CS276: Cryptography September 16, 2015

PRGs ⇒ PRFs, Pseudorandom Permutations, and Feistel Permutations

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1 Introduction

Today we will constuct PRFs (Pseudorandom Functions) from PRGs (Pseudorandomness Generators). We will go on to define Pseudorandom and Strongly Pseudorandom Permutations. We end by introducing an example of a strongly psuedorandom permutation through Fiestel Permutations.

2 Constructing PRFs from PRGs

We begin by recalling the following definition and property from the previous class. The latter will be instrumental in our proof to follow.

Definition 1 *PRF $\mathcal{F}$ is* **pseudorandom** *if $\forall$ppt D:*

$$|\Pr\left[D^{\mathcal{F}_k}(1^k)=1\right]-\Pr\left[D^{\mathcal{U}_k}(1^k)=1\right]|$$

is negligible in k.

Proposition 2 *For a PRG G acting on inputs of size k with expansion factor l, we have:*

$$G(U_k^1)...G(U_k^p)\cong U_{pl(k)}$$

We now begin the construction. Let G be a PRG with expansion factor $l(k)=2k$. Denote G_0, G_1 as the first,last k bits of G's output, respectively (i.e. $G(s)=G_0(s)||G_1(s)$ where $|s|=|G_0(s)|=|G_1(s)|=k$).

Now, define we will define our pseudorandom function $\mathcal{F}$ as follows. Let f_s be defined by:

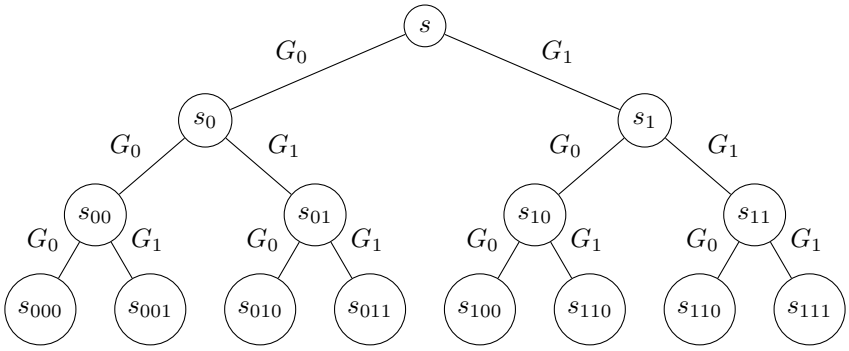
$$\forall s\in\{0,1\}^k\ f_s(x_1...x_k):=G_{x_k}(G_{x_{k-1}}(...G_{x_1}(s)...))$$

And so:

$$\mathcal{F}=\{\mathcal{F}_k\}_{k\in\mathbb{N}} \text{ with } \mathcal{F}_k=\{f_s|s\leftarrow\{0,1\}^k\}$$

Theorem 3 *$\mathcal{F}$ is a PRF.*

Proof: We begin by pictorally representing the f_s. We can think of f_s as a labelled tree of k levels with the root being labelled as s. For a node s_m at level i, it has two children in level $i+1$; the left child is labelled with the result of applying G_0 on s_m and the right child is labelled with the result of appying G_1 on s_m. The leaves of this tree are then the outputs of f_s. We see this in the following example for $k=4$:



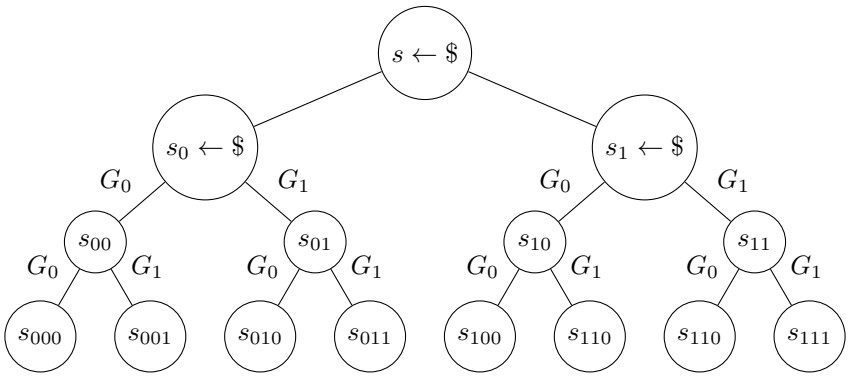
For example, the leaf s_{011} is associated with output of the call:

