CS281A/Stat241A Homework Assignment 1 (due September 15, 2009)

1. **(Directed graphical models and conditional probabilities)**
   A standard test for prostate cancer, the prostate specific antigen (PSA) test, has a sensitivity of around 95% and a specificity of around 55%. (The sensitivity is the conditional probability of a positive test result given that the disease is present; the specificity is the conditional probability of a negative test result given that the disease is absent.) The probability of developing prostate cancer is age-dependent. Suppose that:

   - The probability that a sixty-year-old patient has prostate cancer is 2%, and this probability increases with age.
   - The probability that a sixty-year-old patient has a more benign prostate disease, BPH, is 8%, and this probability increases with age.
   - Given the patient's age the event that he has BPH is conditionally independent of the event that he has prostate cancer.
   - Age and presence or absence of BPH do not affect the sensitivity or specificity of the PSA test.
   - Independent of age, the probability of an enlarged prostate is 10% for healthy patients, 85% for patients with BPH, 60% for patients with prostate cancer, and 85% for patients with both diseases.

   (a) Draw the graph of a directed graphical model that describes the joint probability distribution. List the constraints on entries in the conditional probability tables that have been specified in the information presented above.

   (b) If a sixty-year-old patient tests positive on the PSA test, what is the probability that he has prostate cancer?

   (c) If the patient also has an enlarged prostate, what is the probability that he has prostate cancer?

   (d) If a sixty-year-old patient has an enlarged prostate, what is the probability that he has BPH?

2. **(Directed versus undirected graphical models)**
   Consider the following random variables. X_1, X_2 and X_3 represent the outcomes of three (independent) fair coin tosses. X_4 is the indicator function of the event that $X_1 = X_2$, and X_5 is the indicator function of the event that $X_2 = X_3$.

   (a) Specify a directed graphical model (give the directed acyclic graph and local conditionals) that describes the joint probability distribution.

   (b) Specify an undirected graphical model (give the graph and clique potentials) that describes the joint probability distribution.

   (c) In both cases, list any conditional independencies that are displayed by this probability distribution but are not implied by the graph.

   (d) If the coins were biased, would your answer to (2c) change?

3. **(Eliminate algorithm)**
   Consider the directed graph G_1 of Figure 1.

   (a) What is the corresponding moral graph?

   (b) What is the reconstituted graph that results from invoking the UndirectedGraphEliminate algorithm on the moral graph with the ordering $(8, 7, 2, 4, 6, 5, 3, 1)$?

   (c) What is the reconstituted graph that results from invoking the UndirectedGraphEliminate algorithm on the moral graph with the ordering $(8, 5, 6, 7, 4, 3, 2, 1)$?

   (d) Suppose you wish to use the Eliminate algorithm to calculate $p(x_1|x_8)$. (Suppose that each $X_i \in \{0, 1\}$ and that the local conditionals do not exhibit any special symmetries.) Which of the orderings listed in (3b) and (3c) is preferable? Why?

