:

$$\begin{aligned} \min_x \quad & f_0(x) - (1/t) \sum_{i=1}^{m} \log(-f_i(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$



for t>0, -(1/t) log(-u) is a smooth approximation of *I*_(u)

approximation improves for *t* → ∞, better conditioned for smaller *t*

Barrier Method

- Given: strictly feasible x, t=t$^{(0)}$ > 0, μ > 1, tolerance ϵ > 0
- Repeat

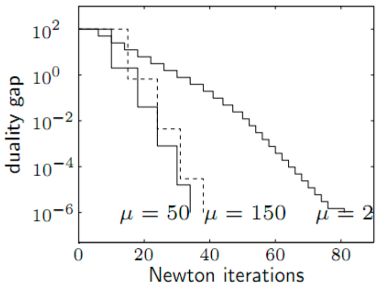
1. *Centering Step.* Compute x*(t) by solving

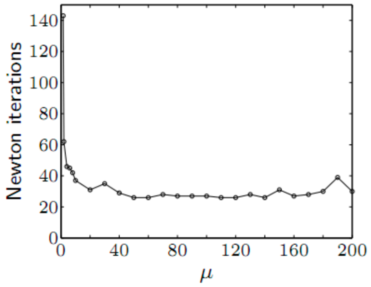
$$\begin{aligned} \min_x \quad & f_0(x) - (1/t) \sum_{i=1}^{m} \log(-f_i(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

   starting from x
2. *Update.* x := x*(t).
3. *Stopping Criterion.* Quit if m/t < ϵ
4. *Increase t.* t := μ t

Example 1: Inequality Form LP

inequality form LP ($m = 100$ inequalities, $n = 50$ variables)



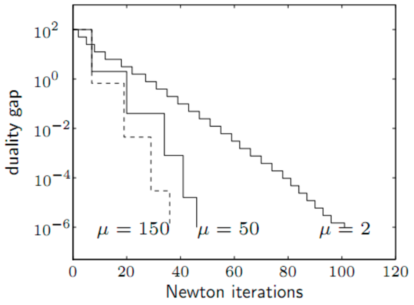


- starts with x on central path ($t^{(0)} = 1$, duality gap 100)
- terminates when $t = 10^8$ (gap 10^{-6})
- centering uses Newton's method with backtracking
- total number of Newton iterations not very sensitive for $\mu \geq 10$

Example 2: Geometric Program

geometric program ($m = 100$ inequalities and $n = 50$ variables)

$$\begin{array}{ll} \text{minimize} & \log\left(\sum_{k=1}^{5} \exp(a_{0k}^T x + b_{0k})\right) \\ \text{subject to} & \log\left(\sum_{k=1}^{5} \exp(a_{ik}^T x + b_{ik})\right) \leq 0, \quad i = 1, \ldots, m \end{array}$$

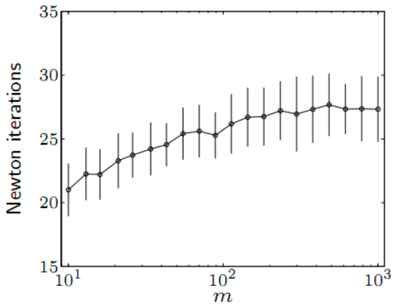


Example 3: Standard LPs

family of standard LPs ($A \in \mathbf{R}^{m \times 2m}$)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$

$m = 10, \ldots, 1000$; for each m, solve 100 randomly generated instances



number of iterations grows very slowly as m ranges over a $100 : 1$ ratio

Initalization

- Basic phase I method:

Initialize by first solving:

$$\begin{array}{ll} \min_{x,s} & s \\ \text{s.t.} & f_i(x) \le s, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- Easy to initialize above problem, pick some x such that Ax = b, and then simply set s = $\max_i f_i(x)$
- Can stop early---whenever s < 0