$$f_s(011) = G_1(G_1(G_0(s))) = G_1(G_1(s_0)) = G_1(s_{01}) = s_{011}$$

Now, we proceed with the proof. Assume (towards contradiction) that there exists PPT D such that D distinuishes $\mathcal{F}$ from $\mathcal{U}$ with probability $\delta(k)$ non-negligible. We proceed by a hybrid argument in order to create a distinguisher for G.

Remark 4 *It is tempting to use the leaves of the tree to construct the hybrid. Namely, we would fill a number of slots with randomness, and fill the remaining slots with the leaves from tree above. However, we quickly see that there are 2^k leaves, so the hybrids would need to be at least 2^k in size which is far too large. Instead, we consider the following argument.*

In light of the above, we will consider hybrids of *levels* in the tree instead of hybrids of leaves. More specifically, $H_k^{(i)}$ is defined by selecting the first i levels in the tree uniformly at random from $\mathcal{U}_k$ (i.e. for any $r < i$, we have that the value associated with the node s_r is produced from (re)sampling from $\mathcal{U}_k$ for a total of $|r|$ times). For all levels greater than i, we apply G_0 or G_1 to the result from the previous level as described in the tree above. Pictorally, can observe the example $H_3^{(1)}$ as represented as below ($s_r \leftarrow \$$ means "sample from $\mathcal{U}_k$):



We can now make the following observation.

Observation 5 $H_k^{(0)} = \mathcal{F}_k, H_k^{(k)} = \mathcal{U}_K$

Using the standard procedure for a hybrid argument, we note that there exists i such that D distinguishes $H_k^{(i)}$ and $H_k^{(i+1)}$ with probability greater than $\frac{\delta(k)}{k}$ for some $\delta(k) \in \text{poly}(k)$.

We now would like to construct a D' that attacks G (actually, in this case, G applied on q inputs of length k) and succeeds with non-negligible probability. We first note that D asks at most $q(k)$ queries. On input $z^{(1)}, ..., z^{(q)}$, D' will first pick $j \leftarrow \{1, 2, ..., q\}$. D' will then run D and simulate its oracle by using $z^{(j)}$ to answer the queries which D dispatches. We outline how $z^{(j)}$ is used to answer the queries from D below.

Given some query q_i, D' will answer by using the following procedure based on traversing the tree mentioned at the beginning of this proff. The first j levels of the trees will be traversed according to the first j bits of q_i. More specifically, for $t < j$, D' takes the left branch from the t-th level if the t-th bit in q_i is 0 and take the right branch otherwise. From the j-th level to the $(j+1)$-th level, we label the two children of the j-th node (i.e. the children are at the $(j+1) - th$ level) with $z^{(j)}$ (or $z^{(j+l)}$ if we had already performed $l-1$ labellings at this level) if the pair of children at the $(j+1)$-th level are unlabelled. Otherwise, we simply use the labelling already provided. We then use this string when computing the values going downards in the tree and where the paths are selected by the remaining $k-j$ bits in q_i. For example, if $j = 1$ and $q_1 = 0110$, we would traverse down the left branch and then the right brach in the tree (corresponding to 01 in q_1). If there is no labelling on the children of this node, we would then label them as $z^{(j)}$. From this point, we can return the result of the query as $G_0(G_1(z^{(j)}))$ because the last 2 bits of q_1 are 10. Then for the query $q_2 = 0110$, we hit the same labelling at level $j+1$, so we return $G_0(G_0(z^{(j)}))$ as the answer to q_2.

Now, from here we see that:

$$z^{(j)} \sim \mathcal{U}_{ql(k)} \Rightarrow D' \text{ sees } H_k^{(i+1)}$$
$$z^{(j)} \sim G(\mathcal{U}_k) \Rightarrow D' \text{ sees } H_k^{(i)}$$

And by the hybrid argument and Property 2, above, the theorem follows. □

3 Pseudorandom Permutations

Now we turn our attention to pseudorandom permutations.