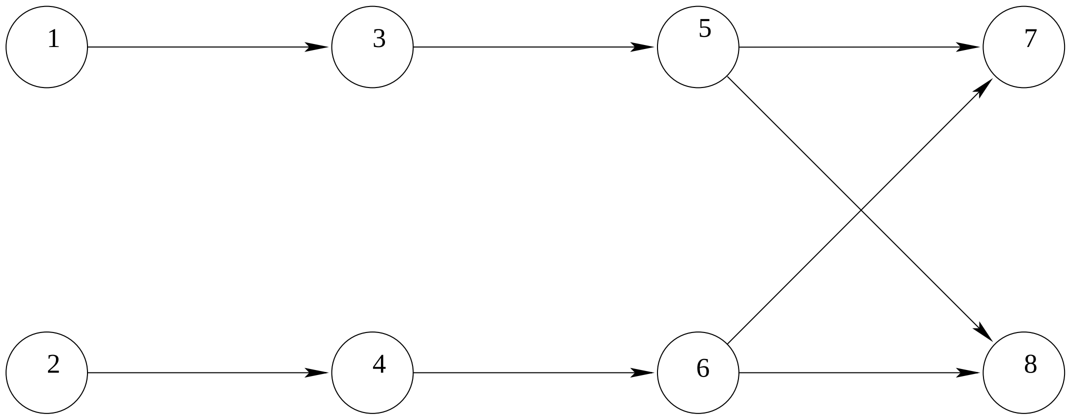
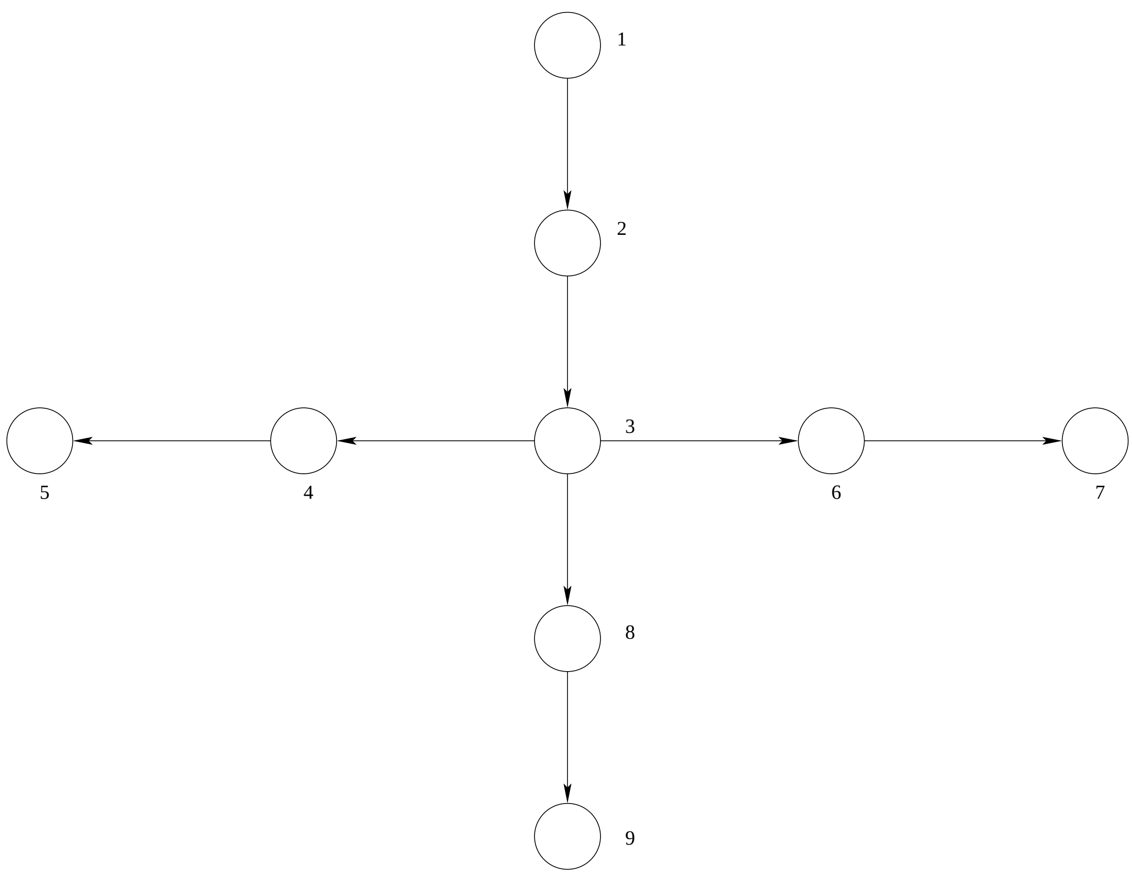


Figure 1: A directed graph, G_1

4. **(Conditional independence and DAGs)**

(a) For some DAGs, we can reverse an edge to give a DAG that gives identical conditional independence assertions. Consider the following DAGs. In each case, find a maximal sequence of edges to reverse so that the corresponding sequence of DAGs are all distinct and imply the same conditional independence assertions.

i.