Initalization

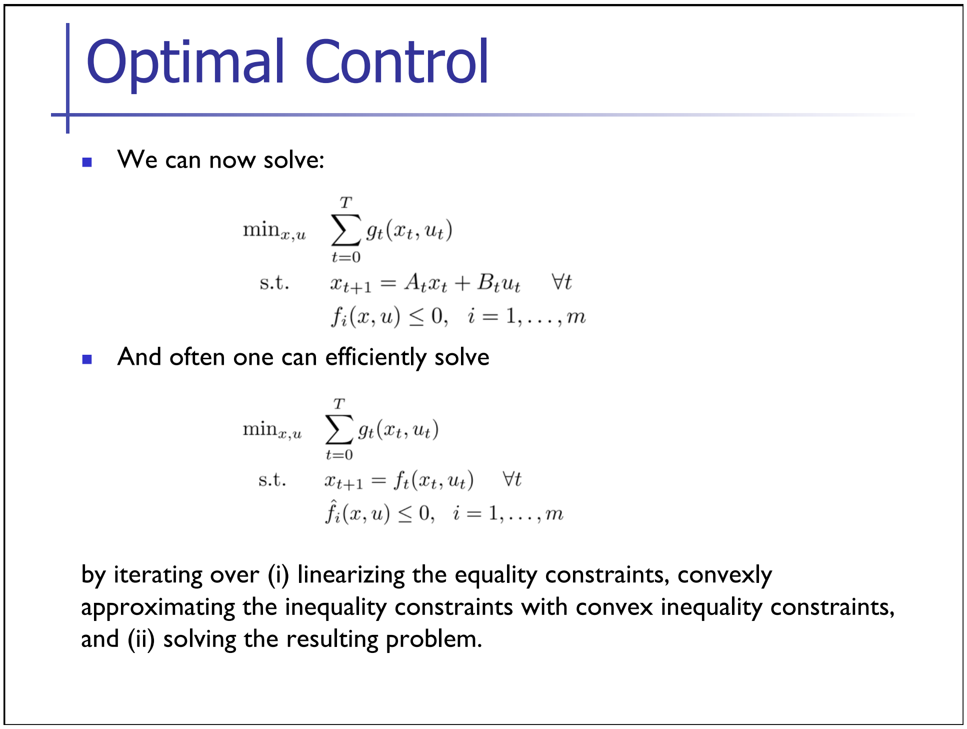
- Sum of infeasibilities phase I method:
- Initialize by first solving:

$$
\begin{array}{ll}
\min_{x,s} & \sum_{I=1}^{m} s_i \\
\text{s.t.} & f_i(x) \leq s_i, \quad i = 1, \ldots, m \\
 & s_i \geq 0, \quad i = 1, \ldots, m \\
 & Ax = b
\end{array}
$$

- Easy to initialize above problem, pick some x such that Ax = b, and then simply set s_i = max(0, f_i(x))
- For infeasible problems, produces a solution that satisfies many more inequalities than basic phase I method

Other methods

- We have covered a primal interior point method
  - one of several optimization approaches
- Examples of others:
  - Primal-dual interior point methods
  - Primal-dual infeasible interior point methods