Definition 6 *A* **permutation ensemble** *is an ensemble $\mathcal{P} = \{\mathcal{P}_k\}_k$ where $\mathcal{P}_k$ is a distribution over permutations $f : \{0,1\}^k \to \{0,1\}^k$.*

Definition 7 *The* **uniform permutation ensemble** *is the ensemble $\Pi = \{\Pi_k\}_k$ where Π_k is uniform over permutations $f : \{0,1\}^k \to \{0,1\}^k$.*

Let $A \stackrel{O}{=} B$ mean "A is oracle indistinguishable with respect to B".

Definition 8 *$\mathcal{P}$ is* **pseudorandom** *if $\mathcal{P} \stackrel{O}{=} \Pi$.*

Lemma 9 $\mathcal{P} \stackrel{O}{=} \Pi \Rightarrow \mathcal{P} \stackrel{O}{=} \mathcal{U}$

Proof: (Sketch) We note that $\Pi \stackrel{O}{=} \mathcal{U}$ because:

$$\Pr\left[D^{\mathcal{U}_k}(1^k) \text{ finds a collision}\right] \leq \frac{time(D)^2}{2^k}$$

From here, we can apply indistinugishability between $\mathcal{P}$ and Π, and the lemma follows. □

Now we will examine the properties required for permutations to be useful in a cryptographic sense.

Definition 10 *A permutation ensemble $\mathcal{P}$ is* **efficiently computable and invertable** *if:*

(1) $\exists ppt\ S$ *such that* $S(1^k) = P_k$

(2) $\exists ppt\ E$ *such that* $\forall f \in S(1^k)$, $E(1^k, f, x) = f(x)$

(3) $\exists dpt\ I$ *such that* $\forall f \in S(1^k)$, $I(1^k, f, y) = f^{-1}(y)$

Now we get to a notion of pseudorandomness for permutations that is stronger than the ones defined earlier.

Definition 11 *$\mathcal{P}$ is* **strongly pseudorandom** *if:*

$$(\mathcal{P}, \mathcal{P}^{-1}) \stackrel{O}{=} (\Pi, \Pi^{-1})$$

and $\forall ppt\ D$,

$$|\Pr\left[D^{\mathcal{P}_k, \mathcal{P}_k^{-1}}(1^k) = 1\right] - \Pr\left[D^{\Pi_k, \Pi_k^{-1}}(1^k) = 1\right]| \text{ is negligible in } k$$

4 Feistel Permutation

We will soon construct a strongly pseudorandom permutation. We begin with the construction of Feistel Permutations, as follows.

Given a function $f : \{0,1\}^k \to \{0,1\}^k$, let $g_f : \{0,1\}^{2k} \to \{0,1\}^{2k}$ be defined by:

$$g_f(x||y) = y||f(y) \oplus x$$

We note that g_f is indeed a permutation. For we have:

$$g_f^{-1}(z||w) = f(z) \oplus w||z$$

and we see that:

$$g_f^{-1}(g_f(x||y)) = g_f^{-1}(y||f(y) \oplus x) = f(y) \oplus (f(y) \oplus x)||y = x||y$$

It should also be clear that both g_f and g_f^{-1} are efficiently computable given f. g_f is called a **Feistel Permutation**. Compositions of g_f's for selections of f's are called **Feistel Networks**.

Now, to get a strong pseudorandom permutation we compose g_f's (for different f's). However, we make the following observations.

Observation 12 *g_f is* **not** *a pseudorandom permutation. We see this by the fact that:*

$$g_f(x||0^k) = 0^k||f(0^k) \oplus x$$

for any $x \in \{0,1\}^k$.

Observation 13 *For any f_1, f_2, we have that $g_{f_2} \circ g_{f_1}$ is also* **not** *a pseudorandom permutation. We see that:*

$$(g_{f_2} \circ g_{f_1})(0^k 0^k) = f_1(0^k)||...$$

and that:

$$(g_{f_2} \circ g_{f_1})(1^k 0^k) = f_1(0^k) \oplus 1^k||...$$

But now we see that $f_1(0^k) \oplus 1^k = \overline{f_1(0^k)}$, and so the fact that $g_{f_2} \circ g_{f_1}$ is not pseudorandom follows.