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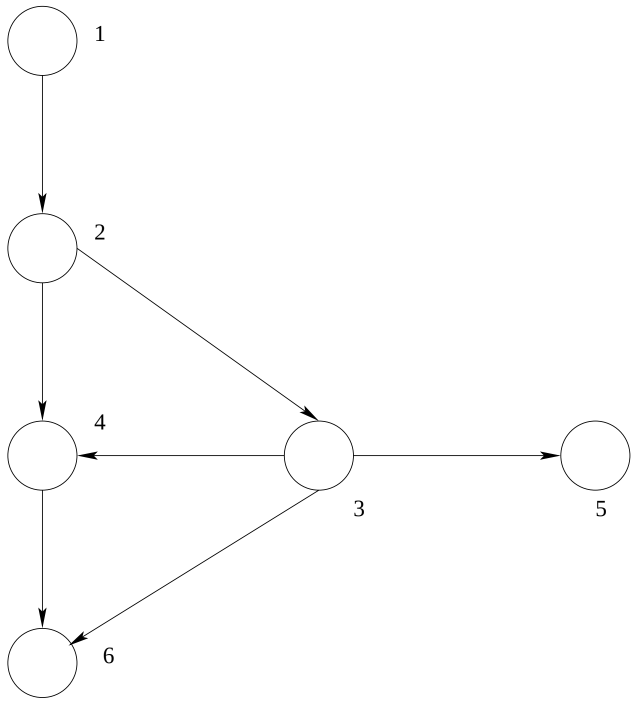
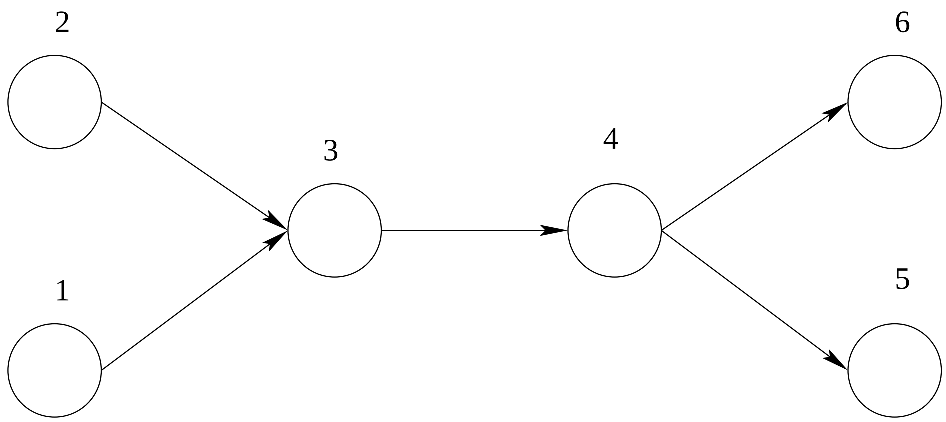


iii.



(b) Say that an edge e in a DAG G is reversible if, when we reverse its orientation, we obtain a DAG G' with identical conditional independence properties. Show that reversibility of an edge $e = (i, j)$ is equivalent to a simple property of the parents π_i, π_j of the vertices incident to the edge.

(c) Show that two DAGs G, G' imply the same conditional independence assertions iff there is a sequence of DAGs $G = G_0, G_1, \ldots, G_n = G'$ so that G_t and G_{t+1} differ only in the reversal of one reversible edge.

(d) Suppose we have two DAGs G, G' that imply the same conditional independence assertions. Write (as pseudocode) an algorithm that takes as input G, G' and produces a sequence of edges $e_1, \ldots, e_n$ so that

- $G_0 = G$.
- G_{t+1} is obtained from G_t by reversing e_{t+1}.
- $G_n = G'$.
- Each e_{t+1} is reversible in G_t, and hence all of the DAGs $G_0, G_1, \ldots, G_n$ imply the same conditional independence assertions.

Show that your algorithm works correctly.