Optimal Control

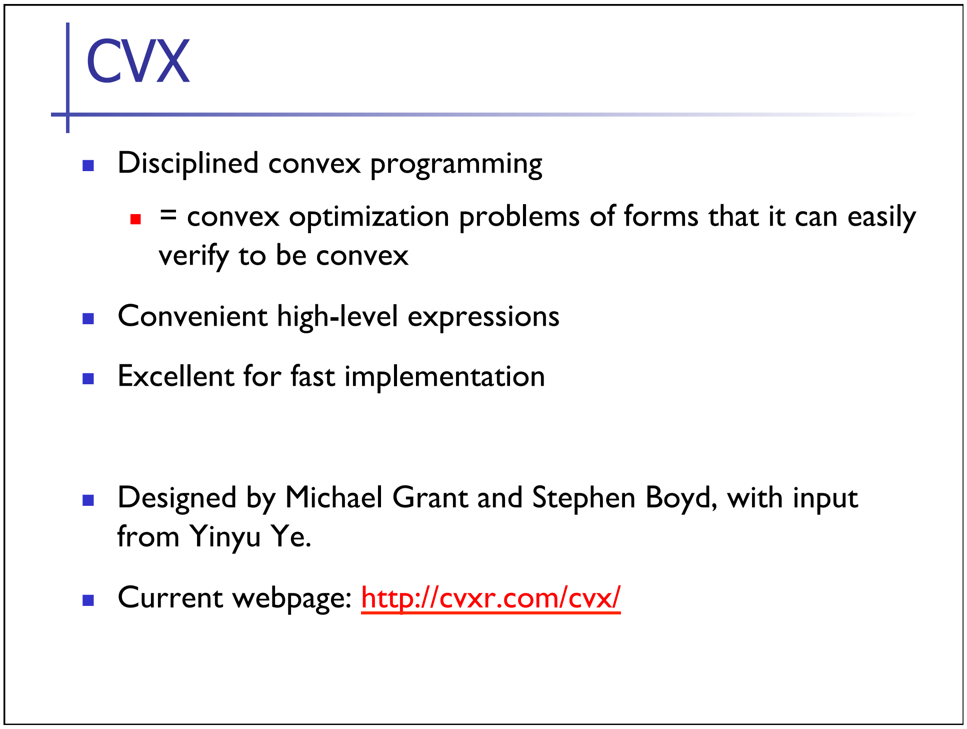
- We can now solve:

$$\begin{aligned} \min_{x,u} \quad & \sum_{t=0}^{T} g_t(x_t, u_t) \\ \text{s.t.} \quad & x_{t+1} = A_t x_t + B_t u_t \quad \forall t \\ & f_i(x, u) \leq 0, \quad i = 1, \ldots, m \end{aligned}$$

- And often one can efficiently solve

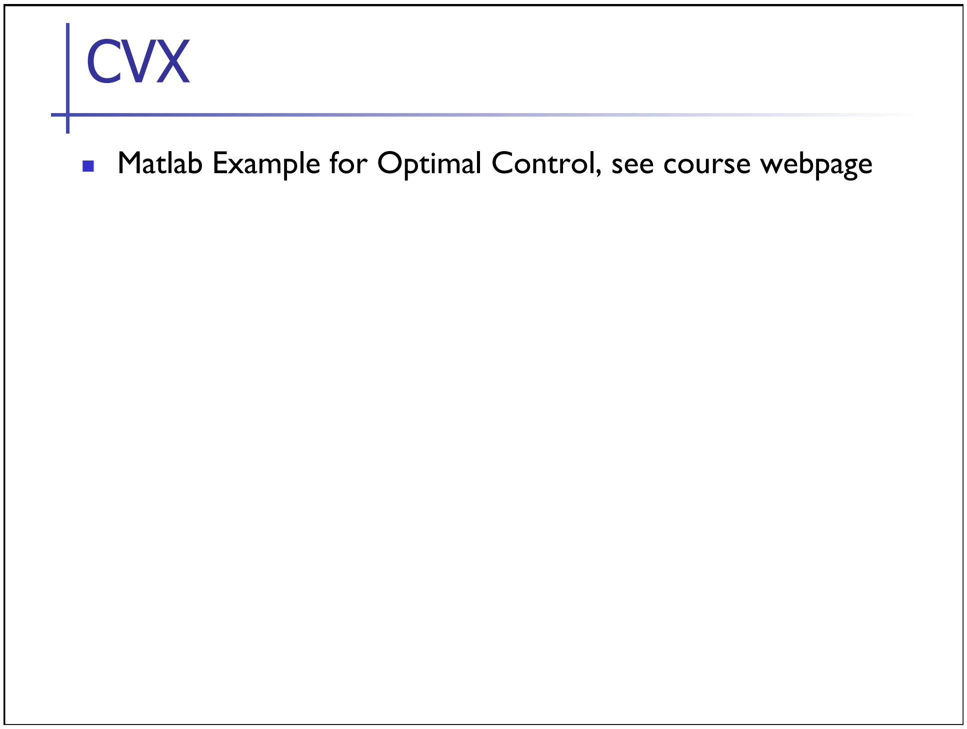
$$\begin{aligned} \min_{x,u} \quad & \sum_{t=0}^{T} g_t(x_t, u_t) \\ \text{s.t.} \quad & x_{t+1} = f_t(x_t, u_t) \quad \forall t \\ & \hat{f}_i(x, u) \leq 0, \quad i = 1, \ldots, m \end{aligned}$$

by iterating over (i) linearizing the equality constraints, convexly approximating the inequality constraints with convex inequality constraints, and (ii) solving the resulting problem.



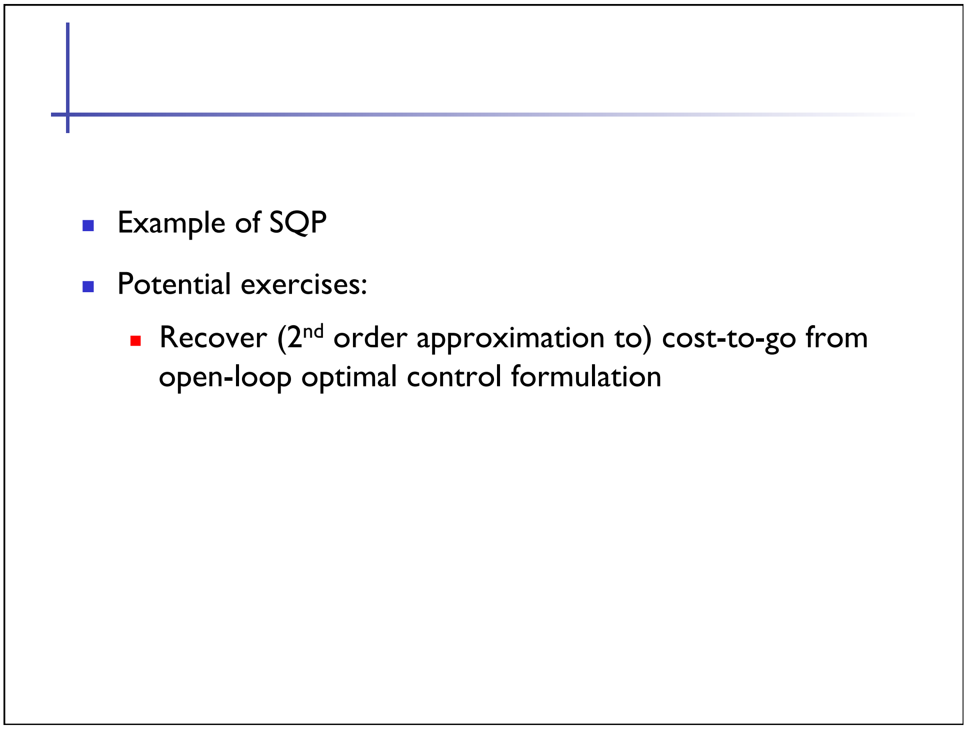
CVX

- Disciplined convex programming
  - = convex optimization problems of forms that it can easily verify to be convex
- Convenient high-level expressions
- Excellent for fast implementation
- Designed by Michael Grant and Stephen Boyd, with input from Yinyu Ye.
- Current webpage: http://cvxr.com/cvx/



CVX

- Matlab Example for Optimal Control, see course webpage



- Example of SQP
- Potential exercises:
  - Recover (2^{nd} order approximation to) cost-to-go from open-loop optimal control formulation