Remark 14 *For any f_1, f_2, f_3, we have that $g_{f_3} \circ g_{f_2} \circ g_{f_1}$ is pseudorandom but not strongly pseudorandom. Showing this is much more difficult than for the above two cases, so it is omitted.*

5 Luby-Rackoff Construction (1988)

Consider some function ensemble $\mathcal{F} = \{\mathcal{F}_k\}_k$. Consider $\mathcal{G} = \{\mathcal{G}_k\}_k$ where:

$$\mathcal{G}_k = \{g_{f_4} \circ g_{f_3} \circ g_{f_2} \circ g_{f_1} | f_1, f_2, f_3, f_4 \in \mathcal{F}_k\}$$

Theorem 15 $\mathcal{F} \stackrel{O}{=} \mathcal{U} \Rightarrow (\mathcal{G}, \mathcal{G}^{-1}) \stackrel{O}{=} (\Pi, \Pi^{-1})$

Proof: Define $\mathcal{R} = \{R_k\}_k$ where $R_k = \{g_{u_4} \circ g_{u_3} \circ g_{u_2} \circ g_{u_1}|_1, u_2, u_3, u_4 \in \mathcal{U}_k\}$. Our proof will consist of two parts:

(S1) $(\mathcal{G}, \mathcal{G}^{-1}) \stackrel{O}{=} (\mathcal{R}, \mathcal{R}^{-1})$

(S2) $(\mathcal{R}, \mathcal{R}^{-1}) \stackrel{O}{=} (\Pi, \Pi^{-1})$

We will prove S1 now and prove S2 tomorrow.

Proof of S1 We will use a hybrid argument, so assume there exists some machine D that can distinguish $\mathcal{G}$ from $\mathcal{R}$ with non-negligible advanage. We proceed by defining 5 hybrids $H_k^{(0)}, H_k^{(1)}, ..., H_k^{(4)}$. We explicitly write out the first two hybrids:

$$H_k^{(0)} = \{g_{f_4} \circ g_{f_3} \circ g_{f_2} \circ g_{f_1} | f_1, f_2, f_3, f_4 \in \mathcal{F}_k\} = \mathcal{G}_k$$
$$H_k^{(1)} = \{g_{f_4} \circ g_{f_3} \circ g_{f_2} \circ g_{u_1} | f_2, f_3, f_4 \in \mathcal{F}_k, u_1 \in \mathcal{U}_k\}$$

The remaining hybrids are defined similarly, and we observe that $H_k^{(4)} = \mathcal{R}_k$.

Now, by an averaging argument, there exists i such that $(H_k^{(i)}, (H_k^{(i)})^{-1})$ and $(H_k^{(i+1)}, (H_k^{(i+1)})^{-1})$

we will now be able to define a machine $\overline{D}^A$ that is able to distinguish neighboring hybrids (where A is an oracle which is either $\mathcal{F}_k$ or $\mathcal{U}_k$).

For simplicity, let us define $\overline{D}^A$ for the case where $i = 2$. The cases where $i = 0, 2, 3, 4$ follow an identical construction. Let $\overline{D}^A(1^k)$ be defined by:

(1) $f_4 \leftarrow \mathcal{F}_k$

(2) $f_3 \leftarrow A$

(3) $u_2, u_1 \leftarrow \mathcal{U}_k$

(4) Construct the Feistel Network $g_{f_4} \circ g_{f_3} \circ g_{u_2} \circ g_{u_1}$

(5) Run D on the Feistel Network constructed in (4)

We can deduce that:

$$A \sim \mathcal{F}_k \Rightarrow \overline{D}^A \text{ sees } (H_k^{(i)}, (H_k^{(i)})^{-1})$$
$$A \sim \mathcal{U}_k \Rightarrow \overline{D}^A \text{ sees } (H_k^{(i+1)}, (H_k^{(i+1)})^{-1})$$

From here, it is straightforward to apply the remainder of the hybrid argument and we obtain our result (S1